MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **93** (2025) 567–574
ISSN: 0340–6253
doi: 10.46793/match.93-2.567T

Maximum Bond Incident Degree Indices of Trees with Given Independence Number

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(Received May 24, 2024)

Abstract

The bond incident degree (BID) indices $\mathcal{T}_f(G)$ of a connected graph G with edge-weight function $f(x,y)$ are defined as

$$\mathcal{T}_f(G) = \sum_{v_i v_j \in E(G)} f(d(v_i), d(v_j)),$$

where $f(x,y) > 0$ is a symmetric real function with $x \geq 1$ and $y \geq 1$ and $d(u)$ is the degree of vertex u in G. In this paper, we prove that extremal tree of order n with given independence number s $(n/2 \leq s \leq n-1)$ having maximum bond incident degree indices $\mathcal{T}_f$ is the spur graph $S_{n,s}$ if edge-weight symmetric function $f(x,y)$ satisfies three conditions: $f(x,y)$ is strictly increasing on x (or y); $f(x,y) - f(x,y-1)$ is increasing on x (or y); $\varphi(x+1,y-1) \geq \varphi(x,y)$ for every $x, y \geq 2$, where $\varphi(x,y) = f(x,y) - f(x-1,y)$.

1 Introduction and notation

The bond incident degree (BID) indices denoted $\mathcal{T}_f(G)$ of a connected graph G with edge-weight function $f(x,y)$ were defined in [2, 9] as

$$\mathcal{T}_f(G) = \sum_{v_i v_j \in E(G)} f(d(v_i), d(v_j)),$$

where $f(x,y) > 0$ is a symmetric real function with $x \geq 1$ and $y \geq 1$.

In this setting extremal results for trees were given by Ali and Dimitrov [1]. Liu, You, Chen and Huang [6] used a unified method to characterize the first two maximum and the first two minimum trees with respect to BID indices, respectively and Gao [4] determined the trees with maximum BID indices, by imposing some general conditions on the edge-weight function $f(x,y)$. Different general conditions were proposed also by Hu, Li, Li, Peng [5], Yao, Liu, Belardo, Yang [10] and Vetrik [8] for deducing extremal graphs in several classes of graphs. In this paper we shall consider edge-weight functions $f(x,y)$ which fulfil the following conditions:

i) $f(x,y)$ is strictly increasing on x (or y);

ii) $f(x,y) - f(x,y-1)$ is increasing on x (or y);

iii) $\varphi(x+1,y-1) \geq \varphi(x,y)$ for every $x,y \geq 2$, where $\varphi(x,y) = f(x,y) - f(x-1,y)$.

The maximum vertex degree of G is denoted by $\Delta(G)$ and $N(u)$ is the set of vertices adjacent with u. The distance between vertices u and v of a connected graph is the length of a shortest path between them. The diameter of G is the maximum distance between vertices of G. If $x \in V(G)$, $G - x$ denotes the subgraph of G obtained by deleting x and its incident edges. A similar notation is $G - xy$, where $xy \in E(G)$. Given a graph G, a subset S of $V(G)$ is said to be an independent set of G if every two vertices of S are not adjacent. The maximum number of vertices in an independent set of G is called the independence number of G and is denoted by $\alpha(G)$. $K_{1,n-1}$ and P_n will denote, respectively, the star and the path on n vertices. Since a tree on n vertices is a bipartite graph, at least one partite set, which is an independent set, has at least $n/2$ vertices, which implies that for any tree T we have $\alpha(T) \geq \lceil n/2 \rceil$ and this bound is reached for example for paths. Also, $\alpha(T) \leq n-1$ and the equality holds only for the star graph. For every $n \geq 2$ and $n/2 \leq s \leq n-1$ the spur $S_{n,s}$ [3] is a tree consisting of $2s - n + 1$ edges and $n - s - 1$ paths of length 2 having a common endvertex; it is obtained from a star $K_{1,s}$ by attaching a pendant edge to $n - s - 1$ pendant vertices of $K_{1,s}$. We have

$\alpha(S_{n,s}) = s$. The bond incident degree index $\mathcal{T}_f(S_{n,s})$ is

$$\mathcal{T}_f(S_{n,s}) = (2s - n + 1)f(s, 1) + (n - s - 1)(f(s, 2) + f(2, 1)). \quad (1)$$

Let $\mathcal{T}_{n,s}$ be the set of trees of order n having independence number s. Note that $\mathcal{T}_{n,n-1} = \{S_{n,n-1} = K_{1,n}\}$ and $\mathcal{T}_{n,n-2} = \{S_{n,n-2}\}$.

2 Main result

The following observation will be useful.

Lemma 2.1 [7] Let T be a tree and $x \in V(T)$, which is adjacent to pendant vertices $v_1, \dots v_r$. If $r \geq 2$ then any maximum independent subset of $V(T)$ contains $v_1, \dots, v_r$.

We shall state a preliminary result which will be used in the proof of the main result of this paper.

Lemma 2.2. If ii) and iii) hold, then we have $f(p, 1) \geq f(p-1, 2)$ for every $p \in N, p \geq 3$.

Proof. For $x = y = 2$ iii) yields $f(3, 1) - f(2, 1) \geq f(2, 2) - f(1, 2)$, or $f(3, 1) \geq f(2, 2)$ since $f(1, 2) = f(2, 1)$. Let $p \geq 4$ and suppose that the property holds for $p - 1$. Then, for $x = p - 1$ and $y = 2$ from iii) we get $f(p, 1) - f(p-1, 1) \geq f(p-1, 2) - f(p-2, 2)$, which implies $f(p, 1) - f(p-1, 2) \geq f(p-1, 1) - f(p-2, 2) \geq 0$ by the induction hypothesis. ■

Theorem 2.3. Let $n \geq 5$, $n/2 \leq s \leq n-1$ and $T \in \mathcal{T}_{n,s}$. If edge-weight function $f(x, y)$ satisfies i)–iii) then $\mathcal{T}_f(T)$ is maximum if and only if $T = S_{n,s}$.

Proof. For $n = 5$ we have two possible values for s: $s = 3$, when $\mathcal{T}_{5,3} = \{P_5, S_{5,3}\}$ and $s = 4$ when $\mathcal{T}_{5,4} = \{S_{5,4} = K_{1,4}\}$. For $s = 3$ we get $\mathcal{T}_f(S_{5,3}) = f(3, 2) + 2f(3, 1) + f(2, 1) > \mathcal{T}_f(P_5) = 2f(2, 2) + 2f(2, 1)$ since this is equivalent to

$$f(3, 2) + 2f(3, 1) > 2f(2, 2) + f(2, 1). \quad (2)$$

By Lemma 2.2 we get $f(3,1) \geq f(2,2)$. By i) we also have $f(3,2) > f(2,1)$, which proves (2). For $s = 4$ $\mathcal{T}_{5,4}$ has a unique member $S_{5,4} = K_{1,4}$, therefore it is extremal.
We shall use induction on n. Let $n \geq 6$ and suppose that the property holds for all trees of order $n-1$ and independence number s with $(n-1)/2 \leq s \leq n-2$. Let T be a tree of order n and independence number s and a path $v_1, v_2, \ldots, v_{d+1}$ of length d in T, where d is the diameter of T. We can suppose that $d \geq 3$ since if we have $d = 2$ it follows $s = n-1$, $T = K_{1,n-1} = S_{n,n-1}$ and the theorem is verified. v_1 and v_{d+1} are pendant vertices of T, hence $T - v_1$ is a tree of order $n-1$.
We shall consider two cases: A. $\alpha(T - v_1) = \alpha(T) = s$ and B. $\alpha(T - v_1) = \alpha(T) - 1 = s - 1$.

A. When $\alpha(T - v_1) = \alpha(T)$ we deduce that $\lceil (n-1)/2 \rceil \leq s \leq n-2$ unless $s = n-1$. In this case $T = K_{1,n-1} = S_{n,n-1}$ and we are done. Let $s \leq n-2$. By Lemma 2.1 we deduce $d(v_2) = 2$ and let $d(v_3) = d_3 \leq \Delta(T) \leq s$.
By the induction hypothesis we get

$$\mathcal{T}_f(T) = \mathcal{T}_f(T - v_1) + f(2,1) + f(d_3,2) - f(d_3,1)$$

$$\leq (2s - n + 2)f(s,1) + (n-s-2)f(s,2) + (n-s-2)f(2,1) + f(2,1)$$

$$+f(d_3,2) - f(d_3,1)$$

using (1). Since $d_3 \leq s$ by ii)we have $f(d_3,2) - f(d_3,1) \leq f(s,2) - f(s,1)$.

It follows that $\mathcal{T}_f(T) \leq (2s-n+1)f(s,1) + (n-s-1)f(s,2) + (n-s-1)f(2,1) = \mathcal{T}_f(S_{n,s})$ and equality holds if and only if $T - v_1 = S_{n-1,s}, d(v_2) = 2$ and $d_3 = s$, which implies that $T = S_{n,s}$.

B. Next we assume that $\alpha(T - v_1) = \alpha(T) - 1 = s - 1$. We have $\lceil (n-1)/2 \rceil \leq s - 1 \leq n-2$ unless $n = 2k, k \in N$ and $s = k$. It follows that $T \in \mathcal{T}_{2k,k}$ and $T - v_1 \in \mathcal{T}_{2k-1,k-1} = \emptyset$ and this case does not apply. Since $v_1, v_2, v_3, \ldots, v_{d+1}$ is a path of maximum length of T, v_3 is the only vertex in $N(v_2)$ having degree $d_3 \geq 2$. By letting $d(v_2) = d_2 \leq s$ we have

$$\mathcal{T}_f(T) = \mathcal{T}_f(T - v_1) + f(d_2, 1) + (d_2 - 2)(f(d_2, 1) - f(d_2 - 1, 1)) \\ + f(d_3, d_2) - f(d_3, d_2 - 1). \quad (3)$$

Condition (ii) implies that the function $(x - 2)(f(x, 1) - f(x - 1, 1))$ is strictly increasing in x for $x \geq 2$, so we get $(d_2 - 2)(f(d_2, 1) - f(d_2 - 1, 1) \leq (s - 2)(f(s, 1) - f(s - 1, 1))$ and equality holds only for $d_2 = s$.We also deduce that $f(d_2, 1) \leq f(s, 1)$. v_2 is adjacent to $d_2 - 1$ pendant vertices and in the graph $T - v_2v_3$ the degree of v_3 is equal to $d_3 - 1$. Each edge v_3x, where $x \neq v_2$ belongs to a path in T ending in a pendant vertex and these vertices are pairwise different. It follows that $d_2 - 1 + d_3 - 1$ is less than or equal to the number of pendant vertices of T, which implies $d_2 - 1 + d_3 - 1 \leq s$, or $d_2 + d_3 \leq s + 2$. We shall prove that

$$f(d_3, d_2) - f(d_3, d_2 - 1) \leq f(s, 2) - f(s - 1, 2) \quad (4)$$

if $d_2 + d_3 \leq s + 2$ and $d_2, d_3 \geq 2$. Indeed, there exist natural numbers x, y such that $f(d_3, d_2) - f(d_3, d_2 - 1) \leq f(x, y) - f(x, y - 1)$, where $x, y \geq 2$ and $x + y = s + 2$ by increasing d_2 or d_3 if $d_2 + d_3 < s + 2$ since ii) holds. Since $f(x, y) - f(x, y - 1) = \varphi(x, y)$, by successively applying iii) we get $f(d_3, d_2) - f(d_3, d_2 - 1) \leq f(s, 2) - f(s, 1)$. By Lemma 2.2 we have $f(s, 1) \geq f(s - 1, 2)$ and (4) is proved.

By the induction hypothesis and (1) we have $\mathcal{T}_f(T - v_1) \leq \mathcal{T}_f(S_{n-1,s-1}) = (2s - n)f(s - 1, 1) + (n - s - 1)(f(s - 1, 2) + f(2, 1))$. Thus we get $\mathcal{T}_f(T) \leq (2s - n)f(s - 1, 1) + (n - s - 1)(f(s - 1, 2) + f(2, 1)) + f(s, 1) + (s - 2)(f(s, 1) - f(s - 1, 1)) + f(s, 2) - f(s - 1, 2) = (s - 1)f(s, 1) + (n - s - 2)f(s - 1, 2) - (n - s - 2)f(s - 1, 1) + f(s, 2) + (n - s - 1)f(2, 1)$. By denoting the last expression by $E(n, s)$, the inequality $E(n, s) \leq \mathcal{T}_f(S_{n,s})$ is equivalent to

$$(n - s - 2)(f(s, 2) - f(s - 1, 2)) \geq (n - s - 2)(f(s, 1) - f(s - 1, 1)). \quad (5)$$

If $s = n - 1$ then $T = K_{1,n-1} = S_{n,n-1}$ and the theorem is true. If $s = n - 2$ then (5) becomes an equality, $T = S_{n,n-2}$ and the theorem is verified. Otherwise $n - s - 2 \geq 1$ or $s \leq n - 3$ and (5) is equivalent to

$f(s,2) - f(s-1,2) \geq f(s,1) - f(s-1,1)$. From ii) it follows that this inequality is valid. If $s \leq n-3$ a necessary condition that $\mathcal{T}_f(T) = \mathcal{T}_f(S_{n,s})$ is that $T - v_1 = S_{n-1,s-1}$. Since $d_2 = s$ it follows that v_1 is adjacent with the center of the star $K_{1,s}$. The inequality $n-s-1 \geq 2$ implies that there exist at least two paths of length two ending in the center of $K_{1,s}$, subgraph of $S_{n,s}$. This yields a contradiction, since we have supposed that the path $v_1, v_2, \ldots, v_{d+1}$ has maximum length in T, but this path has length three. Consequently, if $\alpha(T - v_1) = \alpha(T) - 1$ we have $\mathcal{T}_f(T) \leq \mathcal{T}_f(S_{n,s})$ and the equality holds only if $s = n-1$, when $T = K_{1,n-1} = S_{n,n-1}$ and $s = n-2, T - v_1 = S_{n-1,n-3}, d_2 = s$ and $d_2 + d_3 = s+2$, i. e., $d_3 = 2$ and $T = S_{n,n-2}$. It follows that the equality in case B holds only if $T \in \{S_{n,n-1}, S_{n,n-2}\}$ and the proof is complete.
Notice that only if $n-s-1 \in \{0,1\}$ a pendant vertex adjacent to the center of the star $K_{1,s}$ is the endvertex of a longest path in $S_{n,s}$, and this corresponds to the equality in (5). ■

3 Concluding remarks

We proved that the spur graph $S_{n,s}$ is the unique tree of order n with given independence number s having maximum bond incident degree indices $\mathcal{T}_f(S_{n,s})$ if edge-weight symmetric function $f(x,y)$ satisfies conditions i)-iii).

Several BID indices studied in the literature have edge-weight functions $f(x,y)$ which fulfil conditions i)-iii):

- The function $f(x,y) = (x+y)^\alpha$ has properties i)-iii) for $\alpha \geq 1$. The general sum-connectivity index of a graph G is defined as $\chi_\alpha(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^\alpha$, where α is a real number. For $\alpha = 1$ we obtain the first Zagreb index, and for $\alpha = 2$ we get the first hyper-Zagreb index. Das, Xu and Gutman [3] proved that in the class of trees of order n and independence number s, the spur $S_{n,s}$ maximizes both first and second Zagreb indices and this graph is unique with these properties. By extending this result, the author and Jamil [7] showed that in the same class of trees $\mathcal{T}_{n,s}$, $S_{n,s}$ is the unique graph maximizing general sum-connectivity index $\chi_\alpha(T)$ for $\alpha \geq 1$.

- The function $f(x, y) = g(x) + g(y)$, where $g(x)$ is strictly increasing and convex also satisfies conditions i)-iii); in particular $f(x, y) = x^\alpha + y^\alpha$ for $\alpha \geq 1$ satisfies these conditions. In this case condition iii) is reduced to Jensen' s inequality for convex functions.

- The edge-weight symmetric function $f(x, y) = e^{x+y}$ fulfil mentioned conditions. The corresponding BID index is called the exponential first Zagreb index.

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