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Maximum and Minimum Lanzhou Index of c-Cyclic Graphs*

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Abstract

For a connected graph G with n vertices, the Lanzhou index of G is defined as

$$Lz(G) = \sum_{v \in V(G)} d(v)^2 \, [n - 1 - d(v)],$$

where $d(v)$ is the degree of vertex v in G. The extremal graphs with minimum (respectively, maximum) Lanzhou index has been determined for trees, unicyclic graphs, bicyclic graphs and tricyclic graphs with n vertices, respectively. In this paper, by applying the majorization method, we determine the unique extremal graph with minimum Lanzhou index for c-cyclic graph for $n \geq 3c + 4$ vertices and $c \geq 1$. Besides, we determine the unique extremal graph with maximum Lanzhou index in the class of c-cyclic graph with n vertices for $3 \leq c \leq \frac{n}{13}$, and we also illustrate an example to show that the bound $\frac{n}{13}$ is the best possible. This extends the corresponding results of [4, 9–11, 13].

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1 Introduction

Throughout this paper we consider undirected simple connected graphs. Let G be a graph with vertex set $\mathbf{V}(G)$ and edge set $\mathbf{E}(G)$. A connected graph with $m = n + c - 1$ edges and n vertices is called a c-**cyclic** graph. Especially, when $c = 0, 1, 2$ or 3, then G is called a tree, unicyclic graph, bicyclic graph or tricyclic graph, respectively. As usual, let $d(u)$ and $N(u)$ denote, respectively, the degree and neighbor set of the vertex $u \in V(G)$. A vertex of degree one will be always referred as a **pendent vertex**. For $\mathbf{V}(G) = \{v_1, v_2, \ldots, v_n\}$, if the degree of v_i equals d_i for $1 \leq i \leq n$, then $\pi = (d_1, d_2, \ldots, d_n)$ is called the **degree sequence** of graph G. Sometimes, we write $d_i(G)$ in place of d_i to indicate the dependent of G. Clearly, if G is a c-cyclic graph with degree sequence $\pi = (d_1, d_2, \ldots, d_n)$, then

$$\sum_{i=1}^{n} d_i = 2(n + c - 1). \tag{1}$$

Throughout this paper, we enumerate the degrees in non-increasing order, that is, $d_1 \geq d_2 \geq \cdots \geq d_n$.

The **first Zagreb index** $M_1(G)$ and the **forgotten index** $F(G)$ of graph G is defined as

$$M_1(G) = \sum_{v \in V(G)} d(v)^2 \text{ and } F(G) = \sum_{v \in V(G)} d(v)^3,$$

respectively. The first Zagreb index was defined by Gutman and Trinajstić in [7], while the forgotten index was reintroduced by Furtula and Gutman in [5]. The mathematical and chemical properties of the first Zagreb index haven been studied in [6, 15, 16].

In 2018, Vukičević, Li, Sedlar and Doslić, proposed a new topological index, that is, the **Lanzhou index** $Lz(G)$, for a molecular graph G with n vertices [13], where

$$Lz(G) = (n-1)M_1(G) - F(G) = \sum_{v \in V(G)} d(v)^2 \, [n - 1 - d(v)].$$

In [13], the authors showed that the Lanzhou index behaves better than the existing ones in predicting a chemically relevant property. From the definition, one can easily see that the Lanzhou index is a linear combination of Zagreb and forgotten indices [1].

A **chemical graph** is a connected graph with maximum degree at most four. Determining extreme values or extremal graphs for different topological indices on certain graph classes is very interesting in the reserach of Chemical Graph Theory. In this line, the minimum and maximum Lanzhou indices, respectively, among all connected graphs with n vertices has been determined by Vukičević et. al. [13]. In the same paper, Vukičević et. al. also determined the minimum and maximum Lanzhou indices, respectively, among all trees with n vertices [13]. Later, Liu et. al. [11] determined the minimum and maximum Lanzhou indices in the class of unicyclic graphs and chemical graphs with n vertices, respectively; Liu [10] determined the minimum and maximum Lanzhou indices, respectively, in the class of bicyclic graphs with n vertices; Cui and Zhao [4] identified the minimum Lanzhou indices in the class of tricyclic graphs with n vertices.

By establishing an upper bound to the Lanzhou index for trees with n vertices and fixed maximum degree, Li et. al. [9] also deduced the minimum and maximum Lanzhou indices of unicyclic graphs with n vertices respectively, and they also determined the maximum Lanzhou index for chemical trees with n vertices. Recently, Albalahi et. al. [2] also determined the maximum Lanzhou index of chemical graphs with n vertices and m edges. In this paper, we are concerned with extremal results of Lanzhou index in the class of c-cyclic graphs with n vertices. By employing the majorization method, we determine the unique extremal graph with minimum Lanzhou index in the class of c-cyclic graphs for $n \geq 3c+4$ and $c \geq 1$; and we also identify the unique extremal graph with maximum Lanzhou index among all c-cyclic graphs with n and $3 \leq c \leq \frac{n}{13}$.

Let F_k be the **friendship graph (Dutch windmill graph)**, which is a graph obtained from k triangles that share exactly one vertex. Let H_0 be the c-cyclic graph obtained from F_c by attaching $n-2c-1$ pendant vertices to the unique vertex of degree $2c$ of F_c. The following is one of our main results:

Theorem 1. *Let G be a c-cyclic graph with n vertices. If $n \geq 3c+4$ and $c \geq 1$, then*

$$L_z(G) \geq L_z(H_0) = (n-1)(n-2) + 2c(3n-10),$$

where equality holds if and only if $G = H_0$.

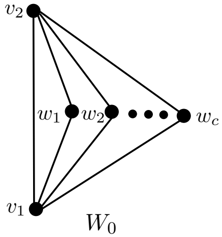

Figure 1. The graph W_0.

For $c \geq 0$, let G_0 be the c-cyclic graph with n vertices, which is obtained from W_0 (see Figure 1) by attaching $\lceil 0.5(n-c-2) \rceil$ and $\lfloor 0.5(n-c-2) \rfloor$ pendent vertices to v_1 and v_2, respectively. The following is the second main result of this paper.

Theorem 2. *If $3 \leq c \leq \frac{n}{13}$ and G is a c-cyclic graph with n vertices, then $Lz(G) \leq Lz(G_0)$, where the equality holds if and only if $G = G_0$.*

Let G be the tricyclic graph with 38 vertices, which is obtained from the complete graph K_4 with four vertices by attaching 11, 11 and 12 pendent vertices to each of three vertices of K_4, respectively. By an elementary computation, we have $Lz(G) = 15496 > 15464 = Lz(G_0)$ for $n = 38$ and $c = 3$. Thus, the bound $\frac{n}{13}$ of Theorem 2 is best possible.

For a graph category $\mathcal{G}$, if $Lz(G)$ is maximum (respectively, minimum) in $\mathcal{G}$, then we call G as a **maximum** (respectively, **minimum**) **extremal graph** of $\mathcal{G}$. Vukičević et. al. [13] showed that G_0 is the unique maximum extremal graph of trees with $n \geq 15$ vertices, Liu et. al. [11] proved that G_0 is the unique maximum extremal graph of unicyclic graphs with $n \geq 28$ vertices, and Liu [10] identified that G_0 is the unique maximum extremal graph of bicyclic graphs with $n \geq 33$ vertices. Combining these results with Theorem 2, we can conclude that: When n is large enough, G_0 is the unique maximum extremal graph of c-cyclic graphs with n vertices.

2 Proof of Theorem 1

The majorization theorem is an important and effective tool to deal with extremal problem of graph spectrum and topological index theory.

Definition 1. [12] Let $\pi = (a_1, a_2, \ldots, a_n)$ and $\pi' = (a'_1, a'_2, \ldots, a'_n)$ be two different non-increasing sequences of nonnegative real numbers, we write $\pi \lhd \pi'$ if and only if $\sum_{i=1}^{j} a_i = \sum_{i=1}^{j} a'_i$, and $\sum_{i=1}^{j} a_i \leq \sum_{i=1}^{j} a'_i$ for all $j = 1, 2, \cdots, n$. The ordering $\pi \lhd \pi'$ is sometimes called **majorization**.

A real valued function $f(x)$ defined on a convex set D is said to be **strictly convex** if

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$$

holds for all $0 < \lambda < 1$ and all $x, y \in D$. The following majorization theorem for a strictly convex function had been discovered long time ago.

Lemma 1. [12] *Let* $\pi = (a_1, a_2, \ldots, a_p)$ *and* $\pi' = (a'_1, a'_2, \ldots, a'_p)$ *be two different non-increasing sequence of non-negative real numbers. If* $\pi \lhd \pi'$ *and* $f(x)$ *is a strictly convex function, then* $\sum_{i=1}^{p} f(a_i) < \sum_{i=1}^{p} f(a'_i)$.

In what follows, we always define $f(x) = x^2(n-1-x)$. Since $f''(x) = 2(n-1-3x)$, $f(x)$ is a strictly convex function for $x \leq \frac{n-1}{3}$.

Corollary 1. *Let* π *and* π' *be two different non-increasing degree sequences with* $\pi \lhd \pi'$. *If* $G \in \Gamma(\pi)$ *and* $G' \in \Gamma(\pi')$, *then* $L_z(G) < L_z(G')$ *holds for* $\Delta(G') \leq \frac{n-1}{3}$.

Proof: Denote by $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ the degree sequences of G and G', respectively. Since $\Delta(G') \leq \frac{n-1}{3}$ and $\pi \lhd \pi'$, we have $d_1 \leq d'_1 \leq \frac{n-1}{3}$ by Definition 1, we have $L_z(G) < L_z(G')$ by Lemma 1, as $f(x)$ is a strictly convex function for $x \leq \frac{n-1}{3}$. ■

Let $q^{(p)}$ denote p copies of the real number q.

Lemma 2. *Let G be a c-cyclic graph with n vertices and degree sequence π, where $c \geq 1$. If $\Delta(G) \leq \frac{n-1}{3}$ and $\pi \neq \pi_1$, then $\pi_1 \lhd \pi$, where $\pi_1 = (3^{(2c-2)}, 2^{(n-2c+2)})$.*

Proof: By contradiction, assume that the result does not hold. Denote by $\pi = (d'_1, d'_2, \ldots, d'_n)$ the degree sequence of G and $\pi_1 = (d_1, d_2, \ldots, d_n)$. By Definition 1, there exists j with $1 \leq j \leq n$ such that $\sum_{i=1}^{j} d_i > \sum_{i=1}^{j} d'_i$.

If $1 \leq j \leq 2(c-1)$, then $\sum_{i=1}^{j} d'_i < 3j$. Thus, $2 \geq d'_j \geq d'_{j+1} \geq \ldots \geq d'_n$. Combining this with $\pi_1 = (3^{(2c-2)}, 2^{(n-2c+2)})$, we have

$$\textstyle\sum_{i=1}^{n} d'_i < 2(n-j) + 3j = 2n + j \leq 2(n+c-1),$$

contrary with $\sum_{i=1}^{n} d'_i = 2(n+c-1)$.

If $2(c-1)+1 \leq j \leq n$, then $\sum_{i=1}^{j} d'_i < 3 \cdot 2(c-1) + 2[j-2(c-1)] = 2(j+c-1)$. Thus, $d'_{j+1} + d'_{j+2} + \cdots + d'_n > 2(n+c-1) - 2(j+c-1) = 2(n-j)$, which implies that $d'_1 \geq d'_2 \geq \ldots \geq d'_j \geq d'_{j+1} \geq 3$. Thus, $2(j+c-1) > \sum_{i=1}^{j} d'_i \geq 3j$, and so $j < 2(c-1)$, a contradiction. ■

Corollary 2. *Let G be a c-cyclic graph with n vertices. If $2 \leq \Delta(G) \leq \frac{n-1}{3}$ and $c \geq 1$, then $L_z(G) \geq 4n^2 + 10nc - 22n - 48c + 48$, with equality if and only if the degree sequence of G is equal to $\pi_1 = (3^{(2c-2)}, 2^{(n-2c+2)})$.*

Proof: Since $2 \leq \Delta(G) \leq \frac{n-1}{3}$, by Lemma 2, $\pi_1 = (3^{(2c-2)}, 2^{(n-2c+2)})$ is the minimum degree sequence in the relationship $\lhd$ among all these degree sequences of c-cyclic graphs with n vertices. The corollary follows from Corollary 1. ■

Remark. A result similar to Corollary 2 has been presented in [3].

Lemma 3. *If $1 \leq c \leq \frac{n-4}{3}$ and $L_z(G)$ is minimum in the class of c-cyclic graphs with n vertices, then G contains at most one vertex of degree greater than $\frac{n-1}{3}$.*

Proof: Suppose that, G contains at least two vertices of degree greater than $\frac{n-1}{3}$. Let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of G, where $d_1 \geq d_2 > \frac{n-1}{3}$.

Suppose that $d(v_1) = d_1$ and $d(v_2) = d_2$. Let $P_{v_1 v_2}$ be a shortest path connecting with v_1 and v_2. Since G is a c-cyclic graph, we have

$|N(v_1) \cap N(v_2)| \leq c+1 \leq \frac{n-1}{3} < d_2$, G contains a vertex $w \neq v_1$ such that $w \in N(v_2) \setminus N(v_1)$ and $w \notin V(P_{v_1 v_2})$, then let $G' = G - v_2 w + v_1 w$. Since $d_1 \geq d_2$, there also exists vertex $w' \neq v_2$ such that $w' \in N(v_1) \setminus N(v_2)$ and $w' \notin V(P_{v_1 v_2})$. Let $G'' = G - v_1 w' + v_2 w'$. By the choice of G and $d_1 + d_2 > \frac{2(n-1)}{3}$, we have $0 \leq L_z(G') + L_z(G'') - 2L_z(G) = 2(2n - 3d_1 - 3d_2 - 2) < 0$, contrary with the choice of G. ■

In the rest of this section, we may always suppose that $L_z(G)$ is minimum in the class of c-cyclic graphs with n vertices, where $c \leq \frac{n-4}{3}$. Bearing Lemma 3 into consideration, G contains at most one vertex of degree greater than $\frac{n-1}{3}$.

Lemma 4. *Let G be a c-cyclic graph with n vertices. If $n \geq 3c+4$ and $\frac{n-1}{3} < \Delta(G) = \Delta < 2c$, then $L_z(G) \geq \Delta^2(n-1-\Delta) + 9(2c-\Delta)(n-4) + 4(n-2c+\Delta-1)(n-3)$.*

Proof: Suppose that the degree sequence of G is $\pi' = (\Delta, d'_2, \ldots, d'_n)$ and denote by $\pi = (\Delta, d_2, \ldots, d_n) = (\Delta, 3^{(2c-\Delta)}, 2^{(n-2c+\Delta-1)})$. Let $\mathbf{a}' = (d'_2, \ldots, d'_n)$ and $\mathbf{a} = (d_2, \ldots, d_n)$. Next, we show that $\mathbf{a} \triangleleft \mathbf{a}'$ for $\pi \neq \pi'$. By contradiction, assume that this is not true. By Definition 1, there exists j with $2 \leq j \leq n$ such that $\sum_{i=2}^{j} d'_i < \sum_{i=2}^{j} d_i$.

If $2 \leq j \leq 2c - \Delta$, then $\sum_{i=2}^{j} d'_i < 3(j-1)$. Thus $2 \geq d'_j \geq \ldots \geq d'_n$, then $\sum_{i=2}^{n} d'_i < 3(j-1) + 2(n-j) = 2n + j - 3 \leq 2n + (2c-\Delta) - 3 = 2(n+c-1) - \Delta - 1$, a contradiction.

If $2c - \Delta + 1 \leq j \leq n$, then $\sum_{i=2}^{j} d'_i < 3(2c-\Delta) + 2(j - 2c + \Delta - 1) = 2(j+c-1) - \Delta$. Thus $\sum_{i=j+1}^{n} d'_i > 2(n+c-1) - \Delta - [2(j+c-1) - \Delta] = 2(n-j)$, which implies that $d'_2 \geq \ldots \geq d'_{j+1} \geq 3$. Now, we have $\sum_{i=2}^{n} d'_i > 3(j-1) + 2(n-j) = 2n + j - 3 \geq 2(n+c-1) - \Delta$, a contradiction.

By Lemma 3, we have $d'_n \leq d'_{n-1} \leq \ldots \leq d'_2 \leq \frac{n-1}{3}$. Thus, the result follows from Lemma 1. ■

Lemma 5. *Let G be a minimum extremal graph in the class of c-cyclic graph with n vertices. If $n \geq 3c+4$ and $\frac{n-1}{3} < \Delta < 2c$, then $L_z(G) > L_z(H_0)$.*

Proof: Since $n \geq 3c+4$ and $\frac{n-1}{3} < \Delta < 2c$, we have $c \geq 2$ and $n \geq 10$.

By Lemma 4, we have

$$L_z(G)-L_z(G_0) \geq -\Delta^3+(n-1)\Delta^2+(24-5n)\Delta+3n^2+4nc-13n-28c+10.$$

Let $f(x) = -x^3+(n-1)x^2+(24-5n)x+3n^2+4nc-13n-28c+10$. Since $f'(x) = -3x^2+2(n-1)x+24-5n$ and $f''(x) = -6x+2(n-1)$, we have $f''(x) < 0$ and $f'(x) \geq f'(2c-1)$ when $\frac{n-1}{3} < x \leq 2c-1$. By an elementary computation, it follows that $f'(2c-1) = -12c^2+(4n+8)c-7n+23 = g(c)$. Since $2 \leq c \leq \frac{n-4}{3}$ and

$$\min\left\{g(2), g\left(\frac{n-4}{3}\right)\right\} = n-9 > 0,$$

then $f'(x) \geq f'(2c-1) > 0$ for $\frac{n-1}{3} < x \leq 2c-1$, this implies that

$$\begin{aligned} f(x) > f\left(\frac{n-1}{3}\right) &= \frac{2}{27}\Big(54(n-7)c+(n-1)(n^2+16n-26)\Big) \\ &> \frac{2}{27}(n-1)(n^2+16n-26) > 0 \end{aligned}$$

for $\frac{n-1}{3} < x \leq 2c-1$. Thus $L_z(G) > L_z(H_0)$, as desired. ■

Lemma 6. *Let G be a minimum extremal graph in the class of c-cyclic graphs with n vertices and degree sequence π. If $\Delta \geq 2c \geq 2$, $n \geq 3c+4$ and $\pi \neq \pi_2$, then $\pi_2 \triangleleft \pi$, where $\pi_2 = (\Delta, 2^{(n+2c-\Delta-1)}, 1^{(\Delta-2c)})$.*

Proof: Let $\pi = (\Delta, d'_2, \ldots, d'_n)$ and $\pi_2 = (\Delta, d_2, \ldots, d_n)$, where $d'_n \leq d'_{n-1} \leq \ldots \leq d'_2 \leq \frac{n-1}{3}$ by Lemma 3. Assume that the result does not hold. Then, there exists j with $2 \leq j \leq n$ such that $\sum_{i=2}^{j} d'_i < \sum_{i=2}^{j} d_i$.

If $2 \leq j \leq n+2c-\Delta$, then $d'_2+\cdots+d'_j < 2(j-1)$, which implies that $d'_j = d'_{j+1} = \cdots = d'_n = 1$. Thus,

$$\Delta+2(j-1) > \Delta+\sum_{i=2}^{j} d'_i = 2(n+c-1)-(n-j),$$

and thus $j > n+2c-\Delta$, a contradiction.

If $n + 2c - \Delta + 1 \leq j \leq n$, then

$$\Delta + \sum_{i=2}^{j} d'_i < \Delta + 2(n + 2c - 1 - \Delta) + [j - (n + 2c - \Delta)] = n + 2c + j - 2,$$

and so

$$\sum_{i=j+1}^{n} d'_i > 2(n + c - 1) - (n + 2c + j - 2) = n - j.$$

This follows that $d'_2 \geq d'_3 \geq \ldots \geq d'_{j+1} \geq 2$, which implies that

$$\begin{aligned} \Delta + \sum_{i=2}^{n} d'_i &> \Delta + 2(j - 1) + (n - j) = n + j + \Delta - 2 \\ &\geq n + (n + 2c - \Delta + 1) + \Delta - 2 = 2n + 2c - 1, \end{aligned}$$

a contradiction. ■

Lemma 7. *Let G be a minimum extremal graph in the class of c-cyclic graphs with n vertices. If $\Delta \geq 2c \geq 2$ and $n \geq 3c + 4$, then $L_z(G) \geq L_z(H_0)$, with equality if and only if $G = H_0$.*

Proof: Let $\pi = (\Delta, d'_2, \ldots, d'_n)$ be the degree sequence of G. By Lemma 3, we have $d'_2 \leq \frac{n-1}{3}$. Combining this with Lemmas 1 and 6, we have $\pi = \pi_2 = (\Delta, 2^{(n+2c-\Delta-1)}, 1^{(\Delta-2c)})$. If $G \neq H_0$, then $\pi \neq (n-1, 2^{(2c)}, 1^{(n-2c-1)})$ and so $\Delta \leq n - 2$.

By an elementary computation, we have

$$\begin{aligned} L_z(G) - L_z(H_0) &= \triangle^2(n - \Delta - 1) + (4n + 8c - 4\triangle - 4)(n - 3) \\ &\quad + (\triangle - 2c)(n - 2) - 8c(n - 3) - (n - 2c - 1)(n - 2) \\ &= (n - 1 - \Delta)(3n + \Delta^2 - 10) > 0, \end{aligned}$$

a contradiction. ■

Proof of Theorem 1: Let G be a minimum extremal graph in the class of c-cyclic graphs with n vertices for $n \geq 3c + 4$ and $c \geq 1$. By Lemmas 5 and 7, it suffices to consider the case of $\Delta \leq \frac{n-1}{3}$. By Corollary 2, we have $L_z(G) \geq 4n^2 + 10nc - 22n - 48c + 48$. Combining this with $n \geq 3c + 4$ and

$c \geq 1$, we have

$$\begin{aligned} L_z(G) - L_z(H_0) &\geq 3n^2 + 4nc - 19n - 28c + 46 \\ &= 3(n-3)^2 + (n-7)(4c-1) + 12 > 0, \end{aligned}$$

contrary with the choice of G. ■

3 Proof of Theorem 2

This section will be dedicated to the proof of Theorem 2. Note that

$$\begin{aligned} &\left(\left\lceil \frac{n+c}{2} \right\rceil\right)^2 \left(n - 1 - \left\lceil \frac{n+c}{2} \right\rceil\right) + \left(\left\lfloor \frac{n+c}{2} \right\rfloor\right)^2 \left(n - 1 - \left\lfloor \frac{n+c}{2} \right\rfloor\right) \\ \geq &\frac{1}{8}\left(n+c+1\right)^2 \left(n-3-c\right) + \frac{1}{8}\left(n+c-1\right)^2\left(n-1-c\right). \end{aligned}$$

Thus,

$$\begin{aligned} Lz(G_0) \geq &\frac{1}{8}\big[(n+c+1)^2(n-c-3) + (n+c-1)^2(n-c-1)\big] \\ &+ 4c(n-3) + (n-2-c)(n-2) \\ = &\frac{1}{4}\big[-c^3 - (n+2)c^2 + (n^2+8n-43)c + (n-1)(n^2+3n-14)\big]. \end{aligned}$$

Throughout this section, we always suppose that G is a maximum extremal graph of c-cyclic graphs with n vertices, and let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of G, where $\mathbf{V}(G) = \{v_1, v_2, \ldots, v_n\}$ and $d(v_i) = d_i = d_i(G)$ holds for $1 \leq i \leq n$. Since G_0 is also a c-cyclic graph with n vertices, we have $Lz(G) \geq Lz(G_0)$. In what follows, we always show that $Lz(G) < Lz(G_0)$ for $G \neq G_0$ to get a contradiction.

For any two different vertices $u, v \in V(G)$, let P_{uv} be an arbitrary path connecting u and v. If there exists $u' \in N(u)$ and $v' \in N(v)$ such that $u', v' \notin V(P_{uv})$, then let $G_1 = G - vv' + uv'$ and $G_2 = G - uu' + vu'$. In this case,

$$2Lz(G) - Lz(G_1) - Lz(G_2) = -2\left(2n - 3d(u) - 3d(v) - 2\right) \geq 0, \text{ and}$$

$$Lz(G) - Lz(G_1) = (d(v) - d(u) - 1)(2n - 3d(u) - 3d(v) - 2) \geq 0.$$

This follows that

$$d(u) + d(v) \geq \frac{2(n-1)}{3}. \tag{2}$$

We claim that $d_1 \leq n-2$. Otherwise, assume that $d_1 = n-1$. Then, $(d_2, d_3, ..., d_n) \trianglelefteq (2c+1, 1^{(n-2)})$. Since $2c+1 \leq \frac{2n+13}{13} < \frac{n-1}{3}$ and $f(x) = x^2(n-1-x)$ is strictly convex for $x < \frac{n-1}{3}$, we have $Lz(G) \leq (2c+1)^2(n-2-2c)+(n-2)^2$ by Lemma 1. Thus, we have $4Lz(G_0)-4Lz(G) \geq 31c^3 - (17n-62)c^2 + (n^2-8n-3)c + (n-1)(n-3)(n+2) = g_1(c)$. Since $g_1''(c) = 2(93c - 17n + 62) \leq \frac{4}{13}(403 - 64n) < 0$, we have

$$g_1(c) \geq \min\left\{g_1(3),\, g_1\left(\frac{n}{13}\right)\right\} > 0,$$

as $g_1(3) = n^3 + n^2 - 182n + 1392 > 0$ and $2197g_1\left(\frac{n}{13}\right) = 2(1088n^3 - 2470n^2 - 5746n + 6591) > 0$, a contradiction. This confirms our claim that $d_1 \leq n-2$.

Since $d_1 \leq n-2$, there exits $u \in V(G) \setminus \{v_1\}$ such that $u \notin N(v_1)$, which also implies that there exists vertex w such that $u \notin P_{v_1w}$ and $uw \in E(G)$. Since $d_1 \geq d(w)$, there exists $v \in N(v_1)$ and $v \notin P_{v_1w}$ such that $vw \notin E(G)$. Thus, $d(v_1) + d(w) \geq \frac{2(n-1)}{3}$ by (2), which confirms that $d_1 \geq \frac{n-1}{3}$.

If there exists vertex $v \in \mathbf{V}(G) \setminus \{v_1, v_2, v_3\}$ such that $v \notin N(v_1) \cup N(v_2) \cup N(v_3)$, then there exists vertex $v' \notin \{v_1, v_2, v_3\}$ such that $vv' \in E(G)$ and $v \notin P_{v'v_3}$. By (2), we have $d_1 + d_2 \geq d_3 + d_4 \geq d_3 + d(v') \geq \frac{2(n-1)}{3}$, as $d_3 \geq d(v')$ and $vv_3 \notin E(G)$. This implies that $2(n+c-1) \geq \frac{4(n-1)}{3} + n - 4$ by (1), contrary with $c \leq \frac{n}{13}$. If $v_3v_1 \notin E(G)$ and $v_2v_3 \notin E(G)$, then there exists $v' \notin \{v_1, v_2, v_3\}$ such that $v_3v' \in E(G)$ and $v_3 \notin P_{v'v_2}$. By (2), we have $d_1 + d_3 \geq d_2 + d_4 \geq d_2 + d(v') \geq \frac{2(n-1)}{3}$, and thus $2(n+c-1) \geq \frac{4(n-1)}{3} + n - 4$, contrary with $c \leq \frac{n}{13}$. Thus, $v_3 \in N(v_1) \cup N(v_2)$. With the similar reason, we have $v_2 \in N(v_1) \cup N(v_3)$ and $v_1 \in N(v_2) \cup N(v_3)$. Now, we can conclude that

$$v \in N(v_1) \cup N(v_2) \cup N(v_3) \text{ holds for any } v \in \mathbf{V}(G). \tag{3}$$

Case 1. $d_1 \le d_3 + 1$**.** Among these $n-3$ vertices of $\mathbf{V}(G) \setminus \{v_1, v_2, v_3\}$, we suppose that there are s_i vertices each of which is adjacent to exactly i vertices of $\{v_1, v_2, v_3\}$, where $1 \le i \le 3$. Since $d_1 + d_2 + d_3 \le s_1 + 2s_2 + 3s_3 + 6 \le n - 3 - s_2 - s_3 + 2s_1 + 3s_3 + 6 = n + 3 + s_2 + 2s_3$ and G contains at most $n - s_2 - s_3 - 3$ pendent vertices, we have

$$\begin{aligned} 2(n+c-1) &\ge n - s_2 - s_3 - 3 + 2s_2 + 3s_3 + d_1 + d_2 + d_3 \\ &\ge 2(d_1 + d_2 + d_3) - 6. \end{aligned} \tag{4}$$

We will show that $d_5 \ge 2$. Otherwise, assume that $d_5 = 1$. Since $c \ge 3$ and G contains at least $n-4$ pendent vertices, we have $3 = c$. By (3), it follows that $n + 2 + 2c - d_1 - d_2 - d_3 = d_4 = 3$ and so $\pi = (d_1, d_2, d_3, 3, 1^{(n-4)})$. Note that $d_1 \le d_3 + 1$ and G contains exactly $n-4$ pendent vertices. If $d_3 \le \frac{n-1}{3}$, then $2(n+c-1) = 2n+4 \le \frac{2(n+2)}{3} + \frac{n-1}{3} + 3 + n - 4 = 2n$, a contradiction. Thus, $d_3 > \frac{n-1}{3}$. Combining this with $\left(\frac{n+5}{3}, \frac{n+5}{3}, \frac{n+5}{3}, 3, 1^{(n-4)}\right) \trianglelefteq \pi$, we have $Lz(G) \le \frac{2}{9}(n+5)^2(n-4) + (n-4)(n-2) + 9(n-4)$ by Lemma 1, as $f(x) = x^2\,(x+1-n)$ is a strictly convex function for $x > \frac{n-1}{3}$. Since $c = 3$, we have $36Lz(G_0) - 36Lz(G) \ge n^3 - 39n^2 - 6n + 368 > 0$, a contradiction. Now, we can conclude that $d_5 \ge 2$.

Since $d_1 \ge \frac{n-1}{3}$, we have $n + 1 + 2c - d_1 - d_2 - d_3 \le n + 2c + 1 - 3d_3 \le n + 2c + 1 - 3d_1 + 3 \le 5 + 2c \le 5 + \frac{2n}{13} < \frac{n-4}{3} \le d_1 - 1 \le d_3$ and $d_5 \ge 2$. This implies that $\pi \trianglelefteq (d_1, d_2, d_3, n+1+2c-d_1-d_2-d_3, 2, 1^{(n-5)})$.

By Lemma 1, we have $Lz(G) \le d_1^2(n-1-d_1) + d_2^2(n-1-d_2) + d_3^2(n-1-d_3) + (n+1+2c-d_1-d_2-d_3)^2(d_1+d_2+d_3-2-2c) + n^2 - 3n - 2$. Denote by $d_3 = x$. By (4), we have

$$\frac{n-4}{3} \le d_1 - 1 \le x \le \frac{c+n+2}{3}. \tag{5}$$

If $d_1 = d_2 = d_3$, then $Lz(G) \le 3x^2(n-1-x) + (n+1+2c-3x)^2(3x-2-2c) + n^2 - 3n - 2 = f_1(x)$. If $d_1 = d_3 + 1 > d_2 = d_3$, then $Lz(G) \le (x+1)^2(n-2-x) + 2x^2(n-1-x) + (n+2c-3x)^2(3x-1-2c) + n^2 - 3n - 2 = f_2(x)$. If $d_1 = d_2 = d_3 + 1$, then $Lz(G) \le 2(x+1)^2(n-2-x) + x^2(n-1-x) + (n+2c-1-3x)^2(3x-2c) + n^2 - 3n - 2 = f_3(x)$. Since

$\frac{n-4}{3} < x \le \frac{c+n+2}{3}$, we have $f_2(x)-f_1(x) = (2c+n-4x)(6c+n-6x+5) > 0$ and $f_3(x) - f_2(x) = (6c+n-6x+2)(2c+n-4x-1) > 0$. Thus,

$$f_1(x) < f_2(x) < f_3(x). \tag{6}$$

Subcase 1.1. $x \le \frac{1}{9}(4c+3n)$**.**

By (6), we have $4Lz(G_0) - 4Lz(G) \ge 4Lz(G_0) - 4f_3(x) = 12(18c+5n-3)x^2 - 96x^3 - 4\,(36c^2+24cn-24c+3n^2-2n-7)x + c^2(31c+31n-34) + (9n^2-8n-35)c + n^3 - 2n^2 - 13n + 38 = f_4(x)$. Since $f_4''(x) = 24(18c+5n-24x-3) \le f_4''(\frac{n-4}{3}) = 24(18c-3n+29) \le \frac{24}{13}(377-21n) < 0$ by (5), we have $f_4'(x) < f_4'(\frac{n-4}{3}) = -4(36c^2-12cn+120c+n^2-20n+97) = 4g_2(c)$. Since $g_2'(c) = 12(n-6c-10) \ge \frac{12}{13}(7n-130) > 0$, we have $169g_2(c) \le 169g_2\left(\frac{n}{13}\right) = -49n^2+1820n-16393 < 0$ and thus $f_4'(x) < 0$.

Since $f_4'(x) < 0$, we have $243f_4(x) \ge 243f_4(\frac{4c+3n}{9}) = 301c^3 - 27(25n-14)c^2 - 27(7n^2-152n+203)c + 27(n-3)(n^2-27n-114) = g_3(c)$. Since $g_3''(c) = 6(301c-225n+126) \le \frac{12}{13}(819-1312n) < 0$, we have $g_3'(c) \le g_3'(3) = -27(7n^2-2n-182) < 0$. Thus, $2197g_3(c) \ge 2197g_3(\frac{n}{13}) = 2(9452n^3 - 540540n^2 - 1441908n + 10143549)$. By the choice of G, we have $39 \le n \le 59$ and so $3 \le c \le 4$.

We claim that $d_2 = d_3$. Otherwise, assume that $d_2 = d_3 + 1$. By (4), we have $x \le \frac{n+c}{3}$, and thus $9f_4(x) \ge 9f_4(\frac{n+c}{3}) = 31c^3 - 9(5n+6)c^2 - 3(n^2-56n+77)c + (n-3)(n^2-27n-114) = g_4(c)$. Since $g_4(3) = n(n^2-39n+66) > 0$ and $3 \le c \le 4$, we have $c = 4$ and $9f_4(x) \ge g_4(4) = n^3 - 42n^2 - 81n + 538$, contrary with $n \ge 52$. Thus, $d_2 = d_3$.

Combining $d_2 = d_3$ with (6), we have $4Lz(G_0) - 4Lz(G) \ge 4Lz(G_0) - 4f_2(x) = 12(18c+5n+5)x^2 - 96x^3 - 4(36c^2+24cn+12c+3n^2+8n-5)x + 31c^3 + 31c^2n + 14c^2 + 9cn^2 + 24cn - 43c + n^3 + 2n^2 - 9n + 30 = f_5(x)$. Since $f_5''(x) = 24(18c+5n-24x+5) \le f_5''(\frac{n-4}{3}) = 24(18c-3n+37) \le \frac{24}{13}(481-21n) < 0$, we have $f_5'(x) \le f_5'(\frac{n-4}{3}) = -4(36c^2-12cn+156c+n^2-26n+163) = g_5(c)$. Since $g_5(3) = -4(n^2-62n+955) < 0$ for $n \ge 39$ and $g_5(4) = -4(n^2-74n+1363) < 0$ for $n \ge 52$, we have $243f_5(x) \ge 243f_5(\frac{4c+3n}{9}) = 301c^3 - 9(75n-122)c^2 - 27(7n^2-104n+307)c + 27(n^3-18n^2-21n+270) = g_6(c)$. Since $g_6(3) = 27(n-2)(n^2-37n-8) > 0$ for $n \ge 39$ and $g_6(4) = 27n^3 - 1242n^2 - 135n + 10966 > 0$ for $n \ge 52$, we get a contradiction to the choice of G.

Subcase 1.2. $\frac{4c+3n}{9} < x \le \frac{c+n+2}{3}$**.** Since $\frac{4c+3n}{9} < \frac{c+n+2}{3}$, we have $3 \le c \le 5$. If $d_1 = d_3 + 1$, then $\frac{4c+3n}{9} < x \le \frac{c+n+1}{3}$ by (4), contrary with $c \ge 3$. Thus, $d_1 = d_2 = d_3$. By (6), we have $Lz(G_0) - Lz(G) \ge 4Lz(G_0) - 4f_1(x) = 12(18c + 5n + 13)x^2 - 96x^3 - 12(6c + n + 5)(2c + n + 1)x + 31c^2(c + n + 2) + c(9n^2 + 56n - 3) + (n + 5)(n^2 + n + 6) = f_6(x)$.

Since $f_6''(x) = 24(18c + 5n - 24x + 13) \le f_6''(\frac{n-4}{3}) = 72(6c - n + 15) \le \frac{72}{13}(195 - 7n) < 0$ by (5), we have $f_6'(x) \le f_6'(\frac{n-4}{3}) = 31c^3 - 9(5n + 6)c^2 - 3(n^2-56n+85)c+n^3-30n^2-33n+278 = g_7(c)$. Since $g_7''(c) = 6(31c-15n-18) \le \frac{12}{13}(-82n-117) < 0$, we have $g_7'(c) \le g_7'(3) = -3(n^2+34n-86) < 0$. Thus, $2197g_7(c) \ge 2197g_7(\frac{n}{13}) = 2(568n^3 - 19110n^2 - 57798n + 305383) > 0$, a contradiction.

Case 2. $d_3 + 1 < d_1$**.**

We will show that

$$v \in N(v_1) \cup N(v_2) \text{ holds for any } v \in \mathbf{V}(G) \setminus \{v_1, v_2\}. \tag{7}$$

Otherwise, there exists vertex $v \in \mathbf{V}(G) \setminus \{v_1, v_2, v_3\}$ such that $v \in N(v_3)$ and $v \notin N(v_1) \cup N(v_2)$ by (3). By (2), we have $d_1 + d_3 \ge d_2 + d_3 \ge \frac{2(n-1)}{3}$. Since $d_3 + 1 < d_1$ and the construction of G_1, we have $d_2 + d_3 = d_1 + d_3 = \frac{2(n-1)}{3}$. Combining this with $c \le \frac{n}{13}$, it follows that $d_1+d_2 \ge d_1+d_3 = d_2+d_3 = \frac{2(n-1)}{3} > d_1+d_4$ and $d_1 = d_2$, which implies that $d_1+d_2+d_3 \ge n-1$. If v is adjacent to another vertex $v' \ne v_3$, then $d_1+d_4 \ge d_2+d_4 \ge d_2+d(v') \ge \frac{2(n-1)}{3}$ by (2) (as $v' \in N(v_1) \cup N(v_2) \cup N(v_3)$ by (3)), a contradiction. Thus, v is a pendent vertex. Since $G_3 = G - vv_3 + v_1v$ is also a c-cyclic graph with $Lz(G) = Lz(G_3)$ and $d_3(G_3)+d_2(G_3) = d_2+d_3-1 < \frac{2(n-1)}{3}$ (as $d_3 > d_4$), v is the unique (pendent) vertex adjacent to v_3, which is not adjacent to v_1 or v_2. Thus, $2d_1 = d_1 + d_2 \ge n - 3$ and so $d_3 \le \frac{2(n-1)}{3} - \frac{n-3}{2} = \frac{n+5}{6} < \frac{n-7}{2} \le d_1 - 2 = d_2 - 2$. If G contains at least two pendent vertices, then we may suppose that u is a pendent vertex adjacent to $w \in \{v_1, v_2\}$ of G by (3). In this case, $G_4 = G - wu + v_3u$ is also a c-cyclic graph with $Lz(G) = Lz(G_4)$, contrary with the fact that $d_1(G_4) = d_1 > d_2 - 1 = d_2(G_4)$ and $d_1(G_4) = d_1 > d_3 + 2 = d_3(G_4) + 1$. Thus, v is also the unique pendent vertex of G. By (1), we have $2(n + c - 1) = d_1 + d_2 + \cdots + d_n \ge n - 1 + 2(n - 4) + 1$, contrary with $c \leq \frac{n}{13}$. Now, we can conclude that (7) holds.

Next, we claim that $v_1v_2 \in E(G)$. Otherwise, assume that $v_1v_2 \notin E(G)$. Then, there exists vertex v such that $vv_2 \in E(G)$ and $v_2 \notin P_{v_1v}$ (Here, we let P_{v_1v} be a shortest path connecting v and v_1). If $vv_1 \notin E(G)$, then there exists $v' \in P_{vv_1}$ such that $v_1v' \in E(G)$ and $v_1 \notin P_{vv'}$. By (2), we have $d_1+d_2 \geq d_3+d_4 \geq d(v)+d(v') \geq \frac{2(n-1)}{3}$, contrary with $c \leq \frac{n}{13}$. Thus, $vv_1 \in E(G)$. By (2) and $d_1 > d_3+1$, we have $d_1+d(v) = \frac{2(n-1)}{3} \leq d_2+d(v)$, as $v_1v_2 \notin E(G)$. Combining this with (7), we have $d_1 + d_2 \geq n-1$ and $d_1 = d_2$. Combining this with either $d_4 \geq 3$ or $d_4 \geq d_5 \geq 2$ (as $c \geq 3$), we have

$$2(n+c-1) \geq \frac{2(n-1)}{3} + \frac{n-1}{2} + n - 1 \geq \frac{13}{6}(n-1),$$

contrary with $c \leq \frac{n}{13}$. Thus, $v_1v_2 \in E(G)$. Combining this with (7), we can conclude that every vertex of G is adjacent to v_1 or v_2. Suppose that $|N(v_1) \cap N(v_2)| = s$. Then, $d_1 + d_2 = n + s > \frac{2(n-1)}{3}$. Combining this with $d_1 \leq n-2$ and (2), we have $d_1 = \lceil \frac{n+s}{2} \rceil$ and $d_2 = \lfloor \frac{n+s}{2} \rfloor$.

For any vertex $v \in N(v_2) \setminus N(v_1)$ or $v \in N(v_1) \setminus N(v_2)$, if $vv' \in E(G)$ holds for some vertex $v' \notin \{v_1, v_2\}$, then $d_1 + d(v') \geq \frac{2(n-1)}{3}$ by (2). Combining this with either $d_4 \geq 3$ or $d_4 \geq d_5 \geq 2$ (as $c \geq 3$), we have

$$2(n+c-1) \geq \frac{2(n-1)}{3} + \left\lfloor \frac{n+s}{2} \right\rfloor + n - 1 \geq \frac{13}{6}(n-1),$$

contrary with $c \leq \frac{n}{13}$. This implies that v is a pendent vertex holds for any $v \notin (N(v_1) \cap N(v_2)) \cup \{v_1, v_2\}$, that is, G contains exactly $n-s-2$ pendent vertices. Since G is a c-cyclic graph with $c \geq 3$ and $v_1v_2 \in E(G)$, we have $2 \leq s \leq c$ and $d_3 \leq s+1$. Note that $d_1 = \lceil \frac{n+s}{2} \rceil$ and $d_2 = \lfloor \frac{n+s}{2} \rfloor$. Thus,

$$d_1^2(n-1-d_1) + d_2^2(n-1-d_2) \leq \frac{1}{4}(n+s)^2(n-2-s). \tag{8}$$

Next, we will show that $s = c$ and so $G = G_0$.

Subcase 2.1. $2 \leq s \leq \frac{2c}{3}$**.** Since $d_3 \leq s+1 \leq c+1-0.5s \leq c \leq \frac{n}{13} < \frac{n-4}{3} \leq d_1 - 1 \leq d_2$ and $d_1 + d_2 = n+s$, we have $\pi \trianglelefteq (d_1, d_2, c+1-0.5s, c+1-0.5s, 1^{(n-4)})$. Combining this with (8), we have $Lz(G) \leq \frac{1}{4}(n+s)^2(n-2-s) + (c+1-0.5s)^2(2n-4+s-2c) + (n-4)(n-2)$.

Thus, $4Lz(G_0)-4Lz(G)\ge (6c-n+10)s^2-(n-2c-2)(n-6c-10)s+7c^3-9c^2n+30c^2+cn^2-8cn-3c-n-2=h_1(s)$. Note that $h_1''(s)=2(6c-n+10)\le \frac{2}{13}(130-7n)<0$. Thus,

$$h_1(s)\ge \min\left\{h_1(2), h_1\left(\frac{2c}{3}\right)\right\}. \tag{9}$$

Since $9h_1\left(\frac{2c}{3}\right)=15c^3-(37n-118)c^2+3(n-7)(n+7)c-9n-18=g_8(c)$. Since $g_8''(c)=2(45c-37n+118)\le\frac{4}{13}(767-218n)<0$, $g_8(3)=9(n^2-38n+112)>0$ and $2197g_8(\frac{n}{13})=41n^3+1534n^2-44616n-39546>0$, we have $g_8(c)\ge\min\{g_8(3), g_8\left(\frac{n}{13}\right)\}>0$. Combining this with (9), we have $h_1(2)=7c^3-3(3n-2)c^2+(n^2+8n-43)c-2n^2+19n-2=g_9(c)\le 0$.

Since $g_9''(c)=6(7c-3n+2)\le\frac{12}{13}(13-16n)<0$, we have $g_9(c)\ge \min\{g_9(3),g_9(\frac{n}{13})\}$. Note that $g_9(3)=n^2-38n+112>0$ and $2197g_9(\frac{n}{13})=59n^3-2964n^2+34476n-4394>0$. Thus, $g_9(c)>0$, a contradiction.

Subcase 2.2. $\frac{2c}{3}<s\le c-1$**.** Then, $c\ge 4$ and $n\ge 52$. Since $2\le 2(c+1-s)\le 2(c-1)\le\frac{2}{13}(n-13)<\frac{n-4}{3}\le d_1-1\le d_2$ and $d_1+d_2=n+s$, we have $\pi\trianglelefteq(d_1,d_2,2(c+1-s),2^{(s-1)},1^{(n-s-2)})$. Combining this with (8), we have $Lz(G)\le\frac{1}{4}(n+s)^2(n-2-s)+4(c+1-s)^2(n+2s-2c-3)+4(s-1)(n-3)+(n-s-2)(n-2)$. Thus, $4Lz(G_0)-4Lz(G)\ge 3(32c-5n+38)s^2-31s^3-(96c^2-32cn+224c+n^2-24n+88)s+31c^3-17c^2n+110c^2+cn^2-24cn+85c-n-2=h_2(s)$. Since $h_2''(s)=6(32c-5n-31s+38)<2(34c-15n+114)\le\frac{2}{13}(1482-161n)<0$, we have $h_2(s)\ge\min\{h_2(\frac{2c}{3}),h_2(c-1)\}$.

Since $h_2(c-1)=-3c^2-2(n+2)c+(n-7)(n-33)=g_{10}(c)$ and $g_{10}'(c)=-2(3c+n+2)<0$, we have $169g_{10}(c)\ge 169g_{10}(\frac{n}{13})=140n^2-6812n+39039>0$. This implies that $27h_2(\frac{2c}{3})=13c^3-9(7n-34)c^2+9(n^2-24n+79)c-27(n+2)=g_{11}(c)\le 0$. Note that $g_{11}''(c)=6(13c-21n+102)\le 6(102-20n)<0$. Thus, $g_{11}(c)\ge\min\{g_{11}(4),g_{11}(\frac{n}{13})\}$. Since $n\ge 52$, we have $g_{11}(4)=36n^2-1899n+8518>0$ and $169g_{11}(\frac{n}{13})=55n^3-2502n^2+4680n-9126>0$, a contradiction.

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