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A Note on the General Sum–Connectivity Index of a Graph and Its Line Graph

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Abstract

For a real number β, the general sum-connectivity index $\chi_\beta(G)$ of a graph G is defined as $\chi_\beta(G) = \sum_{xy\in E(G)}(d_G(x) + d_G(y))^\beta$, where $d(x)$ denote the degree of a vertex x in G. In Chen (2023), the author present the lower bounds for $\chi_\beta(L(G))$ in terms of $\chi_\beta(G)$ for $\beta \geq 0$ and $\beta < 0$, but the lower bounds are not the sharp. In the paper, we give an improvement of the lower bounds for $\beta \geq 0$, i.e.,

$$\chi_\beta(L(G)) \geq \begin{cases} \chi_\beta(G), & if\ \delta(G) \leq 2, \\ 2(1+\dfrac{2}{\Delta+3})^\beta \chi_\beta(G), & if\ \delta(G) \geq 3, \end{cases}$$

and characterize the extremal graphs. In addition, for $\beta < 0$, we present a small improvement on two special cases.

1 Introduction

Topological indices (or chemical indices) as a tool for compact and effective description of structural formulas used to study and predict the structure-

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property correlation of organic compounds [6], which is widely applied in chemistry, physics, biology, and other fields. Therefore, various topological indices have been studied for several decades. In order to define the concept of branching in molecular species [3, 4], Randić [10] introduced in 1975 a topological index–the connectivity index (now called the Randić index), defined for a simple graph G as

$$R(G) = \sum_{xy \in E(G)} (d_G(x) d_G(y))^{-\frac{1}{2}},$$

where $d_G(x)$ denotes the degree of a vertex x of G. In 1998, Bollobás and Erdös [11] generalized this index to

$$R_\beta(G) = \sum_{xy \in E(G)} (d_G(x) d_G(y))^{\beta},$$

where β is a real number.

Based on the work on Randić index, Zhou and Trinajstić [12] proposed the sum-connectivity index $\chi(G)$ of a graph G:

$$\chi(G) = \sum_{xy \in E(G)} (d_G(x) + d_G(y))^{-\frac{1}{2}}.$$

As a generalization of sum-connectivity index, the general sum-connectivity index $\chi_\beta(G)$ of a graph G was introduced by Zhou and Trinajstić [13], that is

$$\chi_\beta(G) = \sum_{xy \in E(G)} (d_G(x) + d_G(y))^{\beta}.$$

The line graph $L(G)$ of a graph G is a graph whose vertices are the edges of G, with two of vertices being adjacent if the corresponding edges are adjacent in G. The line graph was first introduced in [7], we can refer to [2, 8, 9] for some related result on line graphs.

It is known that the number of benzenoid hydrocarbons is actually huge, thus modeling the physicochemical properties of the unknown ones is important to predict its properties. One of the main applications of topological indices to chemistry is to obtain predictions of certain properties of molecules (see [5]). In this sense, the connectivity index χ_2 has shown good predicting ability in comparison with a benchmark set of other predictors, for example, its correlation coefficient with respect to the enthalpy of vaporization (HVAP) is equal to 0.881, see [14].

Besides, when we study on topological indices, often deal with optimization problems on graphs, i.e., find a graph that maximize or minimize one or more of the topological indices. In general, to obtain quasi-minimizing or quasi-maximizing graphs is a good strategy that is commonly used. Line graphs can be used in solving maximization and minimization problems on hexagonal systems since it can be (approximately) transformed into one on triangular systems. Therefore, the research on the general sum-connectivity index of a graph and its line graph is of great significance.

Very recently, Chen [1] present the lower bounds for $\chi_\beta(L(G))$ in terms of $\chi_\beta(G)$ for $\beta \geq 0$ and $\beta < 0$. The detailed results are summarized as follows:

Theorem 1. *(Chen [1]) Let $\beta \geq 0$ be a real number. If G a connected graph not isomorphic to a path, then*

$$\chi_\beta(L(G)) \geq \begin{cases} \chi_\beta(G), & if\ \delta(G) \leq 2; \\ 2\chi_\beta(G), & if\ \delta(G) \geq 3. \end{cases}$$

Theorem 2. *(Chen [1]) Let $\beta < 0$ be a real number. If G a connected graph of order n not isomorphic to a P_n, then*

(i) If $\delta \geq 3$, then

$$\chi_\beta(L(G)) \geq \begin{cases} 2\Big(\dfrac{\Delta+5}{6}\Big)^\beta \chi_\beta(G),\ if\ \Delta \notin \{4,5,6\}; \\ 2\min\Big\{\dfrac{5}{2}(\dfrac{17}{9})^\beta, 2(\dfrac{15}{8})^\beta, \dfrac{3}{2}(\dfrac{13}{7})^\beta, (\dfrac{11}{6})^\beta\Big\}\chi_\beta(G),\ if\ \Delta = 6; \\ 2\min\Big\{2\Big(\dfrac{7}{4}\Big)^\beta, \dfrac{3}{2}\Big(\dfrac{12}{7}\Big)^\beta, \Big(\dfrac{5}{3}\Big)^\beta\Big\}\chi_\beta(G),\ if\ \Delta = 5; \\ 2\min\Big\{\dfrac{3}{2}\Big(\dfrac{11}{7}\Big)^\beta, \Big(\dfrac{3}{2}\Big)^\beta\Big\}\chi_\beta(G),\ if\ \Delta = 4. \end{cases}$$

(ii) If $\delta \leq 2$, then

$$\chi_\beta(L(G)) \geq \begin{cases} \left(\frac{\Delta+3}{3}\right)^\beta \chi_\beta(G), & \textit{if } G \textit{ contains a pendant path of} \\ & \textit{length } 2 \textit{ or } \Delta = 4; \\ \left(\frac{\Delta+3}{4}\right)^\beta \chi_\beta(G), & \textit{otherwise.} \end{cases}$$

However, we found that the lower bounds in Theorem 1 and 2 are not sharp, and the lower bounds can be further improved. Inspired from this, in the paper, we give an improvement of the lower bounds for $\chi_\beta(L(G))$ when $\beta \geq 0$, and characterize the extremal graphs attaining the bounds. In addition, for $\beta < 0$, we present a small improvement on two special cases.

All graphs considered in this paper are simple and connected. Let $G = (V(G), E(G))$ be a graph of order n and of size m. We use $d_G(x)$ and $N_G(x)$ to denote the degree and the set of neighbors of x in G, respectively. Let $I_G(x) = \{xx_i : xx_i \in E(G)\}$. The minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If there is no confusion, we simply denote the above notation as $d(x)$, $N(x)$, $I(x)$, δ, Δ. Furthermore, P_n, C_n, $K_{1,n-1}$ and K_n represent the path, cycle, star and complete graph of order n, respectively.

2 Bounds for $\chi_\beta(L(G))$ in terms of $\chi_\beta(G)$

A path $P = x_0x_1 \cdots x_k$ of G with length $k \geq 2$ is said to be a 2-extremal path if $d(x_0) \neq 2$, $d(x_k) \neq 2$ and $d(x_1) = \cdots = d(x_{k-1}) = 2$. In particular, P is called pendant if $d(x_0) = 1$ and $d(x_k) \geq 3$, or $d(x_0) \geq 3$ and $d(x_k) = 1$. Let $End_3(P)$ be the set of ends of P with degree at least 3 in G. In addition, we use $\mathcal{P}$ to denote the set of 2-extremal paths in G. We firstly give the relationship between $\chi_\beta(L(G))$ and $\chi_\beta(G)$ when $\beta \geq 0$, and begin with the following lemma.

Lemma 1. *Let $\beta \geq 0$, $\delta \geq 3$ and x is a vertex of a graph G of order n*

with $d(x) \geq 3$, then

$$\sum_{e,f \in I(x)} (d_L(e) + d_L(f))^\beta \geq \Big(1 + \frac{2}{\Delta + 3}\Big)^\beta \sum_{y \in N(x)} (d(x) + d(y))^\beta,$$

with equality if and only if $\Delta = \delta = 3$, i.e. G is a 3-regular graph.

Proof. Assume $N(x) = \{x_1, x_2, \cdots, x_t\}$, where $t = d(x)$, and $t_j = d(x_j)$ for $1 \leq j \leq t$. Then

$$\sum_{y \in N(x)} (d(x) + d(y))^\beta = \sum_{j=1}^{t} (t + t_j)^\beta,$$
$$\sum_{e,f \in I(x)} (d_L(e) + d_L(f))^\beta = \sum_{1 \leq i < j \leq t} (2t + t_i + t_j - 4)^\beta.$$

We only need to prove that

$$\sum_{j=1}^{t} (t_j + t)^\beta \leq (\frac{\Delta + 3}{\Delta + 5})^\beta \sum_{1 \leq i < j \leq t} (t_i + t_j + 2t - 4)^\beta.$$

Let's introduce a function $f(a) = \frac{a+\delta+2t-4}{a+t}$. Then $f(a) = (1 + \frac{\delta+t-4}{a+t})$ is a decreasing function for $a \in [3, \Delta]$, thus we have

$$\frac{t_j + \delta + 2t - 4}{t_j + t} \geq \frac{\Delta + \delta + 2t - 4}{t + \Delta},$$
$$\frac{1}{t_j + t} \geq \frac{\Delta + \delta + 2t - 4}{t + \Delta} \frac{1}{t_j + \delta + 2t - 4}$$
$$\geq \frac{\Delta + \delta + 2t - 4}{t + \Delta} \frac{1}{t_j + t_i + 2t - 4},$$
$$(t_j + t)^\beta \leq (\frac{t + \Delta}{\Delta + \delta + 2t - 4})^\beta (t_i + t_j + 2t - 4)^\beta$$
$$\sum_{j=1}^{t} (t_j + t)^\beta \leq \frac{1}{t-1} \sum_{1 \leq i,j \leq t, i \neq j} (\frac{t + \Delta}{\Delta + \delta + 2t - 4})^\beta (t_i + t_j + 2t - 4)^\beta$$

$$= \frac{2}{t-1} (\frac{t + \Delta}{\Delta + 2t - 1})^\beta \sum_{1 \leq i < j \leq t} (t_i + t_j + 2t - 4)^\beta.$$

Now, let $g_\beta(t)=\frac{2}{t-1}(\frac{t+\Delta}{\Delta+2t-1})^\beta$. Since $3\le t\le\Delta$,

$$g'_\beta(t)=$$
$$\frac{-2}{(t-1)^2}\Big(\frac{t+\Delta}{\Delta+2t-1}\Big)^\beta+\frac{2\beta}{t-1}\Big(\frac{t+\Delta}{\Delta+2t-1}\Big)^{\beta-1}\frac{(\Delta+2t-1)-2(t+\Delta)}{(\Delta+2t-1)^2}$$
$$=\frac{-2}{t-1}\Big(\frac{t+\Delta}{\Delta+2t-1}\Big)^{\beta-1}\Big(\frac{1}{t-1}\frac{t+\Delta}{\Delta+2t-1}+\frac{\beta(\Delta+1)}{(\Delta+2t-1)^2}\Big)<0,$$

$g_\beta(t)$ is a decreasing function. And $g_\beta(t)\le g_\beta(3)=(\frac{\Delta+3}{\Delta+5})^\beta$. Therefore,

$$\sum_{j=1}^{t}(t_j+t)^\beta\le(\frac{\Delta+3}{\Delta+5})^\beta\sum_{1\le i<j\le t}(t_i+t_j+2t-4)^\beta.$$

Moreover, the equality holds if and only if $t=t_i=t_j=\Delta=\delta=3$, i.e. G is a 3-regular graph. ■

Lemma 2. *(Chen [1]) Let $\beta\ge0$ and x is a vertex of a graph G of order n with $d(x)\ge3$,*

$$\sum_{e,f\in I(x)}(d_L(e)+d_L(f))^\beta\ge\sum_{y\in N(x)}(d(x)+d(y))^\beta,$$

with equality if and only if $G\cong K_{1,3}$ and x is the center.

Lemma 3. *(Chen [1]) Let $\beta\ge0$ be a real number. If $P\in\mathcal{P}$ of a connected graph G not isomorphic to a path, then*

$$\sum_{\substack{ef\in E(L(G))\\ e,f\in E(P)}}(d_L(e)+d_L(f))^\beta\ge\sum_{xy\in E(P\setminus End_3(P))}(d(x)+d(y))^\beta.$$

Theorem 3. *Let $\beta\ge0$ and $G\ncong P_n$, then*

$$\chi_\beta(L(G))\ge\begin{cases}\chi_\beta(G), & if\ \ \delta(G)\le2;\\ 2(1+\dfrac{2}{\Delta+3})^\beta\chi_\beta(G), & if\ \ \delta(G)\ge3.\end{cases}$$

Moreover, $\chi_\beta(L(G))=\chi_\beta(G)$ if and only if $G\cong K_{1,3}$ or C_n, and $\chi_\beta(L(G))=2(1+\frac{2}{\Delta+3})^\beta\chi_\beta(G)$ if and only if G is a 3-regular graph.

Proof. We first prove the case $\delta(G) \geq 3$. Observe that

$$\chi_\beta(G) = \sum_{xy \in E(G)} (d(x)+d(y))^\beta = \frac{1}{2} \sum_{x \in V(G)} \sum_{y \in N(x)} (d(x)+d(y))^\beta,$$

and

$$\chi_\beta(L(G)) = \sum_{x \in V(G)} \sum_{e,f \in I(x)} (d_L(e)+d_L(f))^\beta.$$

For each vertex $x \in V(G)$ with $d(x) \geq 3$, by Lemma 1, we have

$$\sum_{e,f \in I(x)} (d_L(e)+d_L(f))^\beta \geq \left(1+\frac{2}{\Delta+3}\right)^\beta \sum_{y \in N(x)} (d(x)+d(y))^\beta.$$

Summing up the above inequalities, we conclude that

$$\chi_\beta(L(G)) \geq 2(1+\frac{2}{\Delta+3})^\beta \chi_\beta(G),$$

with the equality if and only if G is a 3-regular graph.

Now, we prove the case $\delta(G) \leq 2$. If $2 = \delta = \Delta$, then $G \cong C_n$, and $L(G) \cong C_n$, we have $\chi_\beta(L(G)) = \chi_\beta(G)$. Next, we consider $1 \leq \delta < \Delta$. Based on the degree of G, we note that

$$\chi_\beta(G) = \sum_{\substack{x \in V(G) \\ d(x) \geq 3}} \sum_{y \in N(x)} (d(x)+d(y))^\beta - \frac{1}{2} \sum_{\substack{x \in V(G) \\ d(x) \geq 3}} \sum_{\substack{y \in N(x) \\ d(y) \geq 3}} (d(x)+d(y))^\beta$$
$$+ \sum_{P \in \mathcal{P}} \sum_{xy \in E(P \setminus End_3(P))} (d(x)+d(y))^\beta,$$

and

$$\chi_\beta(L(G)) = \sum_{\substack{x \in V(G) \\ d(x) \geq 3}} \sum_{e,f \in I(x)} (d_L(e)+d_L(f))^\beta$$
$$+ \sum_{P \in \mathcal{P}} \sum_{\substack{ef \in E(L(G)) \\ e,f \in E(P)}} (d_L(e)+d_L(f))^\beta. \qquad (*)$$

For each vertex $x \in V(G)$ with $d(x) \geq 3$, by Lemma 2, we have

$$\sum_{e,f \in I(x)} (d_L(e) + d_L(f))^\beta \geq \sum_{y \in N(x)} (d(x) + d(y))^\beta.$$

For $P \in \mathcal{P}$, by Lemma 3, we have

$$\sum_{\substack{ef \in E(L(G)) \\ e,f \in E(P)}} (d_L(e) + d_L(f))^\beta \geq \sum_{xy \in E(P \setminus End_3(P))} (d(x) + d(y))^\beta.$$

Substitute the above two equations into $(*)$, we conclude that

$$\begin{aligned} \chi_\beta(L(G)) \geq & \sum_{\substack{x \in V(G) \\ d(x) \geq 3}} \sum_{y \in N(x)} (d(x) + d(y))^\beta \\ & + \sum_{P \in \mathcal{P}} \sum_{xy \in E(P \setminus End_3(P))} (d(x) + d(y))^\beta. \\ \geq & \sum_{\substack{x \in V(G) \\ d(x) \geq 3}} \sum_{y \in N(x)} (d(x) + d(y))^\beta - \frac{1}{2} \sum_{\substack{x \in V(G) \\ d(x) \geq 3}} \sum_{\substack{y \in N(x) \\ d(y) \geq 3}} (d(x) + d(y))^\beta \\ & + \sum_{P \in \mathcal{P}} \sum_{xy \in E(P \setminus End_3(P))} (d(x) + d(y))^\beta = \chi_\beta(G). \end{aligned}$$

And

$$-\frac{1}{2} \sum_{\substack{x \in V(G) \\ d(x) \geq 3}} \sum_{\substack{y \in N(x) \\ d(y) \geq 3}} (d(x) + d(y))^\beta + \sum_{P \in \mathcal{P}} \sum_{xy \in E(P \setminus End_3(P))} (d(x) + d(y))^\beta = 0$$

i.e., $\chi_\beta(L(G)) = \chi_\beta(G)$ if and only if $G \cong K_{1,3}$. ■

Next, we will discuss the bounds for $\chi_\beta(L(G))$ in terms of $\chi_\beta(G)$ when $\beta < 0$.

Lemma 4. *Let $\beta < 0$ and $P \ncong P_n$. If $P \in \mathcal{P}$ is a pendant path with length of 2, then*

$$\sum_{\substack{ef \in E(L(G)) \\ e,f \in E(P)}} (d_L(e) + d_L(f))^\beta \geq \left(\frac{\Delta + 1}{3}\right)^\beta \sum_{\substack{xy \in E(P) \\ x,y \notin End_3(P)}} (d(x) + d(y))^\beta.$$

Proof. Recall that the definition of pendant path, we assume $P = x_0x_1x_2$, where $d(x_0) \geq 3$, $d(x_1) = 2$, $d(x_2) = 1$. Let $e_1 = x_0x_1$, $e_2 = x_1x_2$. Then

$$\sum_{\substack{xy\in E(P)\\ x,y\notin End_3(P)}} (d(x)+d(y))^\beta = (d(x_1)+d(x_2))^\beta = 3^\beta,$$

$$\sum_{\substack{ef\in E(L(G))\\ e,f\in E(P)}} (d_L(e)+d_L(f))^\beta = (d_L(e_1)+d_L(e_2))^\beta = (d(x_0)+1)^\beta$$

Furthermore, we can conclude that

$$\begin{aligned}&\sum_{\substack{ef\in E(L(G))\\ e,f\in E(P)}} (d_L(e)+d_L(f))^\beta - \Big(\frac{\Delta+1}{3}\Big)^\beta \sum_{\substack{xy\in E(P)\\ x,y\notin End_3(P)}} (d(x)+d(y))^\beta \\ &= (d(x_0)+1)^\beta - \Big(\frac{\Delta+1}{3}\Big)^\beta 3^\beta \\ &\geq (\Delta+1)^\beta - (\Delta+1)^\beta = 0.\end{aligned}$$

The proof is now finished. ■

Lemma 5. *(Carballosa [2]) Let $\beta < 0$, $\delta \leq 2$ and a vertex x of a connected graph G of order n with $d(x) \geq 3$, then*

$$\sum_{e,f\in I(x)} (d_L(e)+d_L(f))^\beta$$

$$\geq \begin{cases} \Big(\frac{\Delta+3}{4}\Big)^\beta \sum\limits_{y\in N(x)} (d(x)+d(y))^\beta, \ if \ \Delta \neq 4; \\ min\Big\{\Big(\frac{7}{4}\Big)^\beta, \frac{3}{2}\Big(\frac{9}{5}\Big)^\beta\Big\} \sum\limits_{y\in N(x)} (d(x)+d(y))^\beta, \ if \ \Delta = 4. \end{cases}$$

Lemma 6. *(Chen [1]) Let $\beta < 0$. If G does not contain a pendant path with length of 2, then*

$$\sum_{\substack{ef\in E(L(G))\\ e,f\in E(P)}} (d_L(e)+d_L(f))^\beta \geq \Big(\frac{\Delta+3}{4}\Big)^\beta \sum_{xy\in E(P\setminus End_3(P))} (d(x)+d(y))^\beta.$$

Theorem 4. *Let $\beta < 0$ and $\delta \leq 2$. If G a connected graph not isomorphic to P_n, then*

$$\chi_\beta(L(G)) \geq \begin{cases} \left(\frac{\Delta+1}{3}\right)^\beta \chi_\beta(G), \ \textit{if } G \textit{ contains a pendant path of length } 2 \\ \qquad\qquad \textit{and } \Delta \geq 5; \\ \left(\frac{\Delta+1.5}{3}\right)^\beta \chi_\beta(G), \ \textit{if } \Delta = 4, \textit{ or contains a pendant path of} \\ \qquad\qquad \textit{length } 2 \textit{ and } \Delta = 3. \end{cases}$$

Proof. Similar to the proof of Theorem 3, we first have

$$\chi_\beta(G) = \sum_{\substack{x\in V(G)\\ d(x)\geq 3}} \sum_{y\in N(x)} (d(x)+d(y))^\beta - \frac{1}{2} \sum_{\substack{x\in V(G)\\ d(x)\geq 3}} \sum_{\substack{y\in N(x)\\ d(y)\geq 3}} (d(x)+d(y))^\beta$$

$$+ \sum_{P\in\mathcal{P}} \sum_{xy\in E(P\setminus End_3(P))} (d(x)+d(y))^\beta,$$

and

$$\chi_\beta(L(G)) = \sum_{\substack{x\in V(G)\\ d(x)\geq 3}} \sum_{e,f\in I(x)} (d_L(e)+d_L(f))^\beta + \sum_{P\in\mathcal{P}} \sum_{\substack{ef\in E(L(G))\\ e,f\in E(P)}} (d_L(e)+d_L(f))^\beta.$$

For each vertex $x \in V(G)$ with $d(x) \geq 3$, since $\min\{(\frac{7}{4})^\beta, \frac{3}{2}(\frac{9}{5})^\beta\} \geq (\frac{\Delta+1.5}{3})^\beta$, thus by Lemma 5, we have

$$\sum_{e,f\in I(x)} (d_L(e)+d_L(f))^\beta \geq \begin{cases} (\dfrac{\Delta+3}{4})^\beta \displaystyle\sum_{y\in N(x)} (d(x)+d(y))^\beta, \ \textit{if } \Delta(G) \neq 4; \\ (\dfrac{\Delta+1.5}{3})^\beta \displaystyle\sum_{y\in N(x)} (d(x)+d(y))^\beta, \ \textit{if } \Delta(G) = 4. \end{cases}$$

If G does not contain a pendant path of length 2, then by Lemma 6, we have

$$\sum_{\substack{ef\in E(L(G))\\ e,f\in E(P)}} (d_L(e)+d_L(f))^\beta \geq \left(\frac{\Delta+3}{4}\right)^\beta \sum_{xy\in E(P\setminus End_3(P))} (d(x)+d(y))^\beta.$$

Otherwise, if G contains a pendant path of length 2 and $\Delta \geq 5$. By Lemma 4, we have

$$\sum_{\substack{ef\in E(L(G))\\ e,f\in E(P)}} (d_L(e)+d_L(f))^\beta$$

$$\geq min\{\Big(\frac{\Delta+3}{4}\Big)^\beta, \Big(\frac{\Delta+1}{3}\Big)^\beta\} \sum_{xy\in E(P\setminus End_3(P))} (d(x)+d(y))^\beta$$

$$= \Big(\frac{\Delta+1}{3}\Big)^\beta \sum_{xy\in E(P\setminus End_3(P))} (d(x)+d(y))^\beta.$$

If G contains a pendant path of length 2 and $\Delta = 3$, we have

$$\sum_{\substack{ef\in E(L(G))\\ e,f\in E(P)}} (d_L(e)+d_L(f))^\beta$$

$$\geq min\{\Big(\frac{\Delta+3}{4}\Big)^\beta, \Big(\frac{\Delta+1.5}{3}\Big)^\beta\} \sum_{xy\in E(P\setminus End_3(P))} (d(x)+d(y))^\beta$$

$$= \Big(\frac{\Delta+1.5}{3}\Big)^\beta \sum_{xy\in E(P\setminus End_3(P))} (d(x)+d(y))^\beta.$$

Combining the above, we conclude that if G contains a pendant path of length 2 and $\Delta \geq 5$, then

$$\chi_\beta(L(G)) \geq min\Big\{\Big(\frac{\Delta+3}{4}\Big)^\beta, \Big(\frac{\Delta+1}{3}\Big)^\beta\Big\}\Big(\sum_{\substack{x\in V(G)\\ d(x)\geq 3}} \sum_{y\in N(x)} (d(x)+d(y))^\beta$$

$$+\sum_{P\in\mathcal{P}} \sum_{xy\in E(P\setminus End_3(P))} (d(x)+d(y))^\beta\Big) \geq \Big(\frac{\Delta+1}{3}\Big)^\beta \chi_\beta(G).$$

If $\Delta = 4$, or G contains a pendant path of length 2 and $\Delta = 3$, then have

$$\chi_\beta(L(G)) \geq \Big(\frac{\Delta+1.5}{3}\Big)^\beta \chi_\beta(G).$$

This completes the proof. ∎

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References

[1] X. Chen, General sum-connectivity index of a graph and its line graph, *Appl. Math. Comput.* **443** (2023) #127779.

[2] W. Carballosa, D. Pestana, J. M. Sigarreta, E. Tourís, Relations between the general sum connectivity index and the line graph, *J. Math. Chem.* **58** (2020) 2273–2290.

[3] O. Araujo, J. A. de la Peña, The connectivity index of a weighted graph, *Lin. Algebra Appl.* **283** (1998) 171–177.

[4] I. Gutman, M. Randić, Algebraic characterization of skeletal branching, *Chem. Phys. Lett.* **47** (1977) 15–19.

[5] I. Gutman, J. Tošović, Testing the quality of molecular structure descriptors. Vertex-degreebased topological indices, *J. Serb. Chem. Soc.* **78** (2013) 805–810.

[6] N. Trinajstić, I. Gutman, Mathematical chemistry, *Croat. Chem. Acta.* **75** (2002) 329–356.

[7] H. Whitney, Congruent graphs and the connectivity of graphs, *Am. J. Math.* **54** (1932) 150–168.

[8] J. Zhang, B. Wu, Randić index of a line graph, *Axioms* **11** (2022) 210.

[9] B. Wu, Wiener index of line graphs, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 699–706.

[10] M. Randić, On the characterization of molecular branching, *J. Am. Chem. Soc.* **97** (1975) 6609–6615.

[11] B. Bollobás, P. Erdős, Graphs of extremal weights, *Ars Comb.* **50** (1998) 225–233.

[12] B. Zhou, N. Trinajstić, On a novel connectivity index, *J. Math. Chem.* **46** (2009) 1252–1270.

[13] B. Zhou, N. Trinajstić, On general sum-connectivity index, *J. Math. Chem.* **47** (2010) 210–218.

[14] D. Vukičević, M. Gašperov, Bond additive modeling 1. Adriatic indices, *Croat. Chem. Acta* **83** (2010) 243–260.