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Sombor Index of a Graph and of Its Subgraphs

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Abstract

Recently a new vertex-degree based molecular structure descriptor was defined as Sombor index. Let G be a simple graph. The Sombor index of G, denoted by $SO(G)$, is $\sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$, where d_v is the degree of v. In this paper we investigate the relation between the Sombor index of a graph and the Sombor index of its subgraphs. As a concequence we find some relations between $SO(G)$ and $SO(\overline{G})$, where $\overline{G}$ is the complement graph of G. In particular, we show that if G is a graph of order n, then $SO(G) + SO(\overline{G}) \leq \frac{n(n-1)^2}{\sqrt{2}}$ and the equality holds if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.

1 Introduction

In this paper all graphs are simple, that is they are finite and undirected, without loops and multiple edges. Let $G = (V(G), E(G))$ be a simple graph. The *order* of G is the number of vertices of G. By $e = uv$ we mean the edge e between u and v. For a vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$ or $deg(v, G)$. A *pendant* vertex is a vertex with degree one and a pendant

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edge is an edge such that one of its end points is pendant vertex. A $k-$*regular* graph is a graph such that every vertex of that has degree k. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *disjoint union* of G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. The graph rG denotes the disjoint union of r copies of G. The *edgeless graph* (or *empty graph*), the *complete graph*, the *cycle*, and the *path* of order n, are denoted by $\overline{K_n}$, K_n, C_n and P_n, respectively. Let t and $n_1, \ldots, n_t$ be some positive integers. By $K_{n_1,\ldots,n_t}$ we mean the *complete multipartite graph* with parts size $n_1, \ldots, n_t$. In particular, the *complete bipartite graph* with part sizes m and n denoted by $K_{m,n}$. The *star* of order n, denoted by S_n, is the complete bipartite graph $K_{1,n-1}$.

In chemical graph theory, there are many topological indices. Recently, a new index, Sombor index, has been defined by Ivan Gutman in [5]. For a graph G, the Sombor index of G, denoted by $SO(G)$, is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

where d_v is the degree of v. For example the Sombor index of the star S_n is $(n-1)\sqrt{(n-1)^2+1}$. There are many papers related to properties of Sombor index, for instance see [1–12] and the references cited therein.

In this paper, we study the relation between the Sombor index of a graph and the Sombor index of its subgraphs. As a concequence, we find some relations between $SO(G)$ and $SO(\overline{G})$, where $\overline{G}$ is the complement graph of G. In particular, we show that if G is a graph of order n, then

$$SO(G) + SO(\overline{G}) \leq \frac{n(n-1)^2}{\sqrt{2}}$$

and the equality holds if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.

2 Main results

Let G be a graph. By $G = (H_1, \ldots, H_k)$ we mean that $H_1, \ldots, H_k$ are some spanning subgraphs of G such that the edge sets of $H_1, \ldots, H_k$ are disjoint ($E(H_i) \cap E(H_j) = \emptyset$ for every $i \neq j$) and $E(G) = E(H_1) \cup \cdots \cup$

$E(H_k)$. For example $K_4 = (C_4, 2K_2)$. In this section we investigate the relation between the Sombor index of a graph and the Sombor index of its subgraphs. For every graph H, by $deg(v, H)$ we mean the degree of a vertex v of H in H. For every vertices u and v of H by $f(uv, H)$ we mean

$$f(uv, H) = \begin{cases} \sqrt{(deg(u, H))^2 + (deg(v, H))^2}, & \text{if } u \text{ and } v \text{ are adjacent;} \\ 0\,, & \text{otherwise.} \end{cases}$$

Therefore

$$SO(H) = \frac{1}{2} \sum_{u,v \in V(H)} f(uv, H). \tag{1}$$

Now we prove one of the main results of this paper.

Theorem 1. *Let G be a connected graph. Assume that $G = (H_1, \ldots, H_k)$. Then*

$$SO(G) \geq SO(H_1) + \cdots + SO(H_k).$$

Moreover, the equality holds if and only if for some $j \in \{1, \ldots, k\}$, $H_j = G$ and the other subgraphs are empty graphs.

Proof. If $k = 1$, then there is nothing to prove. Now suppose that $k \geq 2$. Without loss of the generality we can assume that all spanning subgraphs $H_1, \ldots, H_k$ are non-empty graphs. To complete the proof it suffices to show that if for $i = 1, \ldots, k$, $H_i \neq G$, then

$$SO(G) > SO(H_1) + \cdots + SO(H_k).$$

Thus assume that $H_1 \neq G, \ldots, H_k \neq G$. Let n be the order of G and $V(G) = \{v_1, \ldots, v_n\}$. Since $k \geq 2$, $n \geq 2$. We claim that there exist a vertex v_r and the spanning subgraph H_s such that $deg(v_r, H_s) \neq 0$ and $deg(v_r, H_s) \neq deg(v_r, G)$. In other words, there exist a vertex v_r and the spanning subgraph H_s such that $1 \leq deg(v_r, H_s) \leq deg(v_r, G) - 1$. By contradiction, assume that for $i = 1, \ldots, n$ and $j = 1, \ldots, k$,

$$deg(v_i, H_j) = 0 \;\text{ or }\; deg(v_i, H_j) = deg(v_i, G). \tag{2}$$

We note that for every vertex v of G,

$$deg(v,G) = deg(v,H_1) + \cdots + deg(v,H_k). \tag{3}$$

Since $deg(v_i,H_j) = 0$ or $deg(v_i,H_j) = deg(v_i,G)$, by (3) we conclude that for every vertex v_i there exists a spanning subgraph H_j, such that $deg(v_i,H_j) = deg(v_i,G)$ and $deg(v_i,H_l) = 0$ for $l \neq j$. Without loss of the generality assume that $deg(v_1,H_1) = deg(v_1,G)$ and $deg(v_1,H_l) = 0$ for $l \neq 2$. Now we show that $H_1 = G$.

Consider a vertex $v \neq v_1$ in G. Since G is connected, there is a path, say $v_1 v_{j_1} \cdots v$, in G between v_1 and v. Since $deg(v_1,H_1) = deg(v_1,G)$, the neighbors of v_1 in G and the neighbors of v_1 in H are the same. This shows that $deg(v_{j_1},H_1) \neq 0$, and so by (2), $deg(v_{j_1},H_1) = deg(v_{j_1},G)$. Using this procedure we find that $deg(v,H_1) = deg(v,G)$. This shows that $H_1 = G$, a contradiction. Thus the claim is proved. Therefore there exist a vertex v_r and the spanning subgraph H_s such that $1 \leq deg(v_r,H_s) \leq deg(v_r,G) - 1$. This shows that v_r is not an isolated vertex of H_s. Suppose that v_t is a neighbor of v_r in H. Since $deg(v_r,H_s) \leq deg(v_r,G) - 1$, we find that

$$f(v_r v_t, G) > f(v_r v_t, H_s). \tag{4}$$

On the other hand for every vertices u and v of G, $f(uv,G) \geq f(uv,H_i)$, for $i = 1,\ldots,k$. Hence by (1) and (4),

$$SO(G) = \frac{1}{2} \sum_{1 \leq i,j \leq n} f(v_i v_j, G) > \sum_{l=1}^{k} \frac{1}{2} \sum_{1 \leq i,j \leq n} f(v_i v_j, H_l) = \sum_{l=1}^{k} SO(H_l).$$

Thus $SO(G) > \sum_{l=1}^{k} SO(H_l)$ and the proof is complete. ■

Remark. Since $SO(H \cup rK_1) = SO(H)$ (for every graph H), Theorem 1 holds when $H_1,\ldots,H_k$ are some subgraphs of G (not necessary spanning subgraphs) such that the edge sets of $H_1,\ldots,H_k$ are disjoint ($E(H_i) \cap E(H_j) = \emptyset$ for every $i \neq j$) and $E(G) = E(H_1) \cup \cdots \cup E(H_k)$.

Remark. Let G be a graph and $e = uv$ be an edge of G. Clearly, $f(e,G) \geq \sqrt{2}$. This shows that $SO(G) \geq \sqrt{2}m$, (where m is the number of edges of

G) and the equality holds if and only if $G \cong rK_2 \cup sK_1$.

Using Remark 2 we find the next result.

Theorem 2. *Let G be a graph of order n. Then*

$$SO(G) + SO(\overline{G}) \geq \frac{n(n-1)}{\sqrt{2}}$$

and the equality holds if and only if $1 \leq n \leq 2$.

We conclude the paper by the following result.

Theorem 3. *Let G be a graph of order n. Then*

$$SO(G) + SO(\overline{G}) \leq \frac{n(n-1)^2}{\sqrt{2}}$$

and the equality holds if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.

Proof. Clearly $K_n = (G, \overline{G})$. Thus by Theorem 1,

$$SO(G) + SO(\overline{G}) \leq SO(K_n) = \frac{n(n-1)^2}{\sqrt{2}},$$

and the equality holds if and only if $G \cong K_n$ or $\overline{G} \cong K_n$. ■

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