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Construction of Orderenergetic Graphs

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Abstract

The energy $\mathcal{E}(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. If the energy of a graph G of order n is equal to its order, then G is said to be orderenergetic. In this paper, we give two methods to construct orderenergetic graphs. Infinitely many connected non-complete multipartite orderenergetic graphs can be constructed by using regular graphs.

1 Introduction

In this paper we are concerned with simple undirected graphs, without self-loops and weighted edges. Let G be such a graph of order n, with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E(G)$. The complement graph of G is denoted by $\overline{G}$. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be the simple undirected graphs. Then the union $G = G_1 \cup G_2$ of G_1 and G_2 is defined as $G = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. The join $G = G_1 \vee G_2$

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of G_1 and G_2 is defined as $G = \overline{\overline{G_1} \cup \overline{G_2}}$. The degree of a vertex v_i is the number of edges adjacent to the vertex v_i. A graph is called regular, if degree of each vertex is same number. Following the standard terminology, we use the $K_{n_1,n_2,\dots,n_t}$, C_n, nG to denote the complete multipartite graph of order $n_1 + n_2 + \cdots + n_t$, the cycle of order n, and n copies of graph G.

Let $A(G)$ be the $(0,1)-$adjacency matrix of a graph G and its $(i,j)-$ element is

$$A(G)_{ij} = \begin{cases} 1 & \text{if} \quad v_iv_j \in E(G) \\ 0 & \text{if} \quad v_iv_j \notin E(G) \\ 0 & \text{if} \qquad i = j. \end{cases}$$

The eigenvalues of $A(G)$ are denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$ and then the spectrum $Sp(G)$ of graph G is $Sp(G) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. The energy of graph G is defined by $\mathcal{E}(G)$ and is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| \,.$$

This concept was first introduced by Gutman [11] as a way to model the total $\pi-$electron energy of a molecule. Details and more properties on graph energy can be found in the book [17] and the most recent works [1, 5–8, 10, 19, 21]. A graph G of order n is said to be borderenergetic if its energy equals the energy of the complete graph K_n, i.e., if $\mathcal{E}(G) = 2(n-1)$. The definition was first proposed in [13]. The borderenergetic graph attracted much attention and are being studied in these papers [9, 12–14, 20].

Recently, S. Akbari et al. [2] proposed the concept of orderenergetic graphs, namely graphs of order n satisfying $\mathcal{E}(G) = n$. It is shown in [2] that there are infinitely many connected orderenergetic graphs, all non-singular connected graphs only the path P_2 is orderenergetic and there is no orderenergetic graph with nullity $\eta = 1$. It is known that the energy of graph is never an odd integer [3]. Therefore, orderenergetic graphs must have even number of vertices. In [2], the authors use a computer-aided search all orderenergetic connected graphs up to 10 vertices and propose the complete multipartite graph $K_{p,p}, K_{6p,p,p}$ is orderenergetic for all $p \geq 1$. A novel general graph operation was presented in [15]. It is shown in [15] that two different sequences of orderenergetic graphs from a given orderenergetic graph and some orderenergetic graphs from non-orderenergetic graphs are constructed by means of the general graph operation. And then, another general unary graph operation was proposed and several methods for generating orderenergetic graphs using this new operation are given in [16]. It is worth mentioning that these two graph operations can also be used to generate integral and equienergetic graphs. It is shown in [18] that if the cycle C_4 is not an induced subgraph of a graph G with nullity $\eta = 3$, then G is not orderenergetic and there are connected orderenergetic graphs $aK_{p,p} \vee \overline{K_{2p(4a-1)}}$ for integers $a \geq 1, p \geq 1$. In [2], the authors proposed several open problems and conjectures. For example, the following problem:

Problem 1. Find a method for constructing connected orderenergetic graphs, not using the direct product.

In [18], it was shown that there are connected orderenergetic graphs on $10k + 8$ vertices for all $k \geq 0$. Motivated by Problem 1 and this result, we show that there are connected orderenergetic graphs $G = pG_1 \vee \overline{K_q}$ and if $\mathcal{E}(G_1) = n + 2r - 2t + 2$, then $\mathcal{E}(\overline{G_1}) = n$, where G_1 is a $r-$regular graph of order n with t non-negative eigenvalues.

2 Constructing orderenergetic graphs

We now show how to construct $(n-1-r)-$regular $(r > 2)$ orderenergetic graphs by using some $r-$regular graphs.

Lemma 1. *[4] Let G be a $r-$regular graph of order n with spectrum $Sp(G) = \{r, \lambda_2, \ldots, \lambda_n\}$. Then $Sp(\overline{G}) = \{n-1-r, -1-\lambda_2, \ldots, -1-\lambda_n\}$.*

Lemma 2. *[4] If G_1 is an r_1 regular with n_1 vertices and G_2 is r_2 regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \vee G_2$ is given by*

$$\phi(G_1 \vee G_2, x) = \frac{\phi(G_1, x)\phi(G_2, x)}{(x-r_1)(x-r_2)} \left((x-r_1)(x-r_2) - n_1 n_1\right).$$

Theorem 1. *Let G be $r-$regular integral graph of order n with t non-negative eigenvalues. If $\mathcal{E}(G) = n + 2r - 2t + 2$, then $\mathcal{E}(\overline{G}) = n$.*

Proof. From Lemma 1 and $\lambda_1 = r$, the energy of the complement of G is

$$\begin{aligned}
\mathcal{E}(\overline{G}) &= n - 1 - r + \sum_{j=2}^{n} |1 + \lambda_j| \\
&= n - 1 - r + \left(t - 1 + \sum_{j=2}^{t} |\lambda_j| \right) + \sum_{j=t+1}^{n} (-1 - \lambda_j) \\
&= n - 1 - r + t - 1 - r + \sum_{j=1}^{t} |\lambda_j| + \sum_{j=t+1}^{n} |\lambda_j| - (n - t) \\
&= n + t - 2r - 2 + \mathcal{E}(G) + t - n \\
&= 2t - 2r - 2 + n + 2r - 2t + 2 = n
\end{aligned}$$

■

A class of non-complete connected $(n-1-r)-$regular borderenergetic graphs can be constructed in [9].

Theorem 2. *[9] Let r be an even integer. Let $G = pG_1 \cup qK_{r+1}$ be a disconnected $r-$regular graph consisting of p copies of G_1 and q copies of K_{r+1}, where G_1 is a $r-$regular integral graph with $r+2$ vertices, having t non-negative eigenvalues, and satisfying $\mathcal{E}(G_1) = 2r + 4 - 2t + \frac{2r}{p}, p|2r, p \geq 1, q \geq 1$. Then $\overline{G}$ is a connected non-complete borderenergetic graph.*

From Theorem 2, when $n = r+2, p = 2$, the $\mathcal{E}(G_1) = 2r + 4 - 2t + \frac{2r}{p} = n + 2r - 2t + 2$. According to Theorem 1, the graph G_1 is orderenergetic.

The graph G needs not be connected (see Example 1).

Example 1. G^1 is a connected $2-$regular graph with 6 vertices whereas $\overline{G^1}$ is a connected $3-$regular orderenergetic graph, see Fig.1. Note that

$$Sp(G^1) = \{2, 2, -1, -1, -1, -1\} \quad and \quad \mathcal{E}\left(G^1\right) = 8 = 6 + 4 - 4 + 2$$

whereas

$$Sp(\overline{G^1}) = \{3, -3, 0, 0, 0, 0\} \quad and \quad \mathcal{E}\left(\overline{G^1}\right) = 6.$$

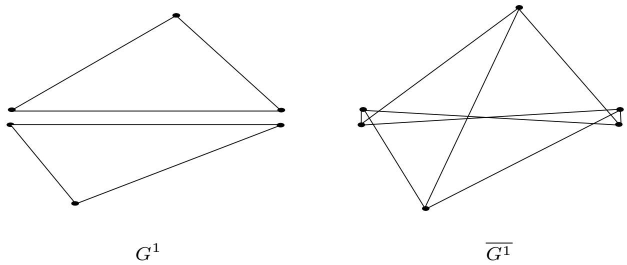

Figure 1. The graphs from Example 1

It is easy to find examples of disconnected orderenergetic graphs. A more interesting task is to construct connected non-complete multipartite orderenergetic graphs. Such a construction is achieved by means of the following theorems:

Theorem 3. *Let G be $r-$regular orderenergetic graph with n vertices then for $m \neq 0$, $G \vee \overline{K_m}$ is orderenergetic if and only if $m = 4n - 2r$.*

Proof. Let $r = \lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of G. As G is orderenergetic of n vertices, $\mathcal{E}(G) = n$ which imply that

$$\sum_{i=2}^{n} |\lambda_i| = n - r. \tag{1}$$

By Lemma 2, the characteristic polynomial of $G \vee \overline{K_m}$ is given by

$$\phi(G \vee \overline{K_m}) = x^{m-1}(x - \lambda_2)(x - \lambda_3) \cdots (x - \lambda_n)(x^2 - rx - nm).$$

Let θ_1 and θ_2 are roots of polynomial $x^2 - rx - nm$. It is easy to observe that θ_1 and θ_2 are of opposite sign. With out loss of generality we assume that $\theta_1 > 0, \theta_2 < 0$. Also,

$$\theta_1 + \theta_2 = r, \tag{2}$$

$$\theta_1 \theta_2 = -nm. \tag{3}$$

Here,

$$Sp(G) = \left\{ \overbrace{0, \cdots, 0}^{m-1}, \lambda_2, \lambda_3, \cdots, \lambda_n, \theta_1, \theta_2 \right\}.$$

Hence,

$$\mathcal{E}(G \vee \overline{K_m}) = \sum_{i=2}^{n} |\lambda_i| + |\theta_1| + |\theta_2| = \sum_{i=2}^{n} |\lambda_i| + \theta_1 - \theta_2 = n - r + \theta_1 - \theta_2.$$

If $G \vee \overline{K_m}$ is orderenergetic then

$$\mathcal{E}(G \vee \overline{K_m}) = n + m = n - r + \theta_1 - \theta_2 \Leftrightarrow \theta_1 - \theta_2 = m + r \tag{4}$$

By (2) and (4)

$$\theta_1 = \frac{m + 2r}{2}$$
$$\theta_2 = -\frac{m}{2}.$$

From (3)

$$\begin{aligned} &\theta_1\theta_2 = -nm \\ &\Leftrightarrow m(m + 2r) = 4nm \\ &\Leftrightarrow m^2 + m(2r - 4n) = 0 \\ &\Leftrightarrow m = 4n - 2r\ . \end{aligned}$$

■

Let $G = K_{p,p}, n = 2p, r = p$. Then we have:

Corollary. *(Lemma 3 in [2]) The graph $K_{p,p} \vee \overline{K_{6p}} \cong K_{6p,p,p}$ is orderenergetic.*

Let $G = aK_{p,p}, n = 2ap, r = p$. Then we have:

Corollary. *(Theorem 5 in [18]) The graph $aK_{p,p} \vee \overline{K_{2p(4a-1)}}$ is orderenergetic.*

Furthermore, we have the following result.

Theorem 4. *Let $G = pG_1 \vee \overline{K_q}$ be a connected graph consisting of p copies of G_1 joined $\overline{K_q}$, where G_1 is a connected $r-$regular graph with n*

vertices. Then G is a connected orderenergetic graph if and only if $\mathcal{E}(G_1) = \frac{1}{p}\left(np+q+r-\sqrt{4npq+r^2}\right)$, $p|\left(q+r-\sqrt{4npq+r^2}\right)$, $p \geq 1, q \geq 1$.

Proof. Let $r = \lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of G_1. Then

$$\sum_{i=2}^{n} |\lambda_i| = \mathcal{E}(G_1) - r. \tag{5}$$

By Lemma 2, the characteristic polynomial of $pG_1 \vee \overline{K_q}$ is given by

$$\begin{aligned}&\phi(pG_1 \vee \overline{K_q})\\ &= x^{q-1}(x-r)^{p-1}(x-\lambda_2)^p(x-\lambda_3)^p\cdots(x-\lambda_n)^p(x^2-rx-npq).\end{aligned}$$

Let θ_1 and θ_2 are roots of polynomial $x^2 - rx - npq$. It is easy to observe that θ_1 and θ_2 are of opposite sign. With out loss of generality we assume that $\theta_1 > 0, \theta_2 < 0$. Also,

$$\theta_1 = \frac{1}{2}(r+\sqrt{4npq+r^2}), \theta_2 = \frac{1}{2}(r+\sqrt{4npq+r^2}). \tag{6}$$

$$\theta_1 - \theta_2 = \sqrt{4npq+r^2}. \tag{7}$$

Hence,

$$\begin{aligned}\mathcal{E}(G) &= r(p-1) + p\sum_{i=2}^{n}|\lambda_i| + |\theta_1| + |\theta_2|\\ &= r(p-1) + p\sum_{i=2}^{n}|\lambda_i| + \theta_1 - \theta_2\\ &= r(p-1) + p\left(\mathcal{E}(G_1) - r\right) + \theta_1 - \theta_2.\end{aligned}$$

If G is orderenergetic then

$$\begin{aligned}&\mathcal{E}(G) = np+q = r(p-1) + p\left(\mathcal{E}(G_1) - r\right) + \theta_1 - \theta_2\\ \Leftrightarrow\ &\theta_1 - \theta_2 = np+q-r(p-1) - p(\mathcal{E}(G_1) - r)\end{aligned} \tag{8}$$

By (7) and (8)

$$\begin{aligned}\theta_1-\theta_2 &= np+q-r(p-1)-p(\mathcal{E}(G_1)-r)\\ &= \sqrt{4npq+r^2}.\end{aligned}$$

Then $\mathcal{E}(G_1)=\frac{1}{p}\left(np+q+r-\sqrt{4npq+r^2}\right)$. ■

By Theorem 4, it is easy to construct connected orderenergetic graphs by starting from graphs of small order. Here are some examples.

Corollary. *If integers $p\geq 1, q\geq 1$ satisfy $q=7p-2-2\sqrt{12p^2-6p+1}$, then $pC_3\vee\overline{K_q}$ is orderenergetic graph.*

For example, the energy of graph $4C_3\vee\overline{K_{52}}$ is exactly 64.

Corollary. *If integers $p\geq 1, q\geq 1$ satisfy $q=15p-2-2\sqrt{56p^2-14p+1}$, then $p(C_3\cup C_4)\vee\overline{K_q}$ is orderenergetic graph.*

For example, the energy of graph $4(C_3\cup C_4)\vee\overline{K_{116}}$ is exactly 144.

Corollary. *If integers $p\geq 1, q\geq 1$ satisfy $q=20p-6-2\sqrt{96p^2-48p+9}$, then $pG^0\vee\overline{K_q}$ is orderenergetic graph, where G^0 is a connected $6-$regular graph with 8 vertices and $\mathcal{E}\left(G^0\right)=12$, see Fig.2.*

For example, the energy of graph $3G^0\vee\overline{K_{108}}$ is exactly 132.

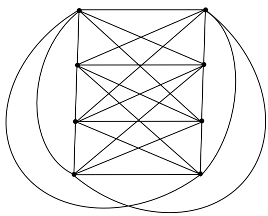

Figure 2. The graph G^0

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