MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **92** (2024) 761–777
ISSN: 0340–6253
doi: 10.46793/match.92-3.761L

The Edge–Hosoya Polynomial of Catacondensed Benzenoid Graphs Associated with its Hosoya Polynomial

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(Received January 11, 2024)

Abstract

This paper reveals the relationship between edge-Hosoya polynomial and the Hosoya polynomial of catacondensed benzenoid graphs. The result shows that, for a catacondensed benzenoid graph, the computations of the edge-Hosoya polynomial can be reduced to that of the Hosoya polynomial.

1 Introduction

Chemical graphs can demonstrate the structures of molecules, where the vertices and edges represent the atoms and the molecular bonds, respectively. Topological indices are the molecular structure descriptors revealing the structure property and structure activity relationships. They play a very important role in quantitative structure–property relations (QSPRs)

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and quantitative structure–activity relations (QSARs) studies. It is worthy to mention that, recently, the topological indices are used to predict the physicochemical properties of the COVID-19 vaccine [1, 8, 11, 15, 16].

Hosoya polynomial is a significant distance-based polynomial and has a strong connection with the Wiener index and hyper-Wiener index of the underlying molecular structure. They enable the identifying of the basic features of molecular topology, such as branching, cyclicity, and centricity (or centrality) and their specific patterns, which are well reflected by the physicochemical properties of chemical compounds. Numan et al. [17] computed the closed formula of Hosoya polynomial for subdivided caterpillar trees, which helps to compute the Wiener index and hyper-Wiener index of uniform subdivision of caterpillar graph.

There are many results on the computations of the Hosoya polynomial for some graphs, such as benzenoid graphs [6, 24, 26], cubes [12], nanotubes [23, 25, 28, 29] and hexagonal graphs [27]. Moreover, the Hosoya polynomial of pent–heptagonal carbon nanosheets and aramids, which admit significant industrical applications, are studied in [19, 20]. We also refer to see [2, 4, 14, 18].

The edge-Hosoya polynomial is defined in [10], which is equivalent to the Hosoya polynomial of the corresponding line graph. In 2017, Tratnik and Pleteršek [21] obtained the explicit formula between the edge-Hosoya polynomial and the Hosoya polynomial of trees. For a catacondensed benzenoid graph, Chen et al. [3] obtained the explicit connection between the edge-Wiener index and the Wiener index. These results demonstrated that Hosoya polynomials are closely correlated to the edge-versions for some kinds of molecular graphs. It motivates us to consider the relationship between the edge-Hosoya polynomial and the Hosoya polynomial for a catacondensed benzenoid graph.

Actually, a catacondensed benzenoid graph can be constructed from a signal hexagon by recursively attaching a 6-cycle. In chemistry, such an operation is known as annelation. Gutman et al. [7] investigated the recursion for the Hosoya polynomial of graphs obtained by attaching a 6-cycle. Afer that, Xu and Zhang [24] generally provided the result on Hosoya polynomials under gated amalgamations, where the graph is obtained by identifying the isomorphic gated subgraphs of two graphs. Correspondingly, Tratnik et al. [22] developed a recursion for the edge-Hosoya polynomial of graphs obtained by attaching a 6-cycle. Recently, Knor and Tratnik [13] provided a general method for computing the edge-Hosoya polynomial of a graph which is obtained by identifying two edges of two connected bipartite graphs.

In this paper, we follow Knor and Tratnik's strategy to reveal the explicit relationship between the edge-Hosoya polynomial and the Hosoya polynomial of catacondensed benzenoid graphs. And this paper is organized as follows. In section 2, the related concepts from chemical graph theory are presented. The main result and some relevance are listed in section 3. Finally, the main result is verified in section 4.

2 Preliminary

For a graph G, the vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The line graph of G, denoted by $L(G)$, is a graph in which each vertex of $L(G)$ corresponds to each edge in G and two vertices in $L(G)$ are adjacent if and only if two corresponding edges in G share a common vertex. For $u, u' \in V(G)$ and $f, f' \in E(G)$, let $d_G(u, u')$ denote the distance between u and u' in G and $d_G(f, f')$ denote the distance between the two corresponding vertices f and f' in $L(G)$. The Hosoya polynomial $H(G, x)$ [9] and the edge-Hosoya polynomial $H_e(G, x)$ [10] of G with respect to the variable x , are defined as

$$H(G, x) = \sum_{\{u,u'\} \subset V(G)} x^{d_G(u,u')},$$

and

$$H_e(G, x) = \sum_{\{f,f'\} \subset E(G)} x^{d_G(f,f')},$$

where the summations $\sum_{\{u,u'\} \subset V(G)}$ and $\sum_{\{f,f'\} \subset E(G)}$ are taken over all pairs of unordered vertices (maybe the same) and of unordered edges (maybe the same) in G, respectively. According to the definition of line graphs, it is obvious that $H_e(G, x) = H(L(G), x)$.

The Wiener index of G is defined by

$$W(G)=\sum_{\{u,u'\}\subset V(G)} d_G(u,u').$$

Obviously, the first derivative of the Hosoya polynomial of G evaluated at $x=1$ is equal to the Wiener index of G, i.e. $W(G)=H'(G,1)$. Moreover, the hyper-Wiener index is defined by

$$WW(G)=\frac{1}{2}\Bigg(W(G)+\sum_{\{u,u'\}\subset V(G)} d_G^2(u,u')\Bigg),$$

and it can be expressed by using the first and the second derivative of $H(G,x)$.

A connected plane graph G is called a *benzenoid graph* if G satisfies:

1. G has no cut-vertices;
2. All inner faces of G are hexagons;
3. For each two hexagons of G, either they are disjoint or they have just one edge in common (and they are then called adjacent);
4. For each three hexagons of G, they do not share any edge in common.

In fact, a molecular graph of benzenoid hydrocarbons is a benzenoid graph, which makes the benzenoid graphs greatly important in theoretical chemistry (see [5] and the references therein).

The *characteristic graph* $T(G)$ of a given benzenoid graph G, is a graph such that each vertex corresponds to each hexagon of G, and two vertices are adjacent in $T(G)$ if and only if the two corresponding hexagons share an edge in G. If $T(G)$ is a tree, G is called *catacondensed.* Moreover, a benzenoid chain is a catacondensed benzenoid graph G such that $T(G)$ is a path. Let CBG_n denote the set of all catacondensed benzenoid graphs with $n(n\geq 2)$ hexagons, and $CBG=\bigcup\limits_{n=2}^{\infty} CBG_n$. A graph in CBG and the corresponding characteristic graph are illustrated in Fig. 1.

For a given $G\in CBG_n$, each hexagon of G may have at most three adjacent hexagons. A hexagon is called to be *terminal* if it has just one

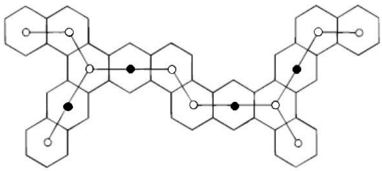

Figure 1. A graph in CBG and corresponding characteristic graph.

adjacent hexagon, and is called to be *branched* if it has three adjacent hexagons. If a hexagon has two adjacent hexagons, then it has exactly two vertices of degree 2. If these two vertices are adjacent, this hexagon is called to be *angularly connected*, otherwise the hexagon is called to be *linearly connected*. In Fig. 1, the linearly connected hexagons are indicated by the solid dots. A benzenoid chain with n hexagons is called *linear chain*, denoted by L_n, if each non-terminal hexagon is linearly connected.

For a given $G \in CBG_n$, each maximal linear chain in G is called a *linear segment* of G. If a linear segment includes a terminal hexagon, it is called a *terminal segment*. For a linear segment S, we use $l(S)$ to denote the number of hexagons in S, which is called the length of S. It is obvious that $1 \leq l(S) \leq n$. If $S_1, \cdots, S_t$ are linear segments of G with lengths $l(S_i) = l_i, 1 \leq i \leq t$, then $n = l_1 + \cdots + l_t - t + 1$. For instance, the graph of Fig. 1 includes nine linear segments, that are of length 2 or of length 3.

3 Main results

In this section, our main result is stated in Theorem 1. Furthermore, some related corollaries are also presented.

Theorem 1. *Let G be a catacondensed benzenoid graph having n hexagons and $S_1, \cdots, S_t$ be linear segments of G with lengths $l(S_i) = l_i (1 \leq i \leq t)$. Then*

$$H_e(G,x) = \frac{(1+2x+2x^2)^2}{x(1+x)^4} H(G,x) + \frac{\sum_{i=1}^{t} \left(x^7 - x^{2l_i+5}\right)}{(1-x)(1+x)^3} - g(n,x),$$

where $g(n,x) = \frac{1}{x(1+x)^4}[2 + 4n + (8+16n)x + (16+32)x^2 + (18+39n)x^3 +$

$(12+33n)x^4+(3+15n)x^5+(-1+n)x^6+(-2-4n)x^7+(-1-2n)x^8]$.

Remark. If G_1 and G_2 are two catacondensed benzenoid graphs with the same segment number set $\{l_1, l_2, ..., l_t\}$, then by combining Theorem 1 and Theorem 1.1 in [26], we can obtain that

$$H_e(G_1,x)-H_e(G_2,x)=\frac{x(1+2x+2x^2)^2}{(1+x)^2}\left(H_{33}(G_1,x)-H_{33}(G_2,x)\right),$$

where $H_{33}(G,x)=\sum\limits_{\substack{\{u,u'\}\subset V(G),\\ deg_G(u)=deg_G(u')=3}} x^{d_G(u,u')}$ is the partial Hosoya polynomial of the graph G. It implies that $H_e(G_1,x)=H_e(G_2,x)$ iff $H_{33}(G_1,x) = H_{33}(G_2,x)$.

Remark. We rewrite the equation in Theorem 1 as follows.

$$\begin{aligned} & x(1+x)^4H_e(G,x) \\ = \; & (1+2x+2x^2)^2H(G,x)+x^8(1+x)^2\sum_{i=1}^{t}\left[1+x^2+\cdots+x^{2(l_i-2)}\right] \\ & -x(1+x)^4g(n,x). \end{aligned}$$

Taking derivatives of the variable x on both sides and setting $x=1$, one can obtain that

$$\begin{aligned} & 48H_e(G,1)+16H_e'(G,1) \\ = \; & 60H(G,1)+25H'(G,1)+36\sum_{i=1}^{t}(l_i-1)+4\sum_{i=1}^{t}(l_i-2)(l_i-1) \\ & -(129+366n). \end{aligned}$$

Moreover, by substituting the three expressions of

$$H_e(G,1)=|E(G)|+\binom{|E(G)|}{2}=\frac{1}{2}(5n+1)(5n+2),$$

$$\sum_{i=1}^{t}(l_i-1)=n-1,$$

and

$$H(G,1)=|V(G)|+\binom{|V(G)|}{2}=\frac{1}{2}(4n+2)(4n+3)$$

into the above equation, we have

$$H'_e(G,1) = \frac{25}{16}H'(G,1) - \frac{120n^2+94n+29}{16} + \frac{1}{4}\sum_{i=1}^{t}(l_i-1)^2,$$

which means that the main result, Theorem 1.4 in [3], is a corollary of Theorem 1. Note that there is a typo in Theorem 1.4 in [3].

4 Proof of Theorem 1

Assumed that $G \in CBG_n$ and G has t linear segments denoted by $S_1, \cdots,$ S_t. Let $l(S_i) = l_i, 1 \leq i \leq t$. In this section, we will prove Theorem 1 by using the mathematical induction for t. Before that, some useful lemmas are listed.

For a graph G and a given vertex $u \in V(G)$, define

$$H(G,u,x) = \sum_{u' \in V(G)} x^{d_G(u,u')}.$$

Similarly, for a given edge $f \in E(G)$, define

$$H_e(G,f,x) = \sum_{f' \in E(G)} x^{d_G(f,f')}.$$

Moreover, define the distance between a vertex a and an edge $e = uv$ in G as $d_G(a,e) = \min\{d_G(a,u), d_G(a,v)\}$ and define

$$H_e(G,a,x) = \sum_{e \in E(G)} x^{d_G(a,e)}.$$

Actually, if $t = 1$, then G reduces to the linear chain L_n, as shown in Fig. 2. The first and the n-th hexagon are called the terminal hexagon of L_n and f_1 is called the terminal edge of L_n. According to the results in [6] , [26] and [22], we have the following.

Lemma 1. *[6, 26] Let u_1, u_2 be two vertices in the terminal hexagon of*

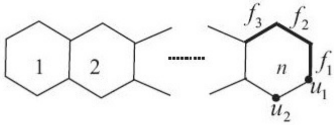

Figure 2. The linear chain L_n.

L_n (shown in Fig. 2). Then

$$\begin{aligned}
H(L_n, u_1, x) =& \frac{(1+x)(1-x^{2n+1})}{1-x}, \\
H(L_n, u_2, x) =& x^2 + x^3 + \frac{(1+x)(1-x^{2n})}{1-x}, \quad \textit{and} \\
H(L_n, x) =& \frac{1}{(1-x)^2}[4n+2+(-3n-3)x+(2n-2)x^2+(-2n-1)x^3 \\
& -2nx^4+nx^5+2x^{2n+2}+2x^{2n+3}].
\end{aligned}$$

Lemma 2. *[22] Let f_1, f_2 and f_3 be the edges in the terminal hexagon of L_n (shown in Fig. 2). Then*

$$\begin{aligned}
H_e(L_n, u_1, x) =& \frac{x^{2n}(2x^2+2x+1)-x^2-2x-2}{x^2-1}, \\
H_e(L_n, f_1, x) =& \frac{x^{2n}(x^3+2x^2+2x)-x^3-x^2-2x-1}{x^2-1} \quad \textit{and} \\
H_e(L_n, x) =& \frac{1}{(1-x^2)^2}[1+5n+(8n-2)x+(2n-8)x^2-(n+10)x^3 \\
& -(7n+5)x^4-(9n+1)x^5+2nx^7+2x^{2n+1}+6x^{2n+2}+ \\
& 10x^{2n+3}+6x^{2n+4}+x^{2n+5}].
\end{aligned}$$

By checking directly, we have

$$\begin{aligned}
& x(1+x)^4 H_e(L_n, x) - (1+2x+2x^2)^2 H(L_n, x) \\
=& \frac{x(1+x)(x^7-x^{2n+5})}{1-x} - x(1+x)^4 g(n, x).
\end{aligned}$$

Then Theorem 1 follows in the case that $t = 1$.

Now consider the case that $t \geq 2$, *i.e.*, G has at least two linear segments,

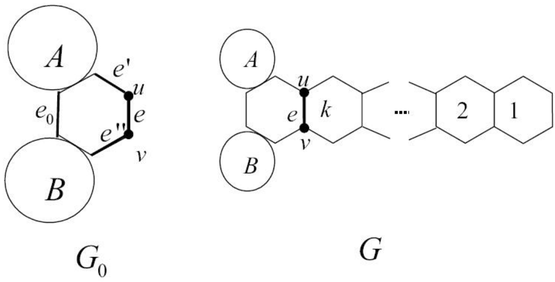

Figure 3. G_0 and G.

say $S_1, \cdots, S_t$. Assume that S_t is a linear segment in G containing a terminal hexagon. Let G_0 be a catacondensed benzenoid graph with segments $S_1, \cdots, S_{t-1}$ (shown in the left hand side of Fig. 3), where A and B are K_2 or graphs in CBG. It is easy to see that G can be constructed by identifying the edge $e = uv \in E(G_0)$ and a terminal edge of L_{l_t-1}, as shown in the right hand side of Fig. 3, where $k = l_t - 1$. This implies that $G_0 \in CBG_{n-k}$.

Recently, Knor and Tratnik [13] developed a method for computing the edge-Hosoya polynomial of graphs by identifying two edges of two connected bipartite graphs, which is stated as follows.

Lemma 3. *[13] Let G be a graph obtained from connected bipartite graphs G_1 and G_2 by identifying the edges $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$ into an edge $e' = uv$. Then*

$$\begin{aligned} H_e(G,x) =& H_e(G_1,x) + H_e(G_2,x) - H_e(G_2,e',x) \\ &+ \sum_{e \in E_u(G_2)} x^{d_{G_2}(e,u)+1} H_e(G_1,u,x) \\ &+ \sum_{e \in E_v(G_2)} x^{d_{G_2}(e,v)+1} H_e(G_1,v,x) \\ &+ \sum_{e \in E_0(G_2)\setminus\{e'\}} x^{d_{G_2}(e,e')-1} H_e(G_1,e',x) \end{aligned}$$

$$+\sum_{e\in E_0(G_2)\setminus\{e'\}}(x-1)x^{d_{G_2}(e,e')-1}.$$

where $E_u(G)=\{e|e\in E(G)\ \ and\ \ d_G(u,e)<d_G(v,e)\}$, $E_0(G)=\{e|e\in E(G)\ \ and\ \ d_G(u,e)=d_{G_2}(v,e)\}$.

By Lemma 3, one can express the edge-Hosoya polynomial of G in Fig. 3 by using the item $H_e(G_0,e,x)$.

Lemma 4. *Let* $n\geq 2$, $G_0\in CBG_{n-k}$ *as shown in Fig. 3, where* $e=uv$ *is an edge in* G_0 *with* $deg_{G_0}(u)=deg_{G_0}(v)=2$, A *and* B *are both* K_2 *or graphs in* CBG. *If* G *is a graph obtained from* G_0 *by identifying* e *with a terminal edge of* L_k, *then*

$$\begin{aligned}H_e(G,x) \quad = \quad & H_e(G_0,x)+H_e(L_k,x)-1+x^3-x^{2k+3}\\ & +\frac{(1+2x+2x^2)(1-x^{2k})\left[H_e(G_0,e,x)-1\right]}{1-x^2}.\end{aligned}$$

Proof. Let $G_1=L_k$ and $G_2=G_0$. As shown in Fig. 3, we have $E_u(G_2)=E(A)\cup\{e'\}$, $E_v(G_2)=E(B)\cup\{e''\}$ and $H_e(L_k,u,x)=H_e(L_k,v,x)$. Moreover, for any edge $g'\in E_u(G_2)$ and $g''\in E_v(G_2)$, we have

$$d_G(e,g')=d_G(u,g')+1,\quad \text{and}\quad d_G(e,g'')=d_G(v,g'')+1.$$

Furthermore, the only edge that belongs to $E_0(G_0)\setminus\{e\}$ is e_0. Therefore, by Lemma 3, we get

$$\begin{aligned}&H_e(G,x)\\ =&H_e(G_0,x)+H_e(L_k,x)-H_e(G_0,e,x)+x^2H_e(L_k,e,x)+x^2(x-1)\\ &+\Bigg(\sum_{f\in E_u(G_0)}x^{d_{G_0}(f,e)}+\sum_{f\in E_v(G_0)}x^{d_{G_0}(f,e)}\Bigg)H_e(L_k,u,x)\\ =&H_e(G_0,x)+H_e(L_k,x)-H_e(G_0,e,x)+x^2H_e(L_k,e,x)+x^2(x-1)\\ &+\big(H_e(G_0,e,x)-x^3-1\big)H_e(L_k,u,x)\\ =&H_e(G_0,x)+H_e(L_k,x)+H_e(G_0,e,x)[H_e(L_k,u,x)-1]\\ &-(x^3+1)H_e(L_k,u,x)+x^2H_e(L_k,e,x)+(x-1)x^2.\end{aligned}$$

Combining Lemma 2 and 3, the result follows by directly calculation. ■

To reveal the relationship between the Hosoya polynomial and the edge-Hosoya polynomial of $G \in CBG$, the following lemmas play a key role.

Lemma 5. *Let $n_0 = n - k \geq 2$ and $G_0 \in CBG_{n_0}$ as shown in Fig. 3, where $e = uv \in E(G_0)$ with $deg_{G_0}(u) = deg_{G_0}(v) = 2$. Then*

$$\sum_{u' \in V(A)} x^{d_{G_0}(u',u)} + \sum_{v' \in V(B)} x^{d_{G_0}(v',v)} = \frac{(1+x)^2 \left[H_e(G_0,e,x) - 1\right] + x^3 - x^5}{1+2x+2x^2}.$$

Proof. Obviously, G_0 can be obtained from a single hexagon including e by stepwise adding hexagons. Denoted by $H_i (1 \leq i \leq n_0 - 1)$ the i-th hexagon added and f_i the edge of H_i identified to an edge of H_{i-1}. Denote the set of the new five edges of H_i by E_i. Then, for $1 \leq i \leq n_0 - 1$,

$$\begin{aligned}
\sum_{e' \in E_i} x^{d_{G_0}(e',e)} &= x^{d_{G_0}(f_i,e)} + 2x^{d_{G_0}(f_i,e)+1} + 2x^{d_{G_0}(f_i,e)+2} \\
&= (1+2x+2x^2) x^{d_{G_0}(f_i,e)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
H_e(G_0,e,x) &= 1 + 2x + 2x^2 + x^3 + \sum_{i=1}^{n_0-1} \sum_{e' \in E_i} x^{d_{G_0}(e',e)} \\
&= 1 + 2x + 2x^2 + x^3 + (1+2x+2x^2) \sum_{i=1}^{n_0-1} x^{d_{G_0}(f_i,e)},
\end{aligned}$$

which implies that

$$\sum_{i=1}^{n_0-1} x^{d_{G_0}(f_i,e)} = \frac{H_e(G_0,e,x) - (1+2x+2x^2+x^3)}{1+2x+2x^2}.$$

Denote by V_i the set of the new four vertices added in the i-th hexagon $(1 \leq i \leq n_0 - 1)$. If $f_i \in E(A)$, then

$$\sum_{w \in V_i} x^{d_{G_0}(w,u)} = x^{d_{G_0}(f_i,e)} + 2x^{d_{G_0}(f_i,e)+1} + x^{d_{G_0}(f_i,e)+2} = (1+x)^2 x^{d_{G_0}(f_i,e)}.$$

According to symmetry, if $f_i \in E(B)$, then

$$\sum_{w\in V_i} x^{d_{G_0}(w,v)} = (1+x)^2 x^{d_{G_0}(f_i,e)}.$$

Without loss of generality, let $g : \cup_{i=1}^{n_0-1} V_i \to \{u,v\}$ be a map such that $d_{G_0}(w,g(w)) = \min\{d_{G_0}(w,u), d_{G_0}(w,v)\}$, for any vertex $w \in \cup_{i=1}^{n_0-1} V_i$. Therefore,

$$\begin{aligned}
& \sum_{u'\in V(A)} x^{d_{G_0}(u',u)} + \sum_{v'\in V(B)} x^{d_{G_0}(v',v)} \\
= \;& 2x + 2x^2 + \sum_{i=1}^{n_0-1} \sum_{w\in V_i} x^{d_{G_0}(w,g(w))} \\
= \;& 2x + 2x^2 + (1+x)^2 \sum_{i=1}^{n_0-1} x^{d_{G_0}(f_i,e)} \\
= \;& \frac{(1+x)^2\left[H_e(G_0,e,x)-1\right] + x^3 - x^5}{1+2x+2x^2}. \qquad \blacksquare
\end{aligned}$$

Consequently, $H(G,x)$ can be written related to $H_e(G_0,e,x)$.

Lemma 6. *Let G be a graph constructed as that in Lemma 4. Then*

$$\begin{aligned}
H(G,x) \;=\;& H(G_0,x) + H(L_k,x) - 2 - x + \frac{(1+x)^2(x^4 - x^{2k+4})}{1+2x+2x^2} \\
& + \frac{(1+x)^3(x - x^{2k+1})\left[H_e(G_0,e,x)-1\right]}{(1-x)(1+2x+2x^2)}.
\end{aligned}$$

Proof. As shown in Fig. 3, let $w \in V(L_k) \setminus \{u,v\}$, $u' \in V(A)$ and $v' \in V(B)$. It is easy to see that

$$d_G(w,u') = d_G(w,u) + d_G(u,u') \quad \text{and} \quad d_G(w,v') = d_G(w,v) + d_G(v,v').$$

Recall that

$$\begin{aligned}
& \sum_{w\in V(L_k)\setminus\{u,v\}} x^{d_G(w,u)} = \sum_{w\in V(L_k)\setminus\{u,v\}} x^{d_G(w,v)} \\
= \;& H(L_k,v,x) - 1 - x = \frac{(1+x)(x-x^{2k+1})}{(1-x)}.
\end{aligned}$$

Combining with Lemma 5, we have

$$
\begin{aligned}
&\sum_{w\in V(L_k)\backslash\{u,v\}}\sum_{u'\in V(A)} x^{d_G(w,u')}+\sum_{w\in V(L_k)\backslash\{u,v\}}\sum_{v'\in V(B)} x^{d_G(w,v')}\\
=&\sum_{w\in V(L_k)\backslash\{u,v\}}\sum_{u'\in V(A)} x^{d_G(w,u)+d_G(u,u')}\\
&+\sum_{w\in V(L_k)\backslash\{u,v\}}\sum_{v'\in V(B)} x^{d_G(w,v)+d_G(v,v')}\\
=&\left[\sum_{w\in V(L_k)\backslash\{u,v\}} x^{d_G(w,v)}\right]\left[\sum_{u'\in V(A)} x^{d_G(u,u')}+\sum_{v'\in V(B)} x^{d_G(v,v')}\right]\\
=&\frac{(1+x)(x-x^{2k+1})}{(1-x)}\left[\frac{(1+x)^2\left[H_e(G_0,e,x)-1\right]+x^3-x^5}{1+2x+2x^2}\right]\\
=&\frac{(1+x)^2(x^4-x^{2k+4})}{(1+2x+2x^2)}+\frac{(1+x)^3(x-x^{2k+1})}{(1-x)(1+2x+2x^2)}\left[H_e(G_0,e,x)-1\right],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&H(G,x)\\
=\;&H(G_0,x)+H(L_k,x)-2-x+\sum_{w\in V(L_k)\backslash\{u,v\}}\sum_{u'\in V(A)} x^{d_G(w,u')}\\
&+\sum_{w\in V(L_k)\backslash\{u,v\}}\sum_{v'\in V(B)} x^{d_G(w,v')}\\
=\;&H(G_0,x)+H(L_k,x)-2-x+\frac{(1+x)^2(x^4-x^{2k+4})}{(1+2x+2x^2)}\\
&+\frac{(1+x)^3(x-x^{2k+1})}{(1-x)(1+2x+2x^2)}\left[H_e(G_0,e,x)-1\right]. \qquad \blacksquare
\end{aligned}
$$

Proof of Theorem 1. The proof will be made by employing mathematical induction on the number of maximal linear segment t in G.

In the case that $t=1$, Theorem 1 has been proved.

In the case that $t\geq 2$, it is supposed that Theorem 1 follows for any graph $G'\in CBG$ with the number of linear segments less than t.

Assume G is obtained from a graph $G_0\in CBG_{n-l_t+1}$ with segments S_1, $\cdots$, S_{t-1} by identifying the edge $e\in E(G_0)$ and a terminal edge in L_{l_t-1}, as shown in Fig. 3, where $k=l_t-1$. From Lemma 4, Lemma 6 and the induction hypothesis,

$$
\begin{aligned}
& x(1+x)^4 H_e(G,x) - (1+2x+2x^2)^2 H(G,x) \\
= \; & \big[x(1+x)^4 H_e(G_0,x) - (1+2x+2x^2)^2 H(G_0,x)\big] \\
& + \big[x(1+x)^4 H_e(L_{l_t-1},x) - (1+2x+2x^2)^2 H(L_{l_t-1},x)\big] \\
& + x(1+x)^4 \left[-1 + x^3 - x^{2(l_t-1)+3}\right] \\
& - (1+2x+2x^2)^2 \left[-2 - x + \frac{(1+x)^2(x^4 - x^{2(l_t-1)+4})}{1+2x+2x^2}\right] \\
= \; & \big[x(1+x)^4 H_e(G_0,x) - (1+2x+2x^2)^2 H(G_0,x)\big] \\
& + \big[x(1+x)^4 H_e(L_{l_t-1},x) - (1+2x+2x^2)^2 H(L_{l_t-1},x)\big] + 2 + 8x \\
& + 16x^2 + 18x^3 + 12x^4 + 3x^5 - x^6 - 2x^7 - x^8 + (1+x)^2 x^{2l_t+4} \\
= \; & \frac{(1+x)}{(1-x)} \sum_{i=1}^{t-1} \left(x^8 - x^{2l_i+6}\right) - x(1+x)^4 g(n-l_t+1,x) \\
& + \frac{(1+x)}{(1-x)} \left(x^8 - x^{2(l_t-1)+6}\right) - x(1+x)^4 g(l_t-1,x) + 2 + 8x \\
& + 16x^2 + 18x^3 + 12x^4 + 3x^5 - x^6 - 2x^7 - x^8 + (1+x)^2 x^{2l_t+4}. \\
= \; & \frac{(1+x)}{(1-x)} \sum_{i=1}^{t} \left(x^8 - x^{2l_i+6}\right) - x(1+x)^4 g(n,x).
\end{aligned}
$$

Then result follows.

***Acknowledgment*:** Our research was financed by the National Natural Science Foundation of China (No. 11901094 and No. 11401102 and No. 12171088). We are grateful to referees for helpful comments.

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