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On Complete Multipartite Orderenergetic Graphs*

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Abstract

Akbari et al. [MATCH Commun. Math. Comput. Chem. 84 (2020) 325–334] defined orderenergetic graphs as those graphs whose energy is equal to their order. They observed that complete tripartite graphs $K_{p,p,6p}$ are orderenergetic for every $p \geq 1$, and stated an expectation that these might be the only complete multipartite orderenergetic graphs with at least three parts. In this note we show the existence of infinitely many other families of such graphs with arbitrarily large number of parts, with $K_{\underbrace{p,\dots,p}_{10\times},40p}$ being an example of such family with 11 parts.

1 Introduction

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices, and let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of its adjacency matrix. The energy of G is defined

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[13] (see also [24]) as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Graphs whose energy may be simply related to their order received a lot of attraction from researchers. Among such graphs we find:

- *hypoenergetic* graphs with $\mathcal{E}(G) < n$ [4, 14, 16, 17, 20, 23, 31];
- *hyperenergetic* graphs with $\mathcal{E}(G) > 2n - 2$ [14, 18, 19, 29, 32];
- *borderenergetic* graphs with $\mathcal{E}(G) = 2n - 2$ and $G \ncong K_n$ [7–9, 11, 12, 15, 25, 27]
- and *orderenergetic* graphs with $\mathcal{E}(G) = n$ [1, 2, 21, 22, 26, 28].

In the paper [1], in which they have introduced orderenergetic graphs, Akbari et al. observed that complete tripartite graphs $K_{p,p,6p}$ are orderenergetic for every $p \geq 1$, and remarked that their computer-based search suggests that these graphs are the only complete multipartite orderenergetic graphs with at least three parts [1, Remark 4]. The main goal of this note is to show that there do exist many other infinite classes of complete multipartite orderenergetic graphs. Our extended computer search led to the new families described in the following theorem.

Theorem 1. *For an integer $p \geq 1$, the following complete multipartite graphs are orderenergetic:*

a) $K_{2p,\,2p,\,9p,\,39p}$

b) $K_{\underbrace{p,\ldots,p}_{5\times},\,5p,\,5p,\,55p}$

c) $K_{\underbrace{p,\ldots,p}_{8\times},\,3p,\,43p}$

d) $K_{\underbrace{p,\ldots,p}_{8\times},\,7p,\,55p}$

Our further study of the conditions for orderenergeticity of complete multipartite graphs of the form

$$K_{\underbrace{p,\dots,p}_{k\times},q}$$

led to the following theorem which ensures the existence of complete multipartite orderenergetic graphs with arbitrarily many parts.

Theorem 2. *Let the sequence* $(a_m, b_m)_{m\geq 1}$ *be defined as*

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix} = \frac{1}{2^m} \begin{pmatrix} 3 & 5 \\ 1 & 3 \end{pmatrix}^m \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Then for each $m \geq 1$ *and each* $p \geq 1$*, the complete multipartite graph*

$$K_{\underbrace{p,\dots,p}_{b_m^2+1},p(2b_m^2+a_mb_m+1)}$$

is orderenergetic.

Details of the computer search and the proof of Theorem 1 are given in Section 2, while Theorem 2 is proved in Section 3. Discussion and concluding remarks are given in Section 4.

2 Computer search and proof of Theorem 1

In our computer search we first used Wolfram Mathematica to generate partitions of all integers up to 70, and then used graph6java [10] to compute energy of the complete multipartite graphs determined by these partitions. Apart from the existing family $K_{p,p,6p}$, this search yielded four new examples of complete multipartite orderenergetic graphs: $K_{2,2,9,39}$, $K_{\underbrace{1,\dots,1}_{8\times},3,43}$, $K_{\underbrace{1,\dots,1}_{5\times},5,5,55}$ and $K_{\underbrace{1,\dots,1}_{8\times},7,55}$. As it turns out, each of these graphs can be turned into a family of graphs simply by multiplying the size of each part by an integer $p \geq 1$, as stated in Theorem 1.

We now need the following well known result.

Lemma 1 ([5, 6])**.** *The characteristic polynomial of the complete multipartite graph* $K_{n_1,n_2,\ldots,n_k}$ *is equal to*

$$\phi(K_{n_1,n_2,\ldots,n_k},\lambda) = \lambda^{n-k}\left(1-\sum_{i=1}^{k}\frac{n_i}{\lambda+n_i}\right)\prod_{j=1}^{k}(\lambda+n_j). \tag{1}$$

We are now in position to prove Theorem 1.

Proof of Theorem 1. We compute the characteristic polynomial and the energy of each family of graphs using (1).

a) We have

$$\begin{aligned}
&\phi(K_{2p,2p,9p,39p},\lambda)\\
&= \lambda^{52p-4}\left(1-\frac{4p}{\lambda+2p}-\frac{9p}{\lambda+9p}-\frac{39p}{\lambda+39p}\right)(\lambda+2p)^2(\lambda+9p)(\lambda+39p)\\
&= \lambda^{52p-4}(\lambda+2p)(\lambda-26p)(\lambda^2+24\lambda p+81p^2)\\
&= \lambda^{52p-4}(\lambda+2p)(\lambda-26p)\left(\lambda+3p(4-\sqrt{7})\right)\left(\lambda+3p(4+\sqrt{7})\right)
\end{aligned}$$

Hence the non-zero eigenvalues of $K_{2p,2p,9p,39p}$ are $26p$, $-2p$, $-3p(4-\sqrt{7})$ and $-3p(4+\sqrt{7})$, which shows that $\mathcal{E}(K_{2p,2p,9p,39p}) = 52p$.

b) After simplification we have

$$\begin{aligned}
&\phi(K_{\underbrace{p,\ldots,p}_{5\times},5p,5p,55p},\lambda)\\
&= \lambda^{70p-8}(\lambda+p)^4(\lambda+5p)(\lambda-35p)(\lambda^2+26\lambda p+55p^2).
\end{aligned}$$

Hence the non-zero eigenvalues of $K_{\underbrace{p,\ldots,p}_{5\times},5p,5p,55p}$ are $35p$, $-5p$, $-p$ with multiplicity 4, $-p(13-\sqrt{114})$ and $-p(13+\sqrt{114})$, which shows that $\mathcal{E}(K_{\underbrace{p,\ldots,p}_{5\times},5p,5p,55p}) = 70p$.

c) After simplification we have

$$\phi(K_{\underbrace{p,\ldots,p}_{8\times},3p,43p},\lambda) = \lambda^{54p-10}(\lambda+p)^7(\lambda-27p)(\lambda^2+20p\lambda+43p^2).$$

Hence the non-zero eigenvalues of $K_{\underbrace{p,\ldots,p}_{8\times},3p,43p}$ are $27p$, $-p$ with multiplicity 7, $-p(10-\sqrt{57})$ and $-p(10+\sqrt{57})$, which shows that $\mathcal{E}(K_{\underbrace{p,\ldots,p}_{8\times},3p,43p}) = 54p$.

d) After simplification we have

$$\phi(K_{\underbrace{p,\ldots,p}_{8\times},7p,55p},\lambda) = \lambda^{70p-10}(\lambda+p)^7(\lambda-35p)(\lambda^2+28\lambda p+99p^2).$$

Hence the non-zero eigenvalues of $K_{\underbrace{p,\ldots,p}_{8\times},7p,55p}$ are $35p$, $-p$ with multiplicity 7, $-p(14-\sqrt{97})$ and $-p(14+\sqrt{97})$, which shows that $\mathcal{E}(K_{\underbrace{p,\ldots,p}_{8\times},7p,55p}) = 70p$.

Thus the graphs $K_{2p,2p,9p,39p}$, $K_{\underbrace{p,\ldots,p}_{5\times},5p,5p,55p}$, $K_{\underbrace{p,\ldots,p}_{8\times},3p,43p}$ and $K_{\underbrace{p,\ldots,p}_{8\times},7p,55p}$ are orderenergetic. ■

3 Pell's equation and proof of Theorem 2

Motivated by the original example $K_{p,p,6p}$, we will prove Theorem 2 by looking for sufficient conditions which ensure that the complete multipartite graph of the form

$$K_{\underbrace{p,\ldots,p}_{k\times},q}$$

is orderenergetic. In this special case, the characteristic polynomial reads

$$\begin{aligned}
&\phi(K_{\underbrace{p,\ldots,p}_{k\times},q},\lambda)\\
&= \lambda^{kp+q-k-1}\left(1-k\frac{p}{\lambda+p}-\frac{q}{\lambda+q}\right)(\lambda+p)^k(\lambda+q)\\
&= \lambda^{kp+q-k-1}[(\lambda+p)(\lambda+q)-kp(\lambda+q)-q(\lambda+p)](\lambda+p)^{k-1}\\
&= \lambda^{kp+q-k-1}[\lambda^2-p(k-1)\lambda-kpq](\lambda+p)^{k-1}.
\end{aligned}$$

Hence the non-zero eigenvalues of $K_{\underbrace{p,\ldots,p}_{k\times},q}$ are $-p$ with multiplicity $k-1$ and two simple eigenvalues

$$\frac{p(k-1)\pm\sqrt{p^2(k-1)^2+4kpq}}{2},$$

of different signs, so that its energy is equal to

$$\mathcal{E}(K_{\underbrace{p,\ldots,p}_{k\times},q}) = p(k-1)+\sqrt{p^2(k-1)^2+4kpq}.$$

The requirement that $K_{\underbrace{p,\ldots,p}_{k\times},q}$ is orderenergetic implies

$$p(k-1)+\sqrt{p^2(k-1)^2+4kpq} = pk+q,$$

or, after simplification and reordering of terms,

$$q^2+2p(1-2k)q-k(k-2)p^2=0.$$

If we consider this equality as a quadratic equation in q, its solutions are

$$q_{1,2}=p\left(2k-1\pm\sqrt{(5k-1)(k-1)}\right).$$

Since q has to be a positive integer, we further require that $(5k-1)(k-1)$ is a perfect square (and we take only $+$ sign instead of $\pm$ to make q positive). As $\gcd(5k-1,k-1)\in\{1,2,4\}$, the product $(5k-1)(k-1)$ is a perfect square if and only if either for some integers a and b,

$$\text{Case I:} \quad 5k-1=a^2 \quad \text{and} \quad k-1=b^2$$

or for some integers c and d,

$$\text{Case II:} \quad 5k-1=2c^2 \quad \text{and} \quad k-1=2d^2.$$

Case I implies $a^2-5b^2=4$, which is an instance of the general Pell's equation with the minimal positive solution $(a_1,b_1)=(3,1)$. According to [3, Eq. (4.4.5)], all its positive solutions (a_m,b_m) are of the form

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix} = \frac{1}{2^m} \begin{pmatrix} 3 & 5 \\ 1 & 3 \end{pmatrix}^m \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

from where we obtain

$$k=b_m^2+1 \quad \text{and} \quad q=p(2b_m^2+a_mb_m+1),$$

which then yields the orderenergetic complete multipartite graph

$$K_{\underbrace{p,\dots,p}_{b_m^2+1},\,p(2b_m^2+a_mb_m+1)},$$

as stated in the theorem.

Case II implies $c^2-5d^2=2$, but this equation has no integer solutions. Namely, modulo 4 the squares of integers are equal to either 0 or 1, so that c^2-5d^2 belongs to $\{0,1,3\}$ modulo 4, and hence it cannot be equal to 2.

At the end, note that the conditions determined in this section are both sufficient and necessary, so that Theorem 2 actually describes all orderenergetic complete multipartite graphs of the form $K_{\underbrace{p,\dots,p}_{k\times},\,q}$.

4 Discussion and concluding remarks

The first few entries of the sequence (a_m, b_m) from Theorem 2 are:

$$\begin{aligned}(a_1, b_1) &= (3, 1),\\ (a_2, b_2) &= (7, 3),\\ (a_3, b_3) &= (18, 8).\end{aligned}$$

The choice $m = 1$ leads to the already known family $K_{p,p,6p}$, while $m = 2$ and $m = 3$ lead, respectively, to the families

$$K_{\underbrace{p,\ldots,p}_{10},40p} \quad \text{and} \quad K_{\underbrace{p,\ldots,p}_{65},273p}.$$

Since $b_m \to \infty$ for $m \to \infty$, we see that Theorem 2 provides examples of orderenergetic complete multipartite graphs that have more than M parts for any fixed M.

The family $K_{\underbrace{p,\ldots,p}_{10},40p}$ further solves [1, Problem 20], which asks for connected orderenergetic graphs (other than balanced complete bipartite graphs) of orders $8t+2$, $8t+4$ and $8t+6$ for some t. Since $K_{\underbrace{p,\ldots,p}_{10},40p}$ has order $50p$, taking $p = 4p'+1$ yields a connected orderenergetic graph of order $200p'+50 = 8t+2$ for $t = 25p'+6$, as required. Similarly, taking $p = 4p'+2$ yields a connected orderenergetic graph of order $200p'+100 = 8t+4$ for $t = 25p'+12$, while taking $p = 4p'+3$ yields a connected orderenergetic graph of order $200p'+150 = 8t+6$ for $t = 25p'+18$, thus completely solving Problem 20. Note that the family $K_{\underbrace{p,\ldots,p}_{65},273p}$ could be similarly used for the same purpose, as its elements have order $338p \equiv 2p \pmod 8$.

Theorem 2 in some part also resolves [1, Problem 19], which asks for a method of constructing connected orderenergetic graphs that does not use the direct product. However, if Problem 19 is more naturally understood as finding graph operations that will construct a new orderenergetic graph from the existing orderenergetic graphs, then one should pay attention to [30], where it was proved that the direct product is the only instance of

NEPS (a rather general graph operation) for which the energy of NEPS can be actually represented as a function of the energy of its factor graphs. This result shows that potential solutions to Problem 19 should be preferably sought among non-NEPS-based graph operations.

One also has to note that it is not possible to extend Theorem 2 to complete multipartite graphs of the form $K_{\underbrace{p,\ldots,p}_{a\times},\underbrace{q,\ldots,q}_{b\times}}$. Following analogous reasoning to that in the proof above, one can get that the energy of complete multipartite graphs of this form is equal to

$$p(a-1)+q(b-1)+\sqrt{[p(a-1)+q(b-1)]^2+4pq(a+b-1)}.$$

However, if both $a \geq 2$ and $b \geq 2$, then

$$\sqrt{[p(a-1)+q(b-1)]^2+4pq(a+b-1)} > p+q,$$

so that such graphs cannot be orderenergetic.

However, the examples of orderenergetic complete multipartite graphs

$$K_{\underbrace{p,\ldots,p}_{8\times},3p,43p} \quad \text{and} \quad K_{\underbrace{p,\ldots,p}_{8\times},7p,55p}$$

from Theorem 1 make it very probable that Theorem 2 could actually be extended to complete multipartite graphs of the form

$$K_{\underbrace{p,\ldots,p}_{k\times},q,r},$$

which we leave as an exercise for the reader.

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