MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **93** (2025) 349–378
ISSN: 0340–6253
doi: 10.46793/match.93-2.349H

Local Stability, Period-Doubling and 1:2 Resonance Bifurcation for a Discretized Chemical Reaction System

Sabina Hrustić[a], Samra Moranjkić[a]*, Zehra Nurkanović[a]

[a] *University of Tuzla, Department of Mathematics, Tuzla, 75000, Bosnia and Herzegovina*

`sabina.hrustic@untz.ba`, `samra.moranjkic@untz.ba`, `zehra.nurkanovic@untz.ba`

(Received June 14, 2024)

Abstract

We investigate a discrete counterpart of planar dynamical system of nonlinear differential equations induced by kinetic differential equations for a two-species chemical reaction. Chemical reactions exhibit a wide range of dynamical behavior. We show how the theoretical analysis provides insight into the potential behavior of chemical reaction systems, determining the areas of parametric space which indicate scenarios for local stability, then for one type of bifurcation co-dimension one and one type of bifurcation co-dimension two. Precisely, we prove the existence of period-doubling bifurcation and 1:2 resonance bifurcation also, by using the center manifold theorem and the technique of normal forms. All mathematical investigations are illustrated with numerical examples, bifurcation diagrams, Lyapunov exponents and phase portraits.

*Corresponding author.

1 Introduction

Over the past four decades the field of nonlinear chemical dynamics has grown significantly. Its applications includes all branches of chemistry as well as areas of mathematics, physics, biology and engineering. In a nonlinear system dynamics is determined by behavior of some key variables, like concentrations, temperature or pressure in a chemical reaction. The possible behaviors of system of chemical reactions vary wildly; there are systems that have a single steady state for all choices of rate constants, systems that have multiple steady states, systems that oscillate and systems that admit chaotic behavior.

Many authors investigated dynamics of system of chemical reactions [3, 11–13, 18]. The first chemical reaction that exhibited temporal oscillations was discovered by Bray in [2] a hundred years ago. Some experimental work has discovered a number of interesting, real systems of chemical oscillators [1, 14, 35]. In 1979, Schnakenberg in [30] proposed a system showing sustained oscillations for a simple model of glycolysis, a metabolic process that converts glucose to provide energy for metabolism.

The first experimental demonstrations of chaos in a chemical system were made on the Belousov-Zhabotinsky (BZ) reaction (see [31]). Chaotic behavior typically emerges from periodic oscillation as a control parameter is varied, often by a period-doubling route, but also by Neimark-Sacker bifurcation, snap-back repellers or bifurcations co-dimension two (see [6, 9, 15, 16, 19–21, 26]). Bifurcations can have significant implications in two-species chemical reaction kinetics, as it can lead to sudden shifts in reaction rates, product distributions, and overall system behavior. Understanding and predicting bifurcations in these types of systems are crucial in fields such as chemical engineering, optimizing industrial processes, designing control strategies to stabilize chemical reactors, enhancing process efficiency, designing more effective drugs with improved pharmacokinetic properties, creating novel materials with specific structures and properties, deciphering complex physiological processes, such as neuronal signaling or gene expression regulation which can lead to new approaches for diagnosing and treating diseases. Very interesting practical implications of bifurcations in industrial chemical processes and biological systems can be found in [10] where Chen *et al.* investigate their impact on novel approaches for the production of the clean fuels: hydrogen and ethanol, and the simulation of the acetylcholine neurocycle in the brain. Kim *et al.* in [22] reported a chaotic model incorporating measurable state variables less than the degrees of freedom of the model and the system was identified with the artificial neural networks.

A typical characteristic for a chemical reaction system is the diffusion phenomenon because of the movements of reactants. Considering this fact, the systems with diffusion are realistic. Reaction-diffusion models have been used over decades to study biological systems. Two concentrations of two substances in some process involving diffusion and interaction, can represent a chemical reaction or interacting populations, for example a predator and its prey. Turing in [32] suggests reaction-diffusion models to explain pattern formation in biological systems for example, mammalian skin spots and stripes, fish skin patterns, snow flakes, and many others. Noufaey in [27] investigated semi-analytical solutions for Schnakenberg system with a reaction–diffusion cell. Yi *et al.* in [33] studied formation for Turing pattern under the influence of diffusion and delay in Schnakenberg type model. Hence, two-species chemical reaction systems are important because they provide a simple and well-understood model system for the study of chemical dynamics, which can be applied to a wide range of real-world problems.

As arguing in [4], where authors give a method to construct many kinds of systems with two internal components leading to a limit cycle, two-species chemical reaction, described by A and B, is given as follows:

$$O \overset{k_1}{\to} B,\ A \overset{k_2}{\to} O,\ B \overset{k_3}{\to} A,\ 2A \overset{k_4}{\to} B,\ 2A + B \overset{k_5}{\to} 3A \tag{1}$$

where O represents the environment and k_1, k_2, k_3, k_4 and k_5 are positive numbers representing reaction rate coefficients. The rate constants k_i, $i = 1, 2, 3, 4, 5$ are additional data that need to be specified in order to have a well-defined state of the system. The schematic diagram of two-species chemical reaction network (1) is visualized in Figure 1. Considering

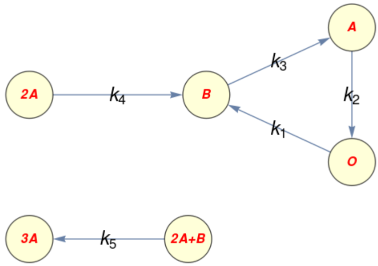


Figure 1. Diagram for chemical reaction network (1)

chemical reaction network (1) and implementing the law of mass action, one has the following two-dimensional induced dynamical system:

$$\begin{aligned} \frac{dx}{dt} &= -k_2x - 2k_4x^2 + k_3y + k_5x^2y \\ \frac{dy}{dt} &= k_1 + k_4x^2 - k_3y - k_5x^2y, \end{aligned} \tag{2}$$

where x and y are dimensionless concentrations of species A and B, respectively. To reduce the number of free parameters while still allowing chaotic dynamics, we set the rate constants $k_2 = a$, $k_3 = b$, $k_1 = c$ and $k_4 = k_5 = 1$. Then system (2) has the following form:

$$\begin{aligned} \frac{dx}{dt} &= -ax - 2x^2 + by + x^2y \\ \frac{dy}{dt} &= c + x^2 - by - x^2y, \end{aligned} \tag{3}$$

where parameters $a, b, c > 0$.

In this paper we analyzed discrete version of model (3) and proved existence of period-doubling bifurcation and 1:2 resonance bifurcation as a borderline case between period-doubling and Naimark-Sacker bifurcation which was examined in [8]. Din *et al.* in [8] analyzed also continuous model (2) where authors examined existence and direction of Hopf bifurcation about positive equilibrium and carried out bifurcation control.

The novel contributions of this paper:

- Period-doubling bifurcation is a complex phenomenon that occurs when a small change in a parameter leads to a large change in the behavior of a dynamical system. It occurs when a stable periodic orbit, such as a limit cycle, loses stability and undergoes a bifurcation, resulting in the emergence of a new periodic orbit with twice the period of the original orbit. This process can repeat itself indefinitely, leading to a cascade of period-doubling bifurcations that can create a fractal structure known as a bifurcation diagram which is an essential tool for understanding the complex behavior of nonlinear systems.

- The discretization of the model is important for performing numerical simulations and analyses. Discretization methods allow us to transform the continuous differential equations into discrete equations that can be solved numerically. This simplifies the study of system's behavior, bifurcations, stability, and other properties using computational techniques.

- In the chemical reaction model, co-dimension-two bifurcations play a significant role for understanding the complex dynamics of the system. Co-dimension-two bifurcations imply the simultaneous occurrence of two different types of bifurcations, resulting in a higher level of complexity in the system's behavior. 1:2 resonance refer to specific relationships between the frequencies of two or more oscillatory components in the system.

The rest of this paper is organized as follows: in Section 2 we conduct the discretization of the system of nonlinear differential equations and investigate a local stability of interior equilibrium point. In Section 3 and 4, we determine parametric conditions for period-doubling bifurcation and 1:2 resonance bifurcation at the fixed point of a two-dimensional map associated to system (5), respectively. Finally, in Section 5, some numerical examples are presented in order to illustrate the theoretical discussions. We chose some parameters to illustrate the dynamics in the neighborhood of 1:2 resonance point, which implies that there exist a period-doubling bifurcation curve and Naimark-Sacker bifurcation curve which intersect at 1:2 resonance point. All visualizations are generated by using Mathematica Wolfram and Dynamica 3.0 (see [23]). We also calculated Lyapunov exponents based on the computational algorithm in [29] since Lyapunov exponents represent good way to test the sensitive dependence, i.e. confirmation of the chaos.

2 Discretization of the system

Discretization is an important tool for studying chemical reaction systems, as it allows the application of mathematical and computational methods to analyze the system. Discretization can also reduce the dimensionality of the system in some cases, making them more manageable for examination and control. Techniques like finite difference or finite element methods, commonly used in simulation, become applicable after discretization. This approach not only reduces computational costs but also enables the use of standard numerical integration techniques. Numerical simulations do not generate continuous curves.

There are several ways to conduct the discretization. One way is to use consistency preserving discretization with nonstandard finite difference scheme as in [8]. But for problems we planned to deal with, much more suitable is Euler's discretization with standard finite difference scheme since it significantly simplifies the system we observe. So, we propose a standard finite difference scheme for discretization of the system (3) as follows

$$\begin{aligned} \frac{x_{n+1}-x_n}{h} &= -ax_n - 2x_n^2 + by_n + x_n^2 y_n, \\ \frac{y_{n+1}-y_n}{h} &= c + x_n^2 - by_n - x_n^2 y_n, \end{aligned} \tag{4}$$

where $0 < h < 1$ is step size for discretization and $a, b, c > 0$. So we have the following system:

$$\begin{aligned} x_{n+1} &= x_n + h(-ax_n - 2x_n^2 + by_n + x_n^2 y_n) \\ y_{n+1} &= y_n + h(c + x_n^2 - by_n - x_n^2 y_n). \end{aligned} \tag{5}$$

Corresponding map associated to the system (5) is of the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + h\left(-ax - 2x^2 + by + x^2 y\right) \\ y + h\left(c + x^2 - by - x^2 y\right) \end{pmatrix}. \tag{6}$$

In order to find equilibrium points of system (5), we solve the following system:

$$\begin{aligned} -X^2Y + aX - bY + 2X^2 &= 0 \\ -c + X^2Y + bY - X^2 &= 0. \end{aligned}$$

The unique positive equilibrium $E(X,Y) = \left(X, \frac{X^2+c}{X^2+b}\right)$ of system (5) is given as

$$E = \left(\frac{-a + \sqrt{a^2+4c}}{2}, \frac{\left(a^2+4c\right)(b+c) + a\,(c-b)\sqrt{a^2+4c}}{2\left(a^2 b + (b+c)^2\right)} \right). \tag{7}$$

Jacobian matrix of the map T at the equilibrium point is given by

$$J_T(h) = \begin{pmatrix} (2XY - 4X - a)\,h + 1 & h\left(X^2+b\right) \\ -2hX\,(Y-1) & -h\left(X^2+b\right)+1 \end{pmatrix},$$

and the corresponding characteristic equation is of the form

$$\phi(\lambda) = \lambda^2 - TrJ_T(h)\lambda + DetJ_T(h) = 0. \tag{8}$$

In order to study the modulus of eigenvalues of the characteristic equation (8) at the positive equilibrium point $E(X,Y)$, we first give the following Lemma, which can be easily proved by the relations between roots and coefficients of the quadratic equation (see [7,25]).

Lemma 1. *Assume that* $\phi(\lambda) = \lambda^2 - TrJ_T\lambda + DetJ_T$ *is a polynomial associated to characteristic equation. Suppose that* $\phi(1) > 0$ *and* λ_1 *and* λ_2 *are two roots of* $\phi(\lambda) = 0$. *Then*

a) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *iff* $\phi(-1) > 0$ *and* $\phi(0) < 1$.

b) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *iff* $\phi(-1) > 0$ *and* $\phi(0) > 1$.

c) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*) iff* $\phi(-1) < 0$.

d) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2| = 1$ *iff* $(TrJ_T)^2 - 4DetJ_T < 0$ *and* $DetJ_T = 1$.

e) $\lambda_1 = -1$ *and* $\lambda_2 \neq -1$ *iff* $\phi(-1) = 0$ *and* $-TrJ_T \neq 2$,

f) $\lambda_1 = \lambda_2 = -1$ *iff* $\phi(-1) = 0$ *and* $-TrJ_T = 2$.

Recall now some definitions of topological types for an equilibrium point (x, y). If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, an equilibrium point (x, y) is called a sink. As we know, a sink is locally asymptotically stable. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, an equilibrium point (x, y) is called a source which is locally unstable. If $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$) an equilibrium point (x, y) is called a saddle. And, finally if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$, an equilibrium point (x, y) is called a non-hyperbolic.

Lemma 2. *Let* a, b, $c, h > 0$ *and equilibrium point* $E(X, Y)$ *of system* (5) *given by* (7)*. Then the positive equilibrium point* E *is:*

I) *a source for* $a + b + 4X - 2XY + X^2 \leq 0$,

II) ***1.*** *if* $\left(a + b + 4X - 2XY + X^2\right)^2 > 4\left(b + X^2\right)(a + 2X)$:

- **a)** *a sink if and only if* $h \in (0, h_1)$,
- **b)** *a saddle point if and only if* $h \in (h_1, h_2)$,
- **c)** *a source if and only if* $h \in (h_2, +\infty)$,
- **d)** *a non-hyperbolic point with* $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *for* $h = h_1$ *or* $h = h_2$,

2. *if* $\left(a + b + 4X - 2XY + X^2\right)^2 = 4\left(b + X^2\right)(a + 2X)$:

- **a)** *a sink if and only if* $h \in (0, h_t)$,
- **b)** *a source if and only if* $h \in (h_t, +\infty)$,
- **c)** *a non-hyperbolic point with* $\lambda_{1,2} = -1$ *for* $h = h_t = h_m$,

3. *if* $0 < \left(a + b + 4X - 2XY + X^2\right)^2 < 4\left(b + X^2\right)(a + 2X)$:

- **a)** *a sink if and only if* $h \in (0, h_t)$,
- **b)** *a source if and only if* $h \in (h_t, +\infty)$,

c) *a non-hyperbolic point with* $\lambda_{1,2} \in \mathbb{C}$, $|\lambda_{1,2}| = 1$ *for* $h = h_t < h_m$,

where

$h_{1,2} = \frac{(a+b+4X-2XY+X^2) \pm \sqrt{(a+b+4X-2XY+X^2)^2 - 4(b+X^2)(a+2X)}}{(b+X^2)(a+2X)}$,

$h_t = \frac{a+b+4X-2XY+X^2}{(b+X^2)(a+2X)}$ *and* $h_m = \frac{4}{a+b+4X-2XY+X^2}$.

Proof. For λ_1 and λ_2 as a roots of characteristic equation (8), it holds

$$TrJ_T(h) = -h\left(a+b+4X-2XY+X^2\right)+2,$$

and

$$DetJ_T(h) = h^2(a+2X)\left(b+X^2\right) - h\left(a+b+4X-2XY+X^2\right)+1.$$

Let us denote $(TrJ_T(h))^2 - 4DetJ_T(h) = h^2\mathcal{K}$. Then

$$\lambda_{1,2} = 1 - h\frac{\left(a+b+4X-2XY+X^2\right) \pm \sqrt{\mathcal{K}}}{2},$$

where

$$\mathcal{K} = \left(a+b+4X-2XY+X^2\right)^2 - 4\left(b+X^2\right)(a+2X). \tag{9}$$

Now, $\phi(1) = h^2(2X+a)\left(X^2+b\right) > 0$ and

$$\phi(-1) = h^2(a+2X)\left(X^2+b\right) - 2h\left(a+b+4X-2XY+X^2\right)+4,$$

$$\phi(-1) = 0 \Leftrightarrow h_{1,2} = \frac{\left(a+b+4X-2XY+X^2\right) \pm \sqrt{\mathcal{K}}}{(b+X^2)(a+2X)}. \tag{10}$$

Notice that $h_{1,2} \in \mathbb{R}$ for $\mathcal{K} \geq 0$ and $0 < h_1 \leq h_2$. Also,

$$\phi(0) = h^2(a+2X)\left(b+X^2\right) - h\left(a+b+4X-2XY+X^2\right)+1,$$

$$\phi(0) = 1 \Leftrightarrow h = h_t = \frac{a+b+4X-2XY+X^2}{(a+2X)(b+X^2)}. \tag{11}$$

If $a+b+4X-2XY+X^2 \leq 0$, then obviously $\phi(-1) > 0$ and $\phi(0) > 1$ which proves part **I**. Suppose now that $a+b+4X-2XY+X^2 > 0$.

Therefore, if $\mathcal{K} > 0$, it directly follows that $\phi(-1) > 0$ and $\phi(0) < 1$ implies $h \in (0, h_1)$, $\phi(-1) > 0$ and $\phi(0) > 1$ implies $h \in (h_2, +\infty)$, $\phi(-1) < 0$ implies $h \in (h_1, h_2)$ and $\phi(-1) = 0$ for $h = h_1$ or $h = h_2$ which completes the proof of the part **II 1**.

If $\mathcal{K} = 0$, then $h_{1,2} = h_t = \frac{a+b+4X-2XY+X^2}{(a+2X)(b+X^2)} = \frac{4}{a+b+4X-2XY+X^2} = h_m$ and from (10) and (11) is obviously $\lambda_{1,2} = -1$, so the conclusions for the part **II 2**. follow immediately.

If $\mathcal{K} < 0$, then $\lambda_{1,2}$ are complex-conjugate. Since $\phi(0) = 1$ implies $h = h_t = \frac{a+b+4X-2XY+X^2}{(a+2X)(b+X^2)}$, then $|\lambda_1| = |\lambda_2| = 1$ for $h = h_t$ and the part **II 3.** is proved. ■

Let us convert the system (5) into the system with equilibrium point at origin by using the translation $u_n = x_n - X, v_n = y_n - Y$. Then the map (6) can be transformed as

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u + h\left(u^2 v - u\left(a - 2X\left(Y-2\right)\right) + u^2\left(Y-2\right) + v\left(X^2+b\right) + 2Xuv\right) \\ v - h\left(u^2 v + u^2\left(Y-1\right) + v\left(X^2+b\right) + 2Xu\left(Y-1\right) + 2Xuv\right) \end{pmatrix}. \tag{12}$$

In this paper we will consider bifurcation of the system (5) in equilibrium point $E(X, Y)$ when conditions **II 1. d)** and **II 2. c)** of Lemma 2 are satisfied. The case **II 3. c)** is considered in [8].

3 Period-doubling bifurcation

Let us consider now the case **II 1. d)** in Lemma 2 since it implies occurrence of a period-doubling bifurcation. For $h = h_1$ or $h = h_2$ equilibrium point is non-hyperbolic with eigenvalues

$$\lambda_1 = -1 \text{ and } \lambda_2 = 3 - h_1\left(a + b + 4X - 2XY + X^2\right) \neq \pm 1.$$

Since h is a parameter of the discretization of the system (5), we will choose smaller value for further consideration. Hence, as a step of discretization, we will use

$$h_1 = \frac{\left(a+b+4X-2XY+X^2\right) - \sqrt{\mathcal{K}}}{(b+X^2)(a+2X)}, \quad \mathcal{K} > 0$$

for $\mathcal{K}$ defined in (9). Let us define the curve $\mathcal{C}_{PDB}$ where exists period-doubling bifurcation:

$$\mathcal{C}_{PDB} = \{(a,b,c,h) : a,b,c > 0 \ \wedge \ h = h_1 \ \wedge \ \mathcal{K} > 0\}. \tag{13}$$

Now, system (12) can be written in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \quad J_T(h_1 + \widehat{h}) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} P_1(u,v) \\ P_2(u,v) \end{pmatrix} \tag{14}$$

where $\widehat{h}$ is very small perturbation in h_1 and

$$\begin{aligned} P_1(u,v) &= (h_1 + \widehat{h})\left(u^2 v + u^2 (Y-2) + 2Xuv\right), \\ P_2(u,v) &= -(h_1 + \widehat{h})\left(u^2 v + u^2 (Y-1) + 2Xuv\right). \end{aligned}$$

Suppose that $(a,b,c,h) \in \mathcal{C}_{PDB}$. In order to make the normal form of system (14), we consider an invertible matrix

$$\Phi = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}$$

where $a_{11} = 1 - h_1 (a - 2X (Y-2))$ and $a_{12} = h_1 \left(X^2 + b\right)$. The basic idea of normal form theory is to employ successive, near identity nonlinear transformations to eliminate, the so called, non-resonant nonlinear terms, and retaining the terms which cannot be eliminated (called resonant terms) to form the normal form and which is sufficient for the study of qualitative behavior of the original system. Hence, considering $h_1(a + 4X - 2XY) = 3 - \lambda_2 - h_1(b + X^2)$, we get

$$\Phi = \begin{pmatrix} h_1 \left(X^2 + b\right) & h_1 \left(X^2 + b\right) \\ 1 - \lambda_2 - h_1 \left(b + X^2\right) & 2 - h_1 \left(b + X^2\right) \end{pmatrix}$$

and

$$\Phi^{-1} = \frac{1}{\lambda_2 + 1} \begin{pmatrix} -\frac{bh_1 + X^2 h_1 - 2}{h_1 (b + X^2)} & -1 \\ \frac{\lambda_2 + bh_1 + X^2 h_1 - 1}{h_1 (b + X^2)} & 1 \end{pmatrix}.$$

Furthermore, taking into account the following similarity transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} h_1\left(X^2+b\right)(s+t) \\ \left(1-\lambda_2\right)s+2t-h_1(s+t)\left(b+X^2\right) \end{pmatrix},$$

system (14) takes the form

$$\Phi \begin{pmatrix} s \\ t \end{pmatrix} = J_T(h_1+\widehat{h})\Phi \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} P_1(s,t) \\ P_2(s,t) \end{pmatrix}.$$

It implies

$$\begin{pmatrix} s \\ t \end{pmatrix} = \Phi^{-1} J_T(h_1+\widehat{h})\Phi \begin{pmatrix} s \\ t \end{pmatrix} + \Phi^{-1} \begin{pmatrix} P_1(s,t) \\ P_2(s,t) \end{pmatrix}.$$

Since

$$\begin{aligned} \Phi^{-1} J_T(h_1+\widehat{h})\Phi &= \begin{pmatrix} -1-\frac{2}{h_1}\widehat{h} & 0 \\ 0 & \lambda_2+\frac{\lambda_2-1}{h_1}\widehat{h} \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} -\frac{2\widehat{h}}{h_1} & 0 \\ 0 & \frac{\lambda_2-1}{h_1}\widehat{h} \end{pmatrix}, \end{aligned}$$

we get

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} f\left(s,t,\widehat{h}\right) \\ g\left(s,t,\widehat{h}\right) \end{pmatrix},$$

where

$$\begin{aligned} f\left(s,t,\widehat{h}\right) &= -\frac{2}{h_1}\widehat{h}s + \frac{\left(h_1+\widehat{h}\right)(s+t)}{\lambda_2+1}G_1, \\ g\left(s,t,\widehat{h}\right) &= \frac{(\lambda_2-1)}{h_1}\widehat{h}t + \frac{\left(h_1+\widehat{h}\right)(s+t)}{\lambda_2+1}G_2, \end{aligned}$$

and

$$\begin{aligned} G_1 =& -h_1^2\left(b+X^2\right)^2\left(2(s+t)-1\right)(s+t) \\ & -2h_1\left(b+X^2\right)(s+t)\left(2X-Y+s\left(\lambda_2-1\right)-2(t-1)\right) \\ & +4X\left(\left(1-\lambda_2\right)s+2t\right), \end{aligned}$$

$$
\begin{aligned}
G_2 = & - h_1^2 \left(b + X^2\right)^2 (s+t)\left((\lambda_2 - 1)(s+t) + 1\right) \\
& - h_1 \left(b + X^2\right)(\lambda_2 - 1)(s+t)\left(2X - Y + s(\lambda_2 - 1) - 2(t-1)\right) \\
& + 2X(\lambda_2 - 1)\left((1 - \lambda_2) s + 2t\right).
\end{aligned}
$$

Due to center manifold theory [17, 24], stability analysis of equilibrium $(u, v) = (0, 0)$ near $\widehat{h} = 0$ can be discussed by investigating reduced equations on a center manifold $\mathcal{W}^C(0,0,0)$

$$
\mathcal{W}^C(0,0,0) = \left\{ \left(s, t, \widehat{h}\right) \in \mathbb{R}_+^3 : t = \mathcal{M}\left(s, \widehat{h}\right), \mathcal{M}(0, \widehat{h}) = 0, D\mathcal{M}\left(s, \widehat{h}\right) = 0 \right\}.
$$

We assume that the center manifold can be approximated by

$$
\mathcal{M}\left(s, \widehat{h}\right) = \mathcal{A}s^2 + \mathcal{B}s\widehat{h} + \mathcal{C}\widehat{h}^2 + \mathcal{D}s^3 + \mathcal{E}s^2\widehat{h} + \mathcal{F}s\widehat{h}^2 + \mathcal{G}\widehat{h}^3,
$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ are unknown coefficients. Then must hold

$$
\mathcal{M}\left(-s + f\left(s, \mathcal{M}\left(s, \widehat{h}\right), \widehat{h}\right), \widehat{h}\right) - \lambda_2 \mathcal{M}\left(s, \widehat{h}\right) - g\left(s, \mathcal{M}\left(s, \widehat{h}\right), \widehat{h}\right) = 0,
$$

hence

$$
\begin{aligned}
& \tfrac{S_{03}}{\lambda_2 + 1} s^3 + \tfrac{S_{12}}{h_1(\lambda_2 + 1)} \widehat{h} s^2 - \tfrac{S_{02}}{\lambda_2 + 1} s^2 - \tfrac{S_{21}}{h_1(\lambda_2 + 1)} \widehat{h}^2 s \\
& - \tfrac{(\lambda_2 - 1)(\mathcal{C} + \mathcal{G} h_1)}{h_1} \widehat{h}^3 - \mathcal{C}(\lambda_2 - 1)\widehat{h}^2 - \mathcal{B}(\lambda_2 + 1)\widehat{h}s = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
S_{03} = & - \mathcal{D}(\lambda_2 + 1)^2 + 2\mathcal{A}h_1(\lambda_2 + 1)\left(h_1\left(b + X^2\right)(2X - Y + 2) + X(\lambda_2 - 1)\right) \\
& + h_1^2\left(b + X^2\right)(\lambda_2 - 1)\left(h_1\left(b + X^2\right) + \lambda_2 - 1\right), \\
S_{12} = & - \mathcal{E}h_1(\lambda_2 - 1)(\lambda_2 + 1) - \mathcal{A}(\lambda_2 + 1)(\lambda_2 - 5) \\
& + h_1\left(h_1^2\left(b + X^2\right)^2 + (\lambda_2 - 1)(2X - Y + 2)h_1\left(b + X^2\right) + 2X(\lambda_2 - 1)^2\right), \\
S_{02} = & \mathcal{A}\left(\lambda_2^2 - 1\right) - h_1^3\left(b + X^2\right)^2 - 2h_1 X(\lambda_2 - 1)^2 \\
& - h_1^2(\lambda_2 - 1)(2X - Y + 2)\left(b + X^2\right), \\
S_{21} = & \mathcal{B}(\lambda_2 + 1)^2 + \mathcal{F}h_1(\lambda_2 + 1)^2 - 2\mathcal{C}h_1^4\left(b + X^2\right)^2 \\
& - 2\mathcal{C}h_1^2\left(h_1\left(b + X^2\right)(\lambda_2 - 1)(2X - Y + 2) + X(\lambda_2 - 1)(\lambda_2 - 3)\right).
\end{aligned}
$$

It implies the following system

$$S_{03}=0, S_{12}=0, S_{02}=0, S_{21}=0,$$
$$(\lambda_2-1)(\mathcal{C}+\mathcal{G}h_1)=0, \mathcal{C}(\lambda_2-1)=0, \mathcal{B}(\lambda_2+1)=0.$$

Now we have $\mathcal{B}=\mathcal{C}=\mathcal{G}=\mathcal{F}=0$,

$$\mathcal{D}=h_1\frac{2(\lambda_2+1)\left(h_1\left(b+X^2\right)(2X-Y+2)+X(\lambda_2-1)\right)\mathcal{A}+h_1\left(b+X^2\right)(\lambda_2-1)\left(h_1\left(b+X^2\right)+\lambda_2-1\right)}{(\lambda_2+1)^2},$$

and for $h_1\neq\frac{2(2X+2-Y)}{5X^2+2aX+b}$, $\lambda_2=3-h_1(a+b+4X-2XY+X^2)$, remaining coefficients can be written as

$$\mathcal{E}=\frac{4\mathcal{A}}{(\lambda_2-1)\,h_1}\neq 0,$$

and

$$\mathcal{A}=\frac{h_1^2\left(b+X^2\right)\left(h_1\left(5X^2+2aX+b\right)+2\left(Y-2X-2\right)\right)}{\lambda_2^2-1}\neq 0.$$

Further, it is satisfied

$$\mathcal{M}\left(0,\widehat{h}\right)=0,$$

so the center manifold can be described as follows

$$\mathcal{W}^C(0,0,0)=\left\{\left(s,t,\widehat{h}\right)\in\mathbb{R}_+^3: t=\mathcal{A}s^2+\mathcal{D}s^3+\mathcal{E}s^2\widehat{h}+o\left(\left(|s|+|\widehat{h}|\right)^4\right)\right\}.$$

Now, we restrict our system to the center manifold:

$$\digamma\left(s,\widehat{h}\right)=-s+f\left(s,\mathcal{M}\left(s,\widehat{h}\right),\widehat{h}\right)=-s+\delta\widehat{h}s+\gamma s^2+o\left(\left(|s|+\left|\widehat{h}\right|\right)^3\right).$$

After some calculations, we get

$$\delta=-\frac{2}{h_1}, \gamma=\frac{\left[\left(5X^2+2aX+b\right)h_1+2(Y-2X-2)\right]\left(b+X^2\right)h_1^2}{4-h_1(a+b+4X-2XY+X^2)}.$$

Now we have

$$\frac{\partial\digamma(0,0)}{\partial\widehat{h}}=0, \frac{\partial^2\digamma(0,0)}{\partial s\partial\widehat{h}}=\delta=-\frac{2}{h_1}\neq 0,$$

$$\frac{\partial^2 F(0,0)}{\partial s^2} = 2\gamma = 2\frac{\left[\left(5X^2+2aX+b\right)h_1+2(Y-2X-2)\right]\left(b+X^2\right)h_1^2}{4-h_1(a+b+4X-2XY+X^2)}.$$

Hence, $\dfrac{\partial^2 F(0,0)}{\partial s^2} = 2\gamma \neq 0$ if $h_1 \neq \frac{2(2X+2-Y)}{5X^2+2aX+b}$.
Period doubling bifurcation theorem in [28] demands that the three following conditions must be satisfied:

(i) $F(0,h) = 0$,

(ii) $F'_s(0,0) = -1$ and

(iii) $\kappa_1 = \left[\frac{\partial^2 F(s,h)}{\partial s \partial h} + \frac{1}{2}\frac{\partial F(s,h)}{\partial h}\frac{\partial^2 F(s,h)}{\partial s^2}\right]_{|(0,0)} = \delta \neq 0,$

$\kappa_2 = \left[\frac{1}{6}\frac{\partial^3 F(s,h)}{\partial s^3} + \left(\frac{1}{2}\frac{\partial^2 F(s,h)}{\partial s^2}\right)^2\right]_{|(0,0)} = \gamma^2 \neq 0$.

Appearance of stable or unstable period two cycle near equilibrium point $(0,0)$ for small $\widehat{h} > 0$ depends of the sign of κ_2.

Theorem 1. *If* $(a,b,c,h) \in \mathbb{C}_{PDB}$ *and* $h_1 \neq \frac{2(2X+2-Y)}{5X^2+2aX+b}$, *there exists a period-doubling bifurcation at* $E(X,Y)$ *of model* (5)*. Furthermore, the period-2 orbits bifurcated from* $E(X,Y)$ *are stable since* $\kappa_2 > 0$ *which is also known as supercritical period-doubling bifurcation.*

4 1:2 resonance bifurcation

Let us consider now case **II) 2. c)** in Lemma 2 where the equilibrium point is non-hyperbolic with eigenvalues $\lambda_1 = \lambda_2 = -1$ since it implies occurrence of the so called 1:2 resonance bifurcation. This is a bifurcation co-dimension 2, so we will take two bifurcation parameters h_t and b_t as solutions of the following system $TrJ_T = -2$ and $DetJ_T = 1$.
For arbitrary but fixed parameters $a, c, X = \frac{-a+\sqrt{a^2+4c}}{2}, Y = \frac{X^2+c}{X^2+b}$ we determine b_t as a root of equation

$$\mathcal{K} = \mathcal{K}(a,b,c,X) = 0, \tag{15}$$

where $\mathcal{K}$ is given by (9), while $h_t = \frac{a+b_t+4X-2XY+X^2}{(a+2X)(b_t+X^2)}$ has the following form $h_t = \frac{b_t^2+\left(2X^2+4X+a\right)b_t+X^2\left(X^2-a\right)}{(a+2X)(b_t+X^2)^2}$. Note that the Jacobian matrix for

(a, b_t, c, h_t) at equilibrium point $E(X,Y) = (X, \frac{X^2+c}{X^2+b})$ is

$$J(h_t, b_t) = \begin{pmatrix} (2XY - 4X - a)\, h_t + 1 & h_t \left(X^2 + b_t\right) \\ -2h_t X (Y-1) & -h_t \left(X^2 + b_t\right) + 1 \end{pmatrix}.$$

We can always select two linearly independent eigenvectors $\mathbf{p}_i \in \mathbb{R}^2$, $i = 1, 2$ of $J(h_t, b_t)$ such that

$$\begin{aligned} J(h_t, b_t)\, \mathbf{p}_1 &= -\, \mathbf{p}_1, \\ J(h_t, b_t)\, \mathbf{p}_2 &= -\, \mathbf{p}_2 + \mathbf{p}_1, \end{aligned}$$

and similarly adjoint eigenvectors $\mathbf{q}_i \in \mathbb{R}^2$, $i = 1, 2$ of the transposed matrix of $J^T(h_t, b_t)$ such that

$$\begin{aligned} J^T(h_t, b_t)\, \mathbf{q}_1 &= -\, \mathbf{q}_1, \\ J^T(h_t, b_t)\, \mathbf{q}_2 &= -\, \mathbf{q}_2 + \mathbf{q}_1, \end{aligned}$$

and $\langle \mathbf{p}_1, \mathbf{q}_1 \rangle = \langle \mathbf{p}_2, \mathbf{q}_2 \rangle = 0$ and $\langle \mathbf{p}_1, \mathbf{q}_2 \rangle = \langle \mathbf{p}_2, \mathbf{q}_1 \rangle = 1$.
The following eigenvectors will be chosen to transform map (12) to the 1:2 resonance normal form at $(h, b) = (h_t, b_t)$. After some tedious calculations we get

$$\mathbf{p}_1 = \begin{pmatrix} 2\frac{2(p_{21}+p_{22})-(2X+a)p_{21}h_t}{(2X+a)h_t} \\ \frac{((2X+a)h_t-2)(2(p_{21}+p_{22})-(2X+a)p_{21}h_t)}{(2X+a)h_t} \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix},$$
$$\mathbf{q}_1 = \begin{pmatrix} -\frac{((2X+a)h_t-2)(2(q_{21}-q_{22})+(2X+a)q_{22}h_t)}{(2X+a)h_t} \\ 2\frac{2(q_{21}-q_{22})+(2X+a)q_{22}h_t}{(2X+a)h_t} \end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix} q_{21} \\ q_{22} \end{pmatrix}.$$

For $p_{21} = q_{22} = 0$ it follows

$$\mathbf{p}_1 = \begin{pmatrix} \frac{4p_{22}}{(2X+a)h_t} \\ \frac{2((2X+a)h_t-2)p_{22}}{(2X+a)h_t} \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ p_{22} \end{pmatrix},$$
$$\mathbf{q}_1 = \begin{pmatrix} -\frac{2((2X+a)h_t-2)q_{21}}{h_t(2X+a)} \\ \frac{4q_{21}}{(2X+a)h_t} \end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix} q_{21} \\ 0 \end{pmatrix}.$$

It obviously holds:

$$\langle \mathbf{p}_1, \mathbf{q}_1 \rangle = \langle \mathbf{p}_2, \mathbf{q}_2 \rangle = 0,$$

$$\langle \mathbf{p}_1, \mathbf{q}_2 \rangle = \langle \mathbf{p}_2, \mathbf{q}_1 \rangle = \frac{4p_{22}q_{21}}{(2X+a)\,h_t} = 1$$

and by choosing $q_{21} = \frac{(2X+a)h_t}{4}$, it implies $p_{22} = 1$. Hence,

$$\mathbf{p}_1 = \begin{pmatrix} \frac{4}{(2X+a)h_t} \\ \frac{2((2X+a)h_t-2)}{(2X+a)h_t} \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\mathbf{q}_1 = \begin{pmatrix} -\frac{(2X+a)h_t-2}{2} \\ 1 \end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix} \frac{(2X+a)h_t}{4} \\ 0 \end{pmatrix}.$$

In order to determine the critical normal form coefficients $\widehat{b}$ and $\widehat{c}$ of the 1:2 resonance bifurcation that determine the non-degeneracy and the scenario of the bifurcation, we need to conduct the following analysis. Denote the nonlinear term of map (12) as $F(U,0)$, $U^T = (u,v)$. Taylor expansion near the origin can be written as

$$F(U,0) = \frac{1}{2}B(U,U) + \frac{1}{6}C(U,U,U)$$

where $B(U,U)$ and $C(U,U,U)$ are multilinear functions. It follows that

$$B(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F(\xi,0)}{\partial \xi_j \partial \xi_k} |_{\xi=0} \; u_j v_k$$

and

$$C(u,v,w) = \sum_{j,k,l=1}^{2} \frac{\partial^3 F(\xi,0)}{\partial \xi_j \partial \xi_k \xi_l} |_{\xi=0} \; u_j v_k w_l.$$

In our case, above mentioned functions are given as

$$B(\mathcal{X},\mathcal{Y}) = 2h_t \begin{pmatrix} (Y-2)\,x_1y_1 + Xx_1y_2 + Xx_2y_1 \\ -(Y-1)\,x_1y_1 - Xx_1y_2 - Xx_2y_1 \end{pmatrix},$$
$$C(\mathcal{X},\mathcal{Y},\mathcal{Z}) = 2h_t \begin{pmatrix} x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1 \\ -x_1y_1z_2 - x_1y_2z_1 - x_2y_1z_1 \end{pmatrix}.$$

Now we need to determine

$$\widehat{\mathbf{b}} = \frac{1}{6}\left\langle \mathbf{q}_1, C\left(\mathbf{p}_1, \mathbf{p}_1, \mathbf{p}_1\right) + 3B\left(\mathbf{p}_1, \mathbf{h}^{20}\right)\right\rangle$$

where $\mathbf{h}^{20} = \left(I_2 - J\left(h_t, b_t\right)\right)^{-1} B\left(\mathbf{p}_1, \mathbf{p}_1\right)$. Since

$$B\left(\mathbf{p}_1, \mathbf{p}_1\right) = 32\begin{pmatrix} \frac{Xh_t(2X+a)-2X+Y-2}{h_t(2X+a)^2} \\ -\frac{Xh_t(2X+a)-2X+Y-1}{h_t(2X+a)^2} \end{pmatrix}, B\left(\mathbf{p}_1, \mathbf{p}_2\right) = 8\begin{pmatrix} \frac{X}{2X+a} \\ \frac{-X}{2X+a} \end{pmatrix},$$

$$C\left(\mathbf{p}_1, \mathbf{p}_1, \mathbf{p}_1\right) = 192\begin{pmatrix} \frac{(2X+a)h_t-2}{h_t^2(2X+a)^3} \\ -\frac{(2X+a)h_t-2}{h_t^2(2X+a)^3} \end{pmatrix},$$

$$\left(I_2 - J\left(h_t, b_t\right)\right)^{-1} = \begin{pmatrix} \frac{1}{h_t(2X+a)} & \frac{1}{h_t(2X+a)} \\ \frac{-2X(Y-1)}{h_t(b_t+X^2)(2X+a)} & \frac{4X+a-2XY}{h_t(b_t+X^2)(2X+a)} \end{pmatrix},$$

$$\mathbf{h}^{20} = 32\begin{pmatrix} -\frac{1}{h_t^2(2X+a)^3} \\ -\frac{4X^3h_t+4X^2(ah_t-1)+aX(ah_t-2)+(Y-1)a}{h_t^2(2X+a)^3(X^2+b_t)} \end{pmatrix},$$

$$B\left(\mathbf{p}_1, \mathbf{h}^{20}\right) =$$

$$128\begin{pmatrix} -\frac{h_tX(2X+a)(5X^2+2Xa+b_t)-2(5X^3+X^2(2(a+1)-Y)+X((1-Y)a+b_t)+(2-Y)b_t)}{h_t^2(2X+a)^4(b_t+X^2)} \\ \frac{h_tX(2X+a)(5X^2+2Xa+b_t)-2(5X^3+X^2(2a+1-Y)+X(a(1-Y)+b_t)+(1-Y)b_t)}{h_t^2(2X+a)^4(b_t+X^2)} \end{pmatrix},$$

we finally get

$$\widehat{\mathbf{b}} = 16\frac{\widehat{\mathbb{B}}}{h_t^2(2X+a)^4(b_t+X^2)} \neq 0,$$

if

$$\begin{aligned} \widehat{\mathbb{B}} \quad &= h_t^2\left(2X+a\right)^2\left(8X^3+3X^2a-ab_t\right)+8\left(b_t+X^2\right) \\ &-2h_t\left(2X+a\right)\left(8X^3+\left(3a-2Y+4\right)X^2+2a\left(1-Y\right)X\right) \\ &-2h_t\left(2X+a\right)\left(4-2Y-a\right)b_t \neq 0. \end{aligned} \qquad (16)$$

Further,

$$\begin{aligned} \widehat{\mathbf{c}} \quad &= \tfrac{1}{2}\left\langle \mathbf{q}_1, C(\mathbf{p}_1, \mathbf{p}_1, \mathbf{p}_2) + 2B(\mathbf{p}_1, \mathbf{h}^{11}) + B(\mathbf{p}_2, \mathbf{h}^{20})\right\rangle \\ &+\tfrac{1}{2}\left\langle \mathbf{q}_2, C(\mathbf{p}_1, \mathbf{p}_1, \mathbf{p}_1) + 3B(\mathbf{p}_1, \mathbf{h}^{20})\right\rangle \end{aligned}$$

where

$$\mathbf{h}^{11} = \left(I_2 - J\left(h_t, b_t\right)\right)^{-1}\left(B\left(\mathbf{p}_1, \mathbf{p}_2\right) + \mathbf{h}^{20}\right).$$

After some calculations we get that the second non-degeneracy condition for the 1:2 resonance bifurcation is

$$\widehat{\mathbf{c}} + 3\widehat{\mathbf{b}} = 16\frac{\phi_0(2X+a)^3 h_t^3 + \phi_1(2X+a)^2 h_t^2 + 8\phi_2(2X+a)h_t + \phi_3}{h_t^3(2X+a)^5(b_t+X^2)^2} \neq 0 \tag{17}$$

where

$$\begin{aligned}
\phi_0 &= 2\left(b_t + X^2\right)\left(12X^3 + 5X^2 a - ab_t\right),\\
\phi_1 &= -52X^5 - 5\left(2Y + 5a - 4\right)X^4 + 2\left(aY + 11a - 24b_t\right)X^3\\
&\quad +2\left(14Yb_t + 4a^2 - b_t\left(11a + 28\right)\right)X^2\\
&\quad +2b_t\left(9Ya - 13a + 2b_t\right)X + 3b_t^2\left(2Y + a - 4\right),
\end{aligned}$$

$$\begin{aligned}
\phi_2 &= 3X^5 + \left(3Y + 2a - 1\right)X^4 + \left(2b_t - 5a + Ya\right)X^3\\
&\quad -\left(Y^2 a + Y\left(4b_t - 3a\right) + 2a\left(a+1\right) - 2b_t\left(a+8\right)\right)X^2\\
&\quad +\left(Ya\left(a - 3b_t\right) - a^2 + b_t\left(7a - b_t\right)\right)X\\
&\quad +b_t\left(Y^2 a + Y\left(b_t - 3a\right) + 2a + b_t\right),\\
\phi_3 &= -16\left(b_t + X^2\right)\left(3X^2 + 2Xa - Ya + a - b_t\right).
\end{aligned}$$

So, we have the following theorem.

Theorem 2. *If b_t is a root of the equation*

$$\left(a + b + 4X - 2XY + X^2\right)^2 - 4\left(b + X^2\right)\left(a + 2X\right) = 0,$$

$h = h_t = \frac{a+b_t+4X-2XY+X^2}{(b_t+X^2)(a+2X)}$ *where* $X = \frac{-a+\sqrt{a^2+4c}}{2}$, $Y = \frac{X^2+c}{X^2+b}$, *and conditions* (16) *and* (17) *are satisfied, then equilibrium point* $E(X,Y)$ *of system* (5) *undergoes 1:2 resonance bifurcation.*

Remark. Two types of degenerate points which can be met in generic two-parameter discrete-time systems, while moving along co-dimension one curves, precisely 1:3 and 1:4 resonance points located on Neimark-Sacker curve, are associated with cases when Jacobian matrix of the system (5) about positive fixed point has complex conjugate eigenvalues $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, and $\pm i$ respectively. Those are also co-dimension two bifurcations where

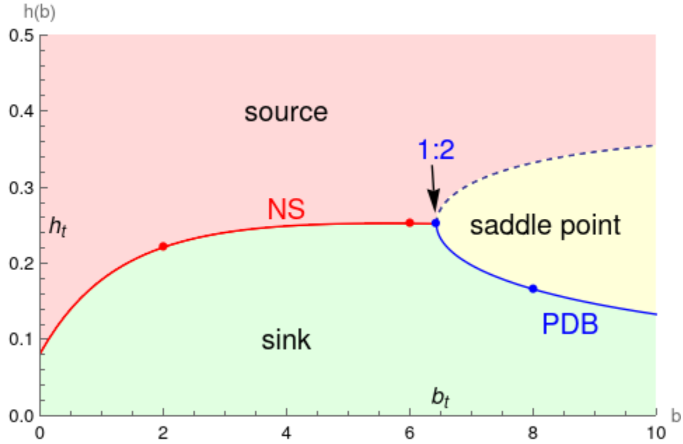


Figure 2. Visual representation of bifurcation distribution near 1:2 resonance point for values of parameters $a = 2$ and $c = 8$ in $(b, h(b))$ plane where $b_t \approx 6.420649595$ and $h_t \approx 0.25293386$.

two independent parameters h_t and b_t can be determined by solving the following conditions respectively: $TrJ_T = -1$ and $DetJ_T = 1$, and $TrJ_T = 0$ and $DetJ_T = 1$. Further, there is a type of degeneracy point corresponding to the case $\lambda_1 = \lambda_2 = 1$, also known as 1:1 resonance point, or $\lambda_1 = 1$, $\lambda_2 = -1$, or special type which appears in the case, so called, Chenciner bifurcation. By using normal form theory and arguing as in [24], conditions for existence above mentioned bifurcations can be determined, but that we leave for some other investigations.

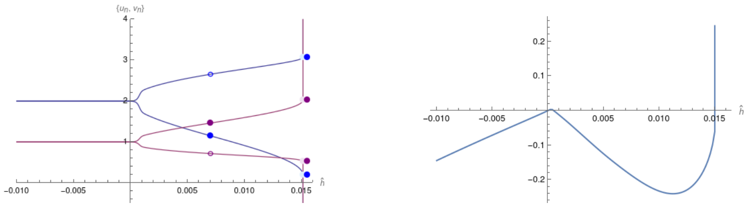
5 Numerical results and illustrations

Consider now the special case of system (5) when $c = b$, i.e. $(a, b, b, h) \in \mathbb{C}_{PDB}$. Then it holds $Y = \frac{X^2+c}{X^2+b} = 1$ and $\sqrt{\mathcal{K}} = \sqrt{(-X^2 + 2X + a - b)^2}$. Since $\mathcal{K} > 0$, it implies $-X^2 + 2X + a - b \neq 0$ and $a - b + 1 \neq 0$. Therefore

$$h_1 = \begin{cases} \frac{2}{X^2+b} & \text{if} \quad a - b + 1 < 0 \\ \frac{2}{2X+a} & \text{if} \quad a - b + 1 > 0 \end{cases} .$$

Now, $h_1 = \frac{2}{X^2+b} < \frac{2}{2X+a} = h_2$ if $X^2 - 2X + b - a > 0$, i.e. $a - b + 1 < 0$ or $h_1 = \frac{2}{2X+a} < \frac{2}{X^2+b} = h_2$ if $X^2 - 2X + b - a < 0$, i.e. $a - b + 1 > 0$. Then it holds $h_1 \neq \frac{2(2X+2-Y)}{5X^2+2aX+b}$.

So, we have $\dfrac{\partial^2 F(0,0)}{\partial s^2} \neq 0$ for $c = b$ and the equilibrium point $E = (X, 1)$ of system (5) exhibits period-doubling bifurcation.



(a) **(b)**

Figure 3. (a) Bifurcation diagram for values of parameters $a = 2$, $b = c = 8$, $h = \frac{1}{6} + \widehat{h}$ and initial conditions $(x_0, y_0) = (2.001, 1.001)$.
(b) Corresponding Lyapunov exponent for the same value of parameters.

Numerically, for values of parameters $a = 2$, $b = c = 8$, $h = \frac{1}{6} + \widehat{h}$ and initial conditions $(x_0, y_0) = (2.001, 1.001)$ we have equilibrium point $E = (2, 1)$ and the minimal period-two solution $\{\ldots, P_1^2, P_2^2, P_1^2, P_2^2, \ldots\}$ where

$$P_1^2 = \left(u_1^2, v_1^2\right) = (2.6579444530118583, 0.7141311825147865),$$
$$P_2^2 = \left(u_2^2, v_2^2\right) = (1.1492888076477241, 1.4620299876429752).$$

On Figure 3 (a), for $\widehat{h} = 0.007$, components of period-two solutions are represented as follows:
u_1^2 with blue circle $(0.007, 2.6579444530118583)$,
v_1^2 with purple circle $(0.007, 0.7141311825147865)$,
u_2^2 with blue point $(0.007, 1.1492888076477241)$,
v_2^2 with purple point $(0.007, 1.4620299876429752)$.
For $a = 2$, $b = c = 8$ it is $h = h_1 = \frac{1}{6}$ and for $\widehat{h} = 0.01512$, $h + \widehat{h} = 0.18179$ there exists the minimal period-four solution

$$\left\{\ldots, Q_1^4, Q_2^4, Q_3^4, Q_4^4, Q_1^4, Q_2^4, Q_3^4, Q_4^4, \ldots\right\},$$

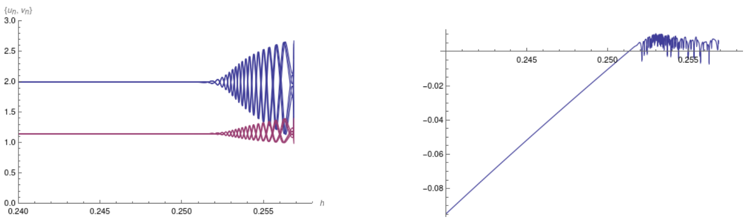
where

$$Q_1^4 = (u_1^4, v_1^4) = (3.072093792829229, 0.5260167014481686),$$
$$Q_2^4 = (u_2^4, v_2^4) = (0.1912394870792422, 2.0285487061653544),$$
$$Q_3^4 = (u_3^4, v_3^4) = (3.0720573954270725, 0.5258714228008146),$$
$$Q_4^4 = (u_4^4, v_4^4) = (0.19081571729926905, 2.0288446871685437).$$

Not all decimals in period-four solution are shown (see Figure 3 (a)). The maximum Lyapunov exponent which indicates the occurrence of periodic orbits, critical bifurcation sets, and chaotic region as $\widehat{h}$ varies, was also plotted (see Figure 3 (b)). Let us recall that Lyapunov exponents can be calculated exactly in a very small number of examples. They are mostly calculated numerically. Lyapunov exponent is calculated by eigenvalues of the limit of the following expression: $(J_0 \cdot J_1 \cdot \ldots \cdot J_n)^{1/n}$ where n tends to infinity, and J_i is the Jacobian of the function at the iterated point (x_i, y_i), i.e.

$$L_i = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_i, \quad i = 1, 2$$

where $\Lambda_{1,2}$ are eigenvalues of Jacobian matrix J_n. For more details see [5, 34].



(a) **(b)**

Figure 4. (a) Bifurcation diagram for value of parameters $a = 2, b = 6.420649595592701, c = 8$ and initial conditions $(x_0, y_0) = (2.0001, 1.001)$.
(b) Corresponding Lyapunov exponent for the same value of parameters (500-505 iterations).

Phase portraits for specifically chosen values of parameters show us some symmetric phenomena (see Figure 5, 6, 7 and 8).

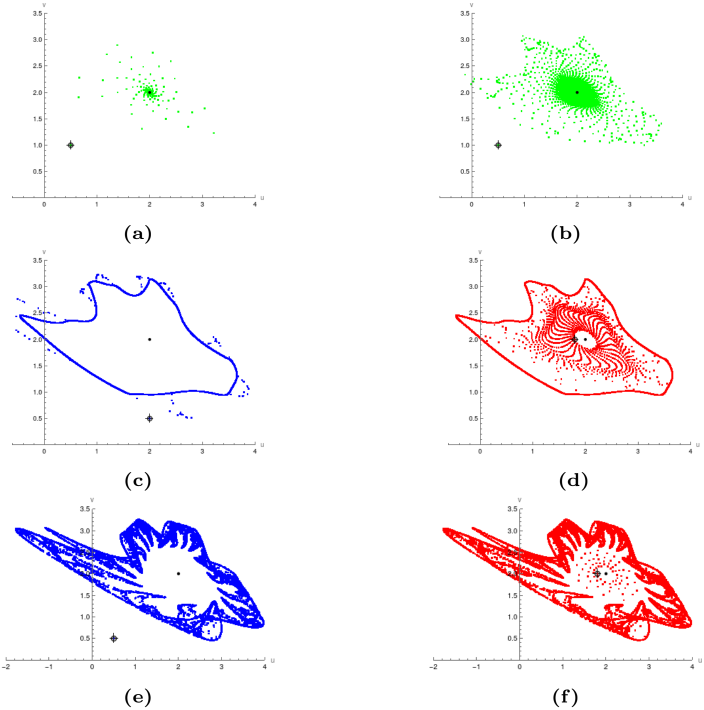


Figure 5. Phase portrait for value of parameters:
$a = 2, b = 2, c = 8, h_1 = \frac{2}{9}$ and:
(a) $\widehat{h} = -0.008$, initial point $(u_0, v_0) = (0.5, 1.0)$,
(b) $\widehat{h} = 0$, initial point $(u_0, v_0) = (0.5, 1.0)$,
(c) $\widehat{h} = 0.001$, initial point $(u_0, v_0) = (2.0, 0.5)$,
(d) $\widehat{h} = 0.001$, initial point $(u_0, v_0) = (1.8, 2.0)$,
(e) $\widehat{h} = 0.005$, initial point $(u_0, v_0) = (0.5, 0.5)$,
(f) $\widehat{h} = 0.001$, initial point $(u_0, v_0) = (1.8, 2.0)$.

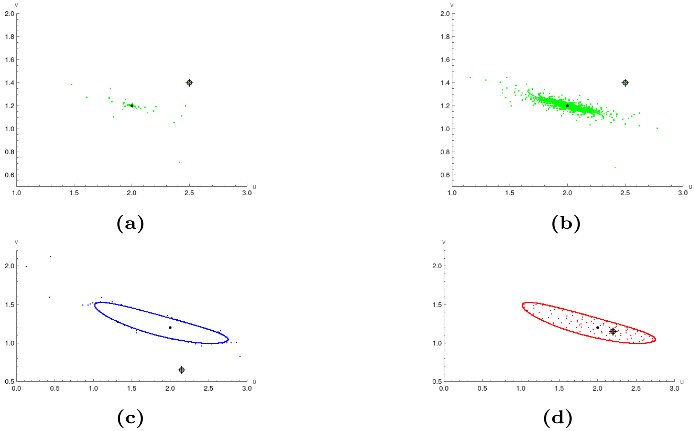


Figure 6. Phase portrait for value of parameters:
$a = 2, b = 6, c = 8, h_1 = \frac{19}{75}$ and:
(a) $\widehat{h} = -0.015$, initial point $(u_0, v_0) = (2.5, 1.4)$,
(b) $\widehat{h} = 0$, initial point $(u_0, v_0) = (2.5, 1.4)$,
(c) $\widehat{h} = 0.004$, initial point $(u_0, v_0) = (2.15, 0.65)$,
(d) $\widehat{h} = 0.004$, initial point $(u_0, v_0) = (2.2, 1.15)$.

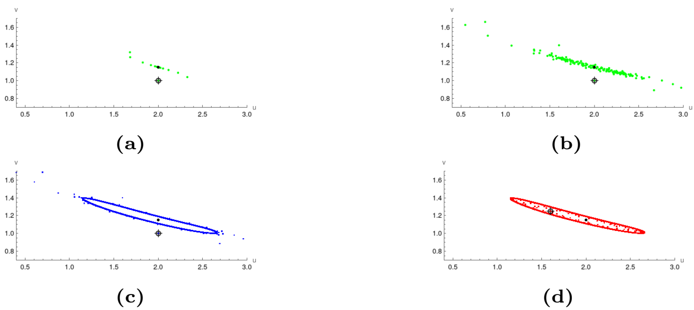


Figure 7. Phase portrait for value of parameters:
$a = 2, b = 6.420649595, c = 8, h_t = 0.25293386$ and:
(a) $\widehat{h} = -0.05$, initial point $(u_0, v_0) = (2.001, 1.001)$,
(b) $\widehat{h} = 0$, initial point $(u_0, v_0) = (2.001, 1.001)$,
(c) $\widehat{h} = 0.0038$, initial point $(u_0, v_0) = (2.001, 1.001)$,
(d) $\widehat{h} = 0.0038$, initial point $(u_0, v_0) = (1.6, 1.25)$.

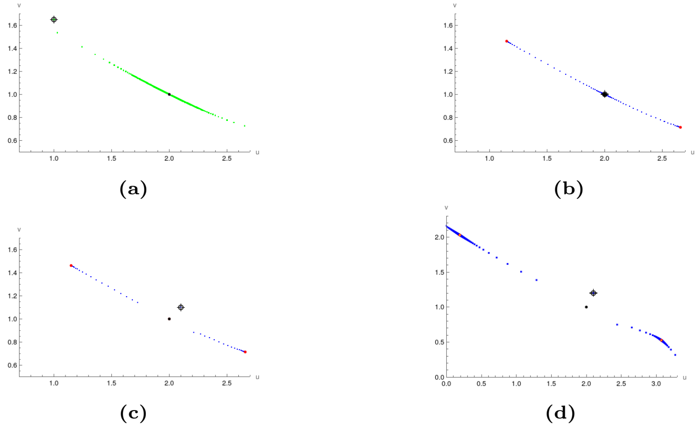


Figure 8. Phase portrait for value of parameters:
$a = 2, b = 8, c = 8, h_1 = \frac{1}{6}$ and:
(a) $\widehat{h} = 0$, initial point $(u_0, v_0) = (1.0, 1.65)$,
(b) $\widehat{h} = 0.007$, initial point $(u_0, v_0) = (2.001, 1.001)$,
(c) $\widehat{h} = 0.007$, initial point $(u_0, v_0) = (2.1, 1.1)$.,
(d) $\widehat{h} = 0.01512$, $(u_0, v_0) = (2.1, 1.2)$.

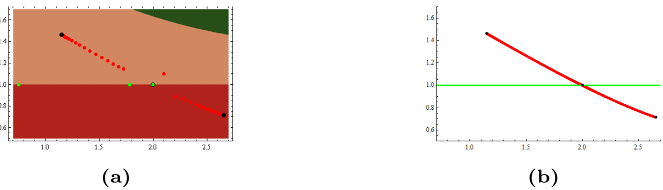


Figure 9. (a) Basin of attractions for values of parameters $a = 2$, $b = 8$, $c = 8$, $h = \frac{1}{6} + 0.007$ and initial condition $(u_0, v_0) = (2.001, 1.001)$.
(b) Stable (green) and unstable (red) manifold for the same values of parameters as in (a).

Some interesting phenomena occurs for values of parameters $a = 34$ and $c = 35$. In that case $\mathcal{K} = (b^2 - 26b + 9)(b - 35)(b - 3) = 0$. For one root of previous equation $b = 13 + 4\sqrt{10}$, it holds $\widehat{b} = 387\sqrt{10} - 12656 \neq 0$ and $\widehat{c} + 3\widehat{b} = \frac{64\sqrt{10} - 325}{729} \neq 0$, so we have 1:2 resonance point. On Figure 10 one can see phase portraits where fixed point loses its stability through pitchfork bifurcation and it becomes stable period-two solution. Then these points change to the two stable closed invariant curves which merges into one stable invariant curve and eventually leads to chaos.

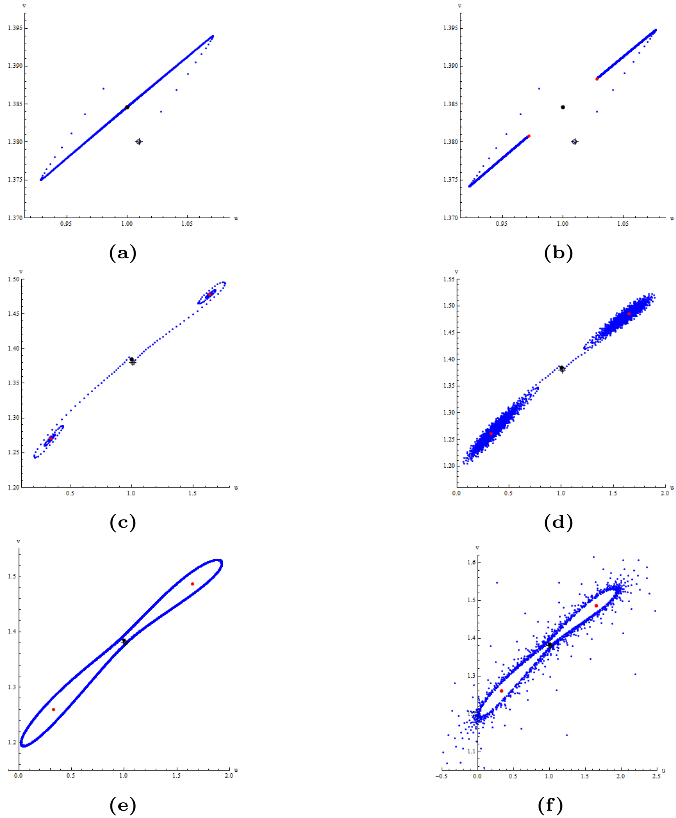


Figure 10. Phase portrait for value of parameters $a = 34, b = 25, c = 35$, $E = (X, Y) = (1, \frac{18}{13})$, $h_1 \approx 0.0629795$ and initial point $(u_0, v_0) = (1.01, 1.38)$ and:
(a) $h = 0.0629 < h_1$,
(b) $h = 0.062981 > h_1$,
(c) $h = 0.0645 > h_1$,
(d) $h = 0.065 > h_1$,
(e) $h = 0.0651 > h_1$ and
(f) $h = 0.06510965 > h_1$.

6 Conclusion

We considered a discrete counterpart of nonlinear differential equations for a two-species chemical reaction and its qualitative behavior. It was studied the local dynamics of the of model and proved that system has a unique positive equilibrium point. Parametric conditions for local asymptotic stability of model (5) are obtained. Furthermore, some of co-dimension-one and co-dimension-two bifurcations are discussed. By using normal form method and bifurcation theory, it is proved that system (5) undergoes period-doubling bifurcation at its positive equilibrium point and co-dimension-two bifurcation associated with 1:2 strong resonances. In the case of 1:2 resonance, the system exposes a resonance pattern where the frequency of one oscillatory component is twice that of another component. This can lead to the repression or amplification of certain oscillations in the system, resulting in a complex behavior. The aim of this type of research is to enable chemists, biologists and other scientists to suppress the chaotic behavior of the model by inspecting the type of bifurcations with the appropriate selection of parameters.

***Acknowledgment*:** The authors are very thankful to the anonymous referees for their useful comments and suggestions, which helped us to refine and enhance this paper.

References

[1] B. P. Belousov, A periodic reaction and its mechanism, *Sb. Ref. Radiat. Med. Medgiz Mosc.* (1959) 145–147.

[2] W. C. Bray, A periodic reaction in homogeneous solution and its relation to catalysis, *J. Am. Chem. Soc.* **43** (1921) 1262–1267.

[3] B. L. Clarke, *Stability of Complex Reaction Networks*, Wiley, Hoboken, 1980.

[4] A. Csaszar, L. Jicsinszky, T. Turanyi, Generation of model reaction leading to limit cycle behavior, *React. Kinet. Catal. Lett.* **18** (1981) 65–71.

[5] N. Das, Determination of Lyapunov exponents in discrete chaotic models, *Int. J. Theor. Appl. Sc.* **4** (2012) 89–94.

[6] Q. Din, Dynamics and Hopf bifurcation of a chaotic chemical reaction model, *MATCH Commun. Math. Comp. Chem.* **88** (2022) 351–369.

[7] Q. Din, T. Donchev, D. Kolev, Stability, bifurcation analysis and chaos control in chlorine dioxide-iodine-malonic acid reaction, *MATCH Commun. Math. Comp. Chem.* **79** (2018) 577–606.

[8] Q. Din, U. Saeed, Stability, discretization, and bifurcation analysis for a chemical reaction system, *MATCH Commun. Math. Comput. Chem.* **90** (2023) 151–174.

[9] Q. Din, M. Shabira, M. Khan, A cubic autocatalator chemical reaction model with limit cycle analysis and consistency preserving discretization, *MATCH Commun. Math. Comput. Chem.* **87** (2022) 441–462.

[10] S. Elnashaie, Z. Chen, P. Garhyan, P. Prasad, A. Mahecha-Botero, Practical implications of bifurcation and chaos in chemical and biological reaction engineering, *Int. J. Chem. React. Eng.* **4** (2006) #1.

[11] P. Erdi, J. Toth, *Mathematical Models of Chemical Reactions. Theory and Applications of Deterministic and Stochastic Models*, Manchester Univ. Press, Manchester, 1989.

[12] M. Feinberg, Lectures on Chemical Reaction Networks, written versions of lectures given at the Mathematics Research Center of the University of Wisconsin, 1979.

[13] M. Feinberg, Chemical oscillations, multiple equilibria, and reaction network structure, *Dynam. Model. React. Syst.* (1980) 59–130.

[14] G. A. Frerichs, T. M. Mlnarik, R. J. Grun, R. C. Thompson, A new pH oscillator: The chlorite-sulfite-sulphuric acid system in a CSTR, *J. Phys. Chem. A* **105** (2001) 829–837.

[15] M. Garić-Demirović, S. Hrustić, S. Moranjkić, M. Nurkanović, Z. Nurkanović, The existence of Li-Yorke chaos in certain predator-prey system of difference equations, *Sarajevo J. Math.* **18** (2022) 45–62.

[16] M. Garić-Demirović, S. Moranjkić, M. Nurkanović, Z. Nurkanović, Stability, Neimark-Sacker bifurcation, and approximation of the invariant curve of certain homogeneous second-order fractional difference equation, *Discr. Dyn. Nat. Soc.* **2020** (2020) #6254013.

[17] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, Cham, 1983.

[18] F. Horn, The dynamics of open reaction systems, *Mathematical Aspects of Chemical and Biochemical Problems and Quantum Chemistry* (Proc. SIAM-AMS Sympos. Appl. Math.) (1974) 125–137.

[19] S. Hrustić, M. R. S. Kulenović, S. Moranjkić, Z. Nurkanović, Global dynamics of perturbation of certain rational difference equation, *Turk. J. Math.* **43** (2019) 894–915.

[20] S. Jašarević Hrustić, M. R. S. Kulenović, M. Nurkanović, Global dynamics and bifurcations of certain second order rational difference equation with quadratic terms, *Qualit. Th. Dynamic. Sys.* **15** (2015) 283–307.

[21] S. Kalabušić, E. Pilav, Bifurcations, permanence and local behavior of the plant-herbivore model with logistic growth of plant biomass, *Qualit. Th. Dynamic. Sys.* **21** (2022) #26.

[22] H. J. Kim, K. S. Chang, A method of model validation for chaotic chemical reaction systems based on neural network, *Korean J. Chem. Eng.* **18** (2001) 623–629.

[23] M. R. S. Kulenović, O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman and Hall/CRC, Boca Raton, 2002.

[24] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer, New York, 1998.

[25] X. Liu, D. Xiao, Complex dynamic behaviors of a discrete-time predator-prey system, *Chaos Soliton Fract.* **32** (2007) 80–94.

[26] B. Li, Q. He, Bifurcation analysis of a two-dimensional discrete Hindmarsh-Rose type model, *Adv. Diff. Eq.* (2019) #124.

[27] K. S. Al Noufaey, Semi-analytical solutions of the Schnakenberg model of a reaction-difusion cell with feedback, *Res. Phys.* **9** (2018) 609–614.

[28] C. Robinson, *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos*, CRC Press, Boca Raton, 1995.

[29] M. Sandri, Numerical calculation of Lyapunov exponents, *The Math. J.* **6** (1996) 78–84.

[30] J. Schnakenberg, Simple chemical reaction systems with limit cycle behavior, *J. Theor. Biol.* **81** (1979) 389–400.

[31] R. A. Smitz, K.R. Graziani, J.L. Hudson, Experimental evidence of chaotic states in the Belousov–Zhabotinskii reaction, *J. Chem. Phys.* **67** (1977) 3040–3044.

[32] A. Turing, The chemical basis of morphogenesis, *Phil. Trans. Roy. Soc. Lond. Biolog. Sci.* **237** (1952) 37–72.

[33] F. Yi, E. A. Gafney, S. Seirin-Lee, The bifurcation analysis of turing pattern formation induced by delay and difusion in the Schnakenberg system, *Discr. Contin. Dyn. Syst. Ser. B* **22** (2017) 647–668.

[34] J. C. Vallejo, M. A. F. Sanjuan, *Predictability of Chaotic Dynamics, A Finite-time Lyapunov Exponents Approach*, Springer, 2017.

[35] A. M. Zhabotinskii, Periodic process of the oxidation of malonic acid in solution, *Biofizika* **9** (1964) 306–311.