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Energy of Vertices of Subdivision Graphs

Harishchandra S. Ramane[a,*], Sheena Y. Chowri[a], Thippeswamy Shivaprasad[a], Ivan Gutman[b]

[a] *Department of Mathematics, Karnatak University, Dharwad - 580003, India*

[b] *Faculty of Science, University of Kragujevac, 34000 Kragujevac, Serbia*

hsramane@yahoo.com, sheenachowri@gmail.com, shivap942@gmail.com, gutman@kg.ac.rs

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Abstract

The energy of a vertex v_i in a graph G is $\mathcal{E}_G(v_i) = |A|_{ii}$, where A is the adjacency matrix of G, and $|A| = (AA^*)^{1/2}$. The graph energy is then $\mathcal{E}(G) = \mathcal{E}_G(v_1) + \mathcal{E}_G(v_2) + \cdots + \mathcal{E}_G(v_n)$. In this paper we calculate the energy of vertices of some subdivision graphs.

1 Introduction

Let G be a simple graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The degree of a vertex $v \in V(G)$ is the number of edges incident to it. If each vertex of G has same degree equal to r, then G is said to be an r-regular graph. As usual, K_n denotes the complete graph on n vertices and $K_{p,q}$ denotes the complete bipartite graph on $p+q$ vertices.

The subdivision graph $S(G)$ of a graph G is obtained from G by inserting a new vertex on each edge of G. Thus if G has n vertices and m edges, then $S(G)$ has $n+m$ vertices and $2m$ edges [6].

*Corresponding author.

The adjacency matrix of a graph G is the square matrix $A = A(G) = [a_{ij}]$ of order n, in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The characteristic polynomial of a graph G is defined as

$$\phi(G : \lambda) = \det(\lambda I - A(G))\,,$$

where I is the identity matrix of order n. The eigenvalues of the adjacency matrix of G are called the eigenvalues of G and are labeled as $\lambda_1, \lambda_2, \ldots, \lambda_n$.

The energy of a graph G is defined as the sum of the absolute values of the eigenvalues of G and is denoted by $\mathcal{E}(G)$ [5]. That is

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|\,.$$

More on the graph energy can be found in [7, 8].

Recently Arizmendi and Juarez-Romero [2] introduced the energy of a vertex. According to them, the energy of a vertex v_i, denoted by $\mathcal{E}_G(v_i)$, is defined as

$$\mathcal{E}_G(v_i) = |A|_{ii}, \quad \text{for } i = 1, 2, \ldots, n\,,$$

where $|A| = (AA^*)^{1/2}$.

Given $\mathcal{E}(G) = \text{trace}(|A(G)|)$, the energy of a graph can be calculated by adding the energies of the vertices of G. That is

$$\mathcal{E}(G) = \sum_{i=1}^{n} \mathcal{E}_G(v_i). \tag{1}$$

In the paper [1], Arizmendi et al. determined the basic properties of the energy of a vertex, including some bounds. In [3] they showed that the energy of a vertex can be calculated by means of a Coulson integral formula. The energy of vertices of some graphs, such as transitive graph, complete graph, hypercube, cycle, complete bipartite graph, friendship graph, dandelion graph, and path are given in [1]. In this paper we obtain the energy of the vertices of the subdivision graph of complete graph, complete bipartite graph, cocktail party graph, and Petersen graph.

We need the following auxiliary results.

Lemma 1. [1] *Let G be a graph with n vertices. Then*

$$\mathcal{E}_G(v_i) = \sum_{j=1}^{n} p_{ij}|\lambda_j|, \qquad i = 1, \dots, n,$$

where λ_j denotes the j-th eigenvalue of the adjacency matrix A and the weights p_{ij} satisfy

$$\sum_{i=1}^{n} p_{ij} = 1 \qquad \text{and} \qquad \sum_{j=1}^{n} p_{ij} = 1.$$

Moreover $p_{ij} = u_{ij}^2$, where $U = (u_{ij})$ is the orthogonal matrix whose columns are the eigenvectors of A.

Lemma 2. [1] *Let G be a graph with n vertices. For $k \in \mathbb{N}$, let $\phi_i(A^k)$ be the k-th moment of A with respect to the linear functional ϕ_i. Then*

$$\phi_i(A^k) = \sum_{j=1}^{n} p_{ij}\lambda_j^k, \qquad i = 1, \dots, n,$$

where the notation is same as in Lemma 1.

Recall that $\phi_i(A^k)$ is equal to the number of $v_i - v_i$ walks in G of length k.

Lemma 3. [1] *Let G be a bipartite graph with partite sets V_1 and V_2. Then*

$$\sum_{u \in V_1} \mathcal{E}_G(u) = \sum_{v \in V_2} \mathcal{E}_G(v)\,.$$

Lemma 4. [4] *Let G be an r-regular graph with n vertices and m edges. Then*

$$\phi(S(G) : \lambda) = \lambda^{m-n}\, \phi(G : \lambda^2 - r)\,.$$

2 Energy of vertex

In this section we determine the energy of the vertices of the subdivision graph of complete graph, complete bipartite graph, cocktail party graph, and Petersen graph.

Theorem 1. *Let $v_1, v_2, \ldots, v_n$ be the vertices of the complete graph K_n, $n \geq 2$. If the vertex set of the subdivision graph $S(K_n)$ is*

$$\left\{v_1, v_2, \ldots, v_n, s_1, s_2, \ldots, s_{\frac{n(n-1)}{2}}\right\}$$

where s_j, $j = 1, 2, \ldots, \frac{n(n-1)}{2}$ are the subdivided vertices, then

$$\mathcal{E}_{S(K_n)}(v_i) = \frac{(n-1)\sqrt{n-2} + \sqrt{2(n-1)}}{n}, \quad i = 1, 2, \ldots, n$$

and

$$\mathcal{E}_{S(K_n)}(s_j) = \frac{2\left((n-1)\sqrt{n-2} + \sqrt{2(n-1)}\right)}{n(n-1)}, \quad j = 1, 2, \ldots, \frac{n(n-1)}{2}.$$

Proof. The characteristic polynomial of K_n is $\phi(K_n : \lambda) = (\lambda+1)^{n-1}(\lambda - (n-1))$ [4] and here $m = \frac{n(n-1)}{2}$. Therefore by Lemma 4, the characteristic polynomial of $S(K_n)$ is

$$\phi(S(K_n) : \lambda) = \lambda^{\frac{n^2-3n}{2}}(\lambda^2 - (n-2))^{n-1}(\lambda^2 - 2(n-1))\,.$$

The distinct eigenvalues of $S(K_n)$ are $\lambda_1 = 0$, $\lambda_2 = \sqrt{n-2}$, $\lambda_3 = -\sqrt{n-2}$, $\lambda_4 = \sqrt{2(n-1)}$ and $\lambda_5 = -\sqrt{2(n-1)}$.

Now we find the weights p_{11}, p_{12}, p_{13}, p_{14}, and p_{15} of the vertex v_1 of $S(K_n)$. By Lemma 2 we have following system of linear equations:

$$\begin{aligned}
p_{11} + p_{12} + p_{13} + p_{14} + p_{15} &= 1\,; \\
p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 + p_{15}\lambda_5 &= 0\,; \\
p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_2^3 + p_{14}\lambda_4^2 + p_{15}\lambda_5^2 &= n-1\,; \\
p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 + p_{15}\lambda_5^3 &= 0\,; \\
p_{11}\lambda_1^4 + p_{12}\lambda_2^4 + p_{13}\lambda_3^4 + p_{14}\lambda_4^4 + p_{15}\lambda_5^4 &= n^2 - n\,.
\end{aligned}$$

That is

$$
\begin{aligned}
&p_{11}+p_{12}+p_{13}+p_{14}+p_{15}=1\,;\\
&\sqrt{n-2}\,p_{12}-\sqrt{n-2}\,p_{13}+\sqrt{2(n-1)}\,p_{14}-\sqrt{2(n-1)}\,p_{15}=0\,;\\
&(n-2)p_{12}+(n-2)p_{13}+2(n-1)p_{14}+2(n-1)p_{15}=n-1\,;\\
&(n-2)^{3/2}\,p_{12}-(n-2)^{3/2}\,p_{13}+(2(n-1))^{3/2}\,p_{14}-(2(n-1))^{3/2}\,p_{15}=0\,;\\
&(n-2)^2\,p_{12}+(n-2)^2\,p_{13}+4(n-1)^2\,p_{14}-4(n-1)^2\,p_{15}=n^2-n\,.
\end{aligned}
$$

Solving the above system of equations we get

$$p_{11}=0\,,\,p_{12}=p_{13}=\frac{n-1}{2n}\,,\,p_{14}=p_{15}=\frac{1}{2n}\,.$$

Therefore, by Lemma 1

$$
\begin{aligned}
\mathcal{E}_{S(K_n)}(v_1) &= p_{11}|\lambda_1|+p_{12}|\lambda_2|+p_{13}|\lambda_3|+p_{14}|\lambda_4|+p_{15}|\lambda_5|\\
&= (0)|0|+\frac{n-1}{2n}\left|\sqrt{n-2}\right|+\frac{n-1}{2n}\left|-\sqrt{n-2}\right|\\
&+ \frac{1}{2n}\left|\sqrt{2(n-1)}\right|+\frac{1}{2n}\left|-\sqrt{2(n-1)}\right|\\
&= \frac{(n-1)\sqrt{n-2}+\sqrt{2(n-1)}}{n}.
\end{aligned}
$$

This means that by symmetry, for all vertices v_i, $i=1,2,\ldots,n$,

$$\mathcal{E}_{S(K_n)}(v_i)=\frac{(n-1)\sqrt{n-2}+\sqrt{2(n-1)}}{n}$$

holds.

Since $S(K_n)$ is a bipartite graph, by Lemma 3,

$$
\begin{aligned}
\sum_{j=1}^{\frac{n(n-1)}{2}}\mathcal{E}_{S(K_n)}(s_j) &= \sum_{i=1}^{n}\mathcal{E}_{S(K_n)}(v_i)\\
\sum_{j=1}^{\frac{n(n-1)}{2}}\mathcal{E}_{S(K_n)}(s_1) &= \sum_{i=1}^{n}\mathcal{E}_{S(K_n)}(v_1)\\
\frac{n(n-1)}{2}\mathcal{E}_{S(K_n)}(s_1) &= n\,\mathcal{E}_{S(K_n)}(v_1)
\end{aligned}
$$

$$\begin{aligned}\mathcal{E}_{S(K_n)}(s_1) &= \frac{2}{n-1}\,\mathcal{E}_{S(K_n)}(v_1)\\ &= \frac{2\left((n-1)\sqrt{n-2}+\sqrt{2(n-1)}\right)}{n(n-1)}\,.\end{aligned}$$

Again, this implies that for all vertices s_j, $j = 1, 2, \ldots, \frac{n(n-1)}{2}$,

$$\mathcal{E}_{S(K_n)}(s_j) = \frac{2\left((n-1)\sqrt{n-2}+\sqrt{2(n-1)}\right)}{n(n-1)}$$

holds. ■

Theorem 2. *Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of the complete bipartite graph $K_{n,n}$. If the vertex set of the subdivision graph $S(K_{n,n})$ is $\{v_1, v_2, \ldots, v_{2n}, s_1, s_2, \ldots, s_{n^2}\}$, where s_j, $j = 1, 2, \ldots, n^2$, are the subdivided vertices, then*

$$\mathcal{E}_{S(K_{n,n})}(v_i) = \frac{n-1}{\sqrt{n}} + \frac{1}{\sqrt{2n}}\,, \quad i = 1, 2, \ldots, 2n$$

and

$$\mathcal{E}_{S(K_{n,n})}(s_j) = \frac{2}{n}\left(\frac{n-1}{\sqrt{n}} + \frac{1}{\sqrt{2n}}\right), \quad j = 1, 2, \ldots, n^2.$$

Proof. The characteristic polynomial of $K_{n,n}$ is $\phi(K_{n,n} : \lambda) = \lambda^{2n-2}(\lambda^2 - n^2)$ [4] and here $m = n^2$. Therefore by Lemma 4, the characteristic polynomial of $S(K_{n,n})$ is

$$\phi(S(K_{n,n}) : \lambda) = \lambda^{n^2-2n+2}(\lambda^2 - n)^{2n-2}(\lambda^2 - 2n)\,.$$

The distinct eigenvalues of $S(K_{n,n})$ are $\lambda_1 = 0$, $\lambda_2 = \sqrt{n}$, $\lambda_3 = -\sqrt{n}$, $\lambda_4 = \sqrt{2n}$, and $\lambda_5 = -\sqrt{2n}$.

Now we find the weights p_{11}, p_{12}, p_{13}, p_{14} and p_{15} of the vertex v_1. By Lemma 2, we have the following system of linear equations:

$$\begin{aligned}p_{11}+p_{12}+p_{13}+p_{14}+p_{15} &= 1\,;\\ \sqrt{n}p_{12}-\sqrt{n}p_{13}+\sqrt{2n}p_{14}-\sqrt{2n}p_{15} &= 0\,;\\ np_{12}+np_{13}+2np_{14}+2np_{15} &= n\,;\end{aligned}$$

$$\begin{aligned} n\sqrt{n}p_{12} - n\sqrt{n}p_{13} + 2n\sqrt{2n}p_{14} - 2n\sqrt{2n}p_{15} &= 0\,; \\ n^2p_{12} + n^2p_{13} + 4n^2p_{14} + 4n^2p_{15} &= n^2 + n\,. \end{aligned}$$

Solving the above system of equations, we get $p_{11} = \frac{1}{2n}$, $p_{12} = p_{13} = \frac{n-1}{2n}$ and $p_{14} = p_{15} = \frac{1}{4n}$. Therefore by Lemma 1

$$\begin{aligned} \mathcal{E}_{S(K_{n,n})}(v_1) &= p_{11}|\lambda_1| + p_{12}|\lambda_2| + p_{13}|\lambda_3| + p_{14}|\lambda_4| + p_{15}|\lambda_5| \\ &= \frac{1}{2n}|0| + \frac{n-1}{2n}\left|\sqrt{n}\right| + \frac{n-1}{2n}\left|-\sqrt{n}\right| \\ &+ \frac{1}{4n}\left|\sqrt{2n}\right| + \frac{1}{4n}\left|-\sqrt{2n}\right| \\ &= \frac{n-1}{\sqrt{n}} + \frac{1}{\sqrt{2n}}\,. \end{aligned}$$

Then by symmetry, also for all other vertices v_i, $i = 2, 3, \ldots, 2n$,

$$\mathcal{E}_{S(K_{n,n})}(v_i) = \frac{n-1}{\sqrt{n}} + \frac{1}{\sqrt{2n}}\,.$$

Since $S(K_{n,n})$ is a bipartite graph, by Lemma 3,

$$\begin{aligned} \sum_{j=1}^{n^2} \mathcal{E}_{S(K_{n,n})}(s_j) &= \sum_{i=1}^{2n} \mathcal{E}_{S(K_{n,n})}(v_i) \\ \sum_{j=1}^{n^2} \mathcal{E}_{S(K_{n,n})}(s_1) &= \sum_{i=1}^{2n} \mathcal{E}_{S(K_{n,n})}(v_1) \\ n^2\,\mathcal{E}_{S(K_{n,n})}(s_1) &= 2n\,\mathcal{E}_{S(K_n)}(v_1) \\ \mathcal{E}_{S(K_n)}(s_1) &= \frac{2}{n}\,\mathcal{E}_{S(K_n)}(v_1) = \frac{2}{n}\left(\frac{n-1}{\sqrt{n}} + \frac{1}{\sqrt{2n}}\right). \end{aligned}$$

Then also for all other vertices s_j, $j = 2, 3, \ldots, n^2$,

$$\mathcal{E}_{S(K_{n,n})}(s_j) = \frac{2}{n}\left(\frac{n-1}{\sqrt{n}} + \frac{1}{\sqrt{2n}}\right)$$

holds. ■

A regular graph H of degree $n-2$ with $n = 2k$, $k \geq 2$, vertices is called cocktail party graph. That is a graph obtained from the complete graph K_{2k} by removing one factor [4]. In a fully analogous manner as Theorems 1 and 2, we can prove:

Theorem 3. *Let* $v_1, v_2, \ldots, v_n$ *be the vertices of the cocktail party graph* H*, where* $n = 2k$*,* $k \geq 2$*. If the vertex set of the subdivision graph* $S(H)$ *is* $\left\{v_1, v_2, \ldots, v_n, s_1, s_2, \ldots, s_{\frac{n(n-2)}{2}}\right\}$*, where* s_j*,* $j = 1, 2, \ldots, \frac{n(n-2)}{2}$*, are the subdivided vertices, then*

$$\mathcal{E}_{S(H)}(v_i) = \frac{k\sqrt{2k-2} + (k-1)\sqrt{2k-4} + \sqrt{4k-4}}{2k}$$

for $i = 1, 2, \ldots, n$*, and*

$$\mathcal{E}_{S(H)}(s_j) = \frac{k\sqrt{2k-2} + (k-1)\sqrt{2k-4} + \sqrt{4k-4}}{2(k^2-k)}$$

for $j = 1, 2, \ldots, n(n-2)/2$*.*

The Petersen graph is the complement of the line graph of K_5.

Theorem 4. *Let* $\{v_1, v_2, \ldots, v_{10}\}$ *be the vertex set of the Petersen graph* P*, and* $\{v_1, v_2, \ldots, v_{10}, s_1, s_2, \ldots, s_{15}\}$ *be the vertex set of its subdivision graph* $S(P)$*. Then*

$$\mathcal{E}_{S(P)}(v_i) \approx 1.64494, \quad i = 1, 2, \ldots, 10$$

and

$$\mathcal{E}_{S(P)}(s_j) \approx 1.09662, \quad j = 1, 2, \ldots, 15\,.$$

Proof. The characteristic polynomial of the Petersen graph P is [4]

$$\phi(P : \lambda) = (\lambda - 3)(\lambda + 2)^4(\lambda - 1)^5.$$

Therefore, by Lemma 4, the characteristic polynomial of $S(P)$ is

$$\phi(S(P) : \lambda) = \lambda^5(\lambda^2 - 6)(\lambda^2 - 1)^4(\lambda^2 - 4)^5.$$

The distinct eigenvalues of $S(P)$ are $\lambda_1 = 0$, $\lambda_2 = \sqrt{6}$, $\lambda_3 = -\sqrt{6}$, $\lambda_4 = 1$, $\lambda_5 = -1$, $\lambda_6 = 2$, and $\lambda_7 = -2$.

Let p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16} and p_{17} be the weights of the vertex v_1 in $S(P)$. Then by Lemma 2, we have the following system of linear equations:

$$\begin{aligned}
p_{11}+p_{12}+p_{13}+p_{14}+p_{15}+p_{16}+p_{17} &= 1\,;\\
\sqrt{6}\,p_{12}-\sqrt{6}\,p_{13}+p_{14}-p_{15}+2p_{16}-2p_{17} &= 0\,;\\
6p_{12}+6p_{13}+p_{14}+p_{15}+4p_{16}+4p_{17} &= 3\,;\\
6\sqrt{6}\,p_{12}-6\sqrt{6}\,p_{13}+p_{14}-p_{15}+8p_{16}-8p_{17} &= 0\,;\\
36\,p_{12}+36\,p_{13}+p_{14}+p_{15}+16\,p_{16}-16\,p_{17} &= 12\,;\\
36\sqrt{6}\,p_{12}-36\sqrt{6}\,p_{13}+p_{14}-p_{15}+32\,p_{16}-32\,p_{17} &= 0\,;\\
216\,p_{12}+216\,p_{13}+p_{14}+p_{15}+64\,p_{16}+64\,p_{17} &= 54\,.
\end{aligned}$$

Solving these equations we get

$$p_{11}=0\,,\; p_{12}=p_{13}=0.05\,,\; p_{14}=p_{15}=0.2\,,\; p_{16}=p_{17}=0.25\,.$$

Therefore by Lemma 1

$$\begin{aligned}
\mathcal{E}_{S(P)}(v_1) &= p_{11}|\lambda_1|+p_{12}|\lambda_2|+p_{13}|\lambda_3|+p_{14}|\lambda_4|+p_{15}|\lambda_5|\\
&+ p_{16}|\lambda_6|+p_{17}|\lambda_7|\\
&= (0)|0|+(0.05)\left|\sqrt{6}\right|+(0.05)\left|-\sqrt{6}\right|+(0.2)|1|+(0.2)|-1|\\
&+ (0.25)|2|+(0.25)|-2|\approx 1.64494
\end{aligned}$$

implying that $\mathcal{E}_{S(P)}(v_i)\approx 1.64494$ holds for all $i=1,2,\ldots,10$.

Since $S(P)$ is a bipartite graph, by Lemma 3,

$$\begin{aligned}
\sum_{j=1}^{15}\mathcal{E}_{S(P)}(s_j) &= \sum_{i=1}^{10}\mathcal{E}_{S(P)}(v_i)\\
\sum_{j=1}^{15}\mathcal{E}_{S(P)}(s_1) &= \sum_{i=1}^{10}\mathcal{E}_{S(P)}(v_1)\\
(15)\mathcal{E}_{S(P)}(s_1) &= (10)\mathcal{E}_{S(P)}(v_1)\\
\mathcal{E}_{S(P)}(s_1) &\approx \frac{10}{15}\cdot 1.64494 = 1.09662
\end{aligned}$$

implying that $\mathcal{E}_{S(P)}(s_j) \approx 1.09662$ holds for all $j = 1, 2, \ldots, 15$. ■

3 Energy of subdivision graphs

Using Eq. (1) and the results of Theorems 1–4, we get the energy of subdivison graph of complete graph, complete bipartite graph, cocktail party graph and of Petersen graph as follows.

$$\begin{aligned}
\mathcal{E}(S(K_n)) &= \sqrt{2(n-1)}\left[2+\sqrt{2(n-1)(n-2)}\right] ;\\
\mathcal{E}(S(K_{n,n})) &= 4\sqrt{n}\left[n-1+\frac{1}{\sqrt{2}}\right] ;\\
\mathcal{E}(S(H)) &= 2\sqrt{2(k-1)}\left[k+\sqrt{(k-1)(k-2)}+\sqrt{2}\right] ;\\
\mathcal{E}(S(P)) &\approx 32.8987 .
\end{aligned}$$

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