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On the Energy and Spread of the Adjacency, Laplacian and Signless Laplacian Matrices of Graphs

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Abstract

In this paper, we explore the connection between the energy and spread of the adjacency, Laplacian, and signless Laplacian matrices for graphs. We then introduce new limitations for the energy and spread of these matrices, based on previous research and our findings.

1 Introduction

Let G be a graph, with its order and size denoted by n and m, respectively. The degree of a vertex v in G, $d_G(v)$, is the number of edges incident to v. The edge degree of an edge e in G, $d_G(e)$, is the number of edges incident

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to e. In this work, we use Δ, δ, and Δ_e to denote the maximum degree, minimum degree, and maximum edge degree of G, respectively.

The chromatic number of a graph G, denoted by $\chi(G)$ or simply χ, is the smallest number of colors necessary to assign to the vertices of G so that adjacent vertices do not have the same color.

Two edges that are not adjacent are called independent edges. The matching number of a graph G, denoted by $\alpha'(G)$ or simply α', is the number of edges in the largest independent set of edges in G.

The clique number of a graph G, denoted by $\omega(G)$ or simply ω, is the number of vertices in the largest complete subgraph of G.

A vertex cover of a graph G is a set of vertices that includes at least one endpoint of every edge in the graph. The vertex cover number of graph G, denoted by $\beta(G)$ or simply β, is the size of a minimum vertex cover of G.

In the following, D and ρ represent the diameter and the number of components of G, respectively.

A graph G has a vertex set denoted by $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of G is represented as $A(G)$, where $a_{ij} = 1$ if there is an edge between vertices v_i and v_j, and $a_{ij} = 0$ otherwise. We obtain the diagonal matrix of G, denoted by $D(G)$, by taking the row sums of $A(G)$, which gives us the degree of each vertex in G. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, while the signless Laplacian matrix of G is denoted by $Q(G) = D(G) + A(G)$. The eigenvalues of $A(G)$, $L(G)$, and $Q(G)$ are known as A-, L-, and Q-eigenvalues, respectively, and are arranged in decreasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n = 0$, and $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$.

Adjacency energy $\mathcal{E}(G)$ [16], Laplacian energy $L\mathcal{E}(G)$ [17], and signless Laplacian energy $Q\mathcal{E}(G)$ [1] of G are defined as:

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|\,, \quad L\mathcal{E}(G) = \sum_{i=1}^{n} \left|\alpha_i - \frac{2m}{n}\right|, \quad Q\mathcal{E}(G) = \sum_{i=1}^{n} \left|q_i - \frac{2m}{n}\right|.$$

For details of the mathematical theory of these, nowadays very popular, graph-spectral invariants see the book [24], the recent papers [7–12], and the references cited therein. Also, the adjacency spread $AS(G)$ [15], Laplacian spread $LS(G)$ [13], and signless Laplacian spread $QS(G)$ [30, 36], of

G are defined as:

$$AS(G) = \lambda_1 - \lambda_n, \ \ LS(G) = \alpha_1 - \alpha_{n-1}, \ \ QS(G) = q_1 - q_n.$$

The spectrum of $A(G)$, $L(G)$, and $Q(G)$ can be represented as follows: $Spec(A(G)) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $Spec(L(G)) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $Spec(Q(G)) = (q_1, q_2, \ldots, q_n)$.

The first Zagreb index $M_1(G)$ [35] and Randić index $R(G)$ [38] of G can be defined as:

$$M_1 = M_1(G) = \sum_{v \in V(G)} d_G(v)^2, \ \ R = R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(v)}}.$$

The adjacency, Laplacian, and signless Laplacian eigenvalues satisfy:

$$\sum_{i=1}^{n} \lambda_i = 0, \qquad \sum_{i=1}^{n} \lambda_i^2 = 2m, \tag{1}$$

$$\sum_{i=1}^{n} \alpha_i = 2m, \quad \sum_{i=1}^{n} \alpha_i^2 = 2m + M_1(G), \tag{2}$$

$$\sum_{i=1}^{n} q_i = 2m, \quad \sum_{i=1}^{n} q_i^2 = 2m + M_1(G). \tag{3}$$

We denote the union of p copies of a graph H by pH. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$. The duplication graph D_pH is a graph with pn vertices. It is obtained from pH by joining vertex v_i to every neighbor of v_i in the j-th copy of G_j where $1 \leq j \leq p$ and $1 \leq i \leq n$.

In a theorem from [32], it is shown that there is a relationship between the eigenvalues of H and D_pH.

Theorem 1. [32] *Suppose G is a graph with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the eigenvalues of the adjacency matrix of the duplication graph D_pH are $p\lambda_i$, where $1 \leq i \leq n$ and 0 with the multiplicity $(p-1)n$.*

In this study, we explore the connection between the energy and spread of the adjacency, Laplacian, and signless Laplacian matrices for graphs. We then introduce new limitations for the energy and spread of these matrices, based on previous research and our findings.

2 Main result

In this section, we will prove our main results which establish the relationship between the energy and spread of the adjacency, Laplacian, and signless Laplacian matrices of graphs. Firstly, we will start by presenting the theorem that plays the main role in achieving our objectives.

Theorem 2. *Let* $a_1, a_2, \ldots, a_n$ *be real numbers such that* $\sum_{i=1}^{n} a_i = 0$, $a_1 = \max_{1\leq i\leq n} a_i$ *and* $a_n = \min_{1\leq i\leq n} a_i$. *Then*

$$\sum_{i=1}^{n} {a_i}^2 \leq \frac{1}{2}\,(a_1 - a_n)\sum_{i=1}^{n} |a_i|. \tag{4}$$

The equality holds if and only if $\{a_i : 1 \leq\ i \leq\ n\} \subseteq \{a_1, 0, a_n\}$.

Proof. Given $\sum_{i=1}^{n} a_i = 0$, we have:

$$\sum_{i=1}^{n} {a_i}^2 = \frac{1}{2}\sum_{i=1}^{n}(2a_i - a_1 - a_n)a_i \leq \frac{1}{2}\sum_{i=1}^{n} |2a_i - a_1 - a_n||a_i|. \tag{5}$$

The equality holds if and only if $(2a_i - a_1 - a_n)a_i \geq 0$ for $i = 1, 2, \ldots, n$.

We also know that $1 \leq\ i \leq\ n$, then $a_n - a_1 \leq 2a_i - a_1 - a_n \leq\ a_1 - a_n$. The equality on the left side holds if and only if $a_i = a_n$, whilst the equality on the right side holds if and only if $a_i = a_1$. Therefore, for $1 \leq\ i \leq\ n$, $|2a_i - a_1 - a_n| \leq\ a_1 - a_n$ and the equality holds if and only if $a_i \in \{a_1, a_n\}$. Hence, for $1 \leq\ i \leq\ n$:

$$|2a_i - a_1 - a_n||a_i| \leq\ (a_1 - a_n)|a_i|. \tag{6}$$

The equality holds if and only if $a_i \in \{a_1, 0, a_n\}$.

By combining Equations (5) and (6), we get:

$$\begin{aligned}\sum_{i=1}^{n} {a_i}^2 &\leq \frac{1}{2}\sum_{i=1}^{n} |2a_i - a_1 - a_n||a_i| \\ &\leq \frac{1}{2}\sum_{i=1}^{n}(a_1 - a_n)|a_i| = \frac{1}{2}\,(a_1 - a_n)\sum_{i=1}^{n} |a_i|.\end{aligned}$$

The equality in (4) holds if and only if the equality in (5), and also for all $1 \leq i \leq n$, the equalities in (6) hold. Therefore, the equality in (4) holds if and only if $\{a_i : 1 \leq i \leq n\} \subseteq \{a_1, 0, a_n\}$, as desired. ■

In the following theorem, we will derive the relationship between the energy and spread of adjacency, Laplacian, and signless Laplacian matrices of graphs.

It is worth noting that Jahanbani and Sheikholeslami proved in [19] that $\mathcal{E}(G) \geq \frac{4m}{AS(G)}$, with equality holding for $G \cong K_n$ and $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Theorem 3. *Let G be graph of order n and size m. Then*

(i) $\mathcal{E}(G) \geq \frac{4m}{AS(G)}$ *and the equality holds if and only if* $\{\lambda_i : 1 \leq i \leq n\} \subseteq \{\lambda_1, 0, \lambda_n\}$. *Moreover, if G is bipartite, then the equality holds if and only if* $G \cong \bigcup\limits_{i=1}^{k} K_{a_i,b_i} \cup r\, K_1$, *where* $n = \sum\limits_{i=1}^{k} (a_i + b_i) + r,\ r \geq 0$ *and* $a_1 b_1 = a_2 b_2 = \cdots = a_k b_k$.

(ii) $L\mathcal{E}(G) \geq \frac{2nM_1(G)+4m(n-2m)}{n\alpha_1}$ *and the equality holds if and only if* $\{\alpha_i : 1 \leq i \leq n\} \subseteq \{\alpha_1, \frac{2m}{n}, 0\}$. *Moreover, the equality holds if* $G \cong K_n$ *or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

(iii) $Q\mathcal{E}(G) \geq \frac{2nM_1(G)+4m(n-2m)}{n\, QS(G)}$ *and the equality holds if and only if* $\{q_i : 1 \leq i \leq n\} \subseteq \{q_1, \frac{2m}{n}, q_n\}$. *Moreover, the equality holds if* $G \cong K_n$ *or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof. (i) Let $a_i = \lambda_i$ for $i = 1, 2, \ldots, n$. Then, by the first part of relation (1), we have $\sum_{i=1}^{n} a_i = 0$. Thus, by using Theorem 2,

$$\sum_{i=1}^{n} \lambda_i^2 \leq \frac{1}{2}(\lambda_1 - \lambda_n) \sum_{i=1}^{n} |\lambda_i| = \frac{1}{2} AS(G)\mathcal{E}(G).$$

Then, by the second part of relation (1), $2m \leq \frac{1}{2} AS(G)\mathcal{E}(G)$. Hence, we conclude that $\mathcal{E}(G) \geq \frac{4m}{AS(G)}$ and the equality holds if and only if $\{\lambda_i : 1 \leq i \leq n\} \subseteq \{\lambda_1, 0, \lambda_n\}$. The first part of the proof is done.

Suppose that equality holds. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0 \geq \lambda_{p+1} \geq \lambda_{p+2} \geq \lambda_n$. By Theorem 2, we have $\{\lambda_i : 1 \leq i \leq n\} \subseteq \{\lambda_1, 0, \lambda_n\}$. Then G has at most three distinct eigenvalues and hence the diameter is at most 2, see [4]. If $d(G) = 1$, then $G \cong K_n$ and hence the equality holds. Otherwise, $d(G) = 2$. In this case there are exactly three distinct eigenvalues in G, that is, $\lambda_1 > 0 > \lambda_n$. Now, we assume that G is a bipartite graph with p components. Then we obtain $\lambda_i = -\lambda_{n-i+1}$ $(1 \leq i \leq p)$ and if $n > 2p$, $\lambda_j = 0$ $(j = p+1, \ldots, n-p)$. Moreover, $\sum\limits_{i=1}^{p} \lambda_i^2 = \sum\limits_{i=p+1}^{n} \lambda_i^2 = m$. Thus we have

$$\sum_{i=1}^{n} \lambda_i^2 = 2m = \lambda_1 \sum_{i=1}^{n} |\lambda_i|,$$

that is, $\sum\limits_{i=1}^{n} (\lambda_1 - |\lambda_i|)\,|\lambda_i| = 0$, that is,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_p = -\lambda_{n-p+1} = -\lambda_{n-p+2} = \cdots = -\lambda_n.$$

First we assume that G is connected. For any connected graph, it is well-known that $\lambda_1 > \lambda_2$. Thus we have $p = 1$ and hence $\lambda_1 = -\lambda_n$, $\lambda_i = 0$ for $i = 2, \ldots, n-1$. Therefore $G \cong K_{a,b}$ $(a+b = n)$.

Next we assume that G is disconnected. Let $G = \bigcup\limits_{i=1}^{k} G_i \cup r\,K_1$, where $r \geq 0$. For each G_i $(1 \leq i \leq k)$, $G_i \cong K_{a_i,b_i}$ with $a_i b_i = m_i$ such that $m_1 = m_2 = \cdots = m_k$. Hence $G \cong \bigcup\limits_{i=1}^{k} K_{a_i,b_i} \cup r\,K_1$, where $n = \sum\limits_{i=1}^{k} (a_i + b_i) + r$, $r \geq 0$, and $a_1 b_1 = a_2 b_2 = \cdots = a_k b_k$.

Conversely, let $G \cong \bigcup\limits_{i=1}^{k} K_{a_i,b_i} \cup r\,K_1$, where $n = \sum\limits_{i=1}^{k} (a_i + b_i) + r$, $r \geq 0$, and $a_1 b_1 = a_2 b_2 = \cdots = a_k b_k$. Then $\lambda_1 = \lambda_2 = \cdots = \lambda_k = -\lambda_{n-k+1} = \cdots = -\lambda_{n-1} = -\lambda_n = \sqrt{a_i b_i}$ $(i = 1, \ldots, k)$ and $\lambda_i = 0$ $(i = k+1, k+2, \ldots, n-k)$. Thus we have $\mathcal{E}(G) = 2k\sqrt{a_1 b_1}$, $m = \sum\limits_{i=1}^{k} a_i b_i = k\,a_1 b_1$ and $AS(G) = 2\lambda_1 = 2\sqrt{a_1 b_1}$. Hence $\mathcal{E}(G) = \frac{4m}{AS(G)}$.

(ii) Let $a_i = \alpha_i - \frac{2m}{n}$ for $i = 1, 2, \ldots, n$. Then, by the first part of relation (2), we can write $\sum_{i=1}^{n} a_i = 0$. Thus, by Theorem 2,

$$\sum_{i=1}^{n}\left(\alpha_i - \frac{2m}{n}\right)^2 \leq \frac{1}{2}\alpha_1 L\mathcal{E}(G).$$

Then, by the second part of relation (2), we have $\frac{nM_1(G)+2m(n-2m)}{n} \leq \frac{1}{2}\alpha_1 L\mathcal{E}(G)$. Therefore, $L\mathcal{E}(G) \geq \frac{2nM_1(G)+4m(n-2m)}{n\alpha_1}$ and the equality holds if and only if $\{\alpha_i : 1 \leq i \leq n\} \subseteq \{\alpha_1, \frac{2m}{n}, 0\}$. Clearly, if $G \cong K_n$, then

$$Spec(L(G)) = (\overbrace{n, \ldots, n}^{n-1\ items}, 0)$$

and so $L\mathcal{E}(G) = \frac{2nM_1(G)+4m(n-2m)}{n\alpha_1} = 2(n-1)$. Moreover, if $G \cong K_{\frac{n}{2},\frac{n}{2}}$, then

$$Spec(L(G)) = \left(n, \overbrace{\frac{n}{2}, \ldots, \frac{n}{2}}^{n-2\ items}, 0\right)$$

and so $L\mathcal{E}(G) = \frac{2nM_1(G)+4m(n-2m)}{n\alpha_1} = n$.

(iii) Let $a_i = q_i - \frac{2m}{n}$ for $i = 1, 2, \ldots, n$. Then, by the first part of relation (3), we have $\sum_{i=1}^{n} a_i = 0$. Thus, by using Theorem 2,

$$\sum_{i=1}^{n}\left(q_i - \frac{2m}{n}\right)^2 \leq \frac{1}{2}\, QS(G)Q\mathcal{E}(G).$$

Hence, by the second part of relation (3), we have $\frac{nM_1(G)+2m(n-2m)}{n} \leq \frac{1}{2}\, QS(G)Q\mathcal{E}(G)$. It concludes that $Q\mathcal{E}(G) \geq \frac{2nM_1(G)+4m(n-2m)}{n\, QS(G)}$ and the equality holds if and only if $\{q_i : 1 \leq i \leq n\} \subseteq \{q_1, \frac{2m}{n}, q_n\}$.

Clearly if $G \cong K_n$, then

$$Spec(Q(G)) = (2n-2, \overbrace{n-2, \ldots, n-2}^{n-1\ items})$$

and so $Q\mathcal{E}(G) = \frac{2nM_1(G)+4m(n-2m)}{n\, QS(G)} = 2(n-1)$. Moreover, if $G \cong K_{\frac{n}{2},\frac{n}{2}}$, then

$$Spec(Q(G)) = \left(n, \overbrace{\frac{n}{2}, \ldots, \frac{n}{2}}^{n-2\ items}, 0\right)$$

and so $Q\mathcal{E}(G) = \frac{2nM_1(G)+4m(n-2m)}{n\,QS(G)} = n$. ■

If we have an $n \times n$ matrix M, and there exists another matrix B such that $MB = BM = I_n$, then we say that M is non-singular. This means that M has no zero eigenvalues. In other words, all of the eigenvalues of M are non-zero.

Corollary 1. *If $A(G)$ is nonsingular, then $\mathcal{E}(G) \geq \frac{4m}{AS(G)}$ with equality if and only if $G \cong K_n$.*

Consider the following questions related to a graph G of order n and size m:

1. If $\mathcal{E}(G) = \frac{4m}{AS(G)}$, then what is the structure of graph G?
2. If $L\mathcal{E}(G) = \frac{2nM_1(G)+4m(n-2m)}{n\alpha_1}$, then what is the structure of graph G?
3. If $Q\mathcal{E}(G) = \frac{2nM_1(G)+4m(n-2m)}{n\,QS(G)}$, then what is the structure of graph G?

Proposition 4. *Let G be a graph with m edges. If $\mathcal{E}(G) = \frac{4m}{AS(G)}$, then $\mathcal{E}(D_nG) = \frac{4|E(D_nG)|}{AS(D_nG)}$.*

Proof. Let $v \in V(G)$. By definition of D_nG, we can say that there are n copies of v in D_nG, all with degree $nd_G(v)$. For any graph H, it is known that $\sum_{v\in V(H)} d_H(v) = 2|E(H)|$. Then, according to Theorem 1, we have

$$\frac{4|E(D_nG)|}{AS(D_nG)} = \frac{2n\sum_{v\in V(G)} nd_G(v)}{nAS(G)} = \frac{4n^2m}{nAS(G)} = n\frac{4m}{AS(G)}, \tag{7}$$

$$\mathcal{E}(D_nG) = \sum_{i=1}^{n} |n\lambda_i| = n\mathcal{E}(G). \tag{8}$$

Therefore, using relations (7) and (8), we can conclude that if $\mathcal{E}(G) = \frac{4m}{AS(G)}$, then $\mathcal{E}(D_nG) = \frac{4|E(D_nG)|}{AS(D_nG)}$. ■

We need to refer to the theorem presented in [26] to prove our upcoming results.

Theorem 5. [26] *Let G be a graph. Then $M_1 \geq \frac{4m^2}{n}$ and the equality holds if and only if G is a regular graph.*

Corollary 2. *Let G be graph. Then*

i) *$L\mathcal{E}(G) \geq \frac{4m}{\alpha_1}$ and the equality holds if and only if G is a regular graph and $\{\alpha_i : 1 \leq i \leq n\} \subseteq \{\alpha_1, \frac{2m}{n}, 0\}$. Moreover, the equality holds if $G \cong K_n$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

ii) *$Q\mathcal{E}(G) \geq \frac{4m}{QS(G)}$ and the equality holds if and only if G is a regular graph and $\{q_i : 1 \leq i \leq n\} \subseteq \{q_1, \frac{2m}{n}, q_n\}$. Moreover, the equality holds if $G \cong K_n$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

Proof. The proof follows from Theorems 3(ii), 3(iii) and 5. ■

Remark 1. Let G be a graph. According to Theorem 3, we reach to the following results:

i) Each upper bound on $AS(G)$ gives a lower bound for $\mathcal{E}(G)$. Also, each upper bound on $\mathcal{E}(G)$ gives a lower bound for $AS(G)$.

ii) Each upper bound on α_1 drives a lower bound for $L\mathcal{E}(G)$. Also, each upper bound on $L\mathcal{E}(G)$ gives a lower bound for α_1.

iii) Each upper bound on $QS(G)$ leads a lower bound for $Q\mathcal{E}(G)$. Moreover, each upper bound on $Q\mathcal{E}(G)$ gives a lower bound for $QS(G)$.

3 Applications

In this section, we will present new bounds for the energy and spread of the adjacency, Laplacian, and signless Laplacian matrices of graphs, building upon previous publications and our results.

Theorem 6. *Let G be a graph with $m \geq 1$. Then*

i) [33] *$\mathcal{E}(G) \leq \sqrt{2mn}$.*

ii) [21] *$\mathcal{E}(G) \leq \frac{2m}{n} + \sqrt{(n-1)(2m - \frac{4m^2}{n^2})}$ for $2m \geq n$, and the equality holds if and only if G is isomorphic to $\frac{n}{2}K_2$ or K_n or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value $\sqrt{(2m - (\frac{2m}{n})^2)/(n-1)}$.*

iii) [18] $\mathcal{E}(G) \leq \frac{m}{R} + \sqrt{(n-1)(2m - (\frac{m}{R})^2}$ *and the equality holds if and only if* $G \cong K_n$.

iv) [18] $\mathcal{E}(G) \leq (\chi - 1) + \sqrt{(n-1)(2m - (\chi-1)^2)}$ *and the equality holds if and only if* $G \cong K_n$.

v) [18] $\mathcal{E}(G) \leq \sqrt[D]{n-1} + \sqrt{(n-1)(2m - (\sqrt[D]{n-1})^2)}$ *and the equality holds if and only if* $G \cong K_n$.

vi) [5] $\mathcal{E}(G) \leq 2m - \frac{2m}{n}(\frac{2m}{n} - 1) - \ln(\frac{n|\det A|}{2m})$, *where* A *is a nonsingular matrix, and the equality holds if and only if* $G \cong K_n$.

vii) [37] $\mathcal{E}(G) \leq 2\alpha'\sqrt{2\Delta_e + 1}$, *where* Δ_e *is an even number, and the equality holds if and only if each component of* G *is isomorphic to* K_1 *or* K_2.

viii) [37] $\mathcal{E}(G) \leq \alpha' \left(\sqrt{2\nu + 2\sqrt{2\nu}} + \sqrt{2\nu - 2\sqrt{2\nu}}\right)$, *where* $\nu = \Delta_e + 1$ *is an even number, and the equality holds if and only if each component of* G *is isomorphic to* K_1 *or* P_3.

ix) [42] $\mathcal{E}(G) \leq \sqrt{\frac{M_1}{n}} + \sqrt{(n-1)(2m - \frac{M_1}{n})}$ *and the equality holds if and only if* G *is isomorphic to* $\frac{n}{2}K_2$ *or* K_n *or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value* $\sqrt{(2m - (\frac{2m}{n})^2)/(n-1)}$.

x) [40] $\mathcal{E}(G) \leq \sqrt{\Delta} + \sqrt{(n-1)(2m - \Delta)}$ *and the equality holds if and only if* $G \cong \frac{n}{2}K_2$.

xi) [40] $\mathcal{E}(G) \leq 2\cos(\frac{\pi}{n+1}) + \sqrt{(n-1)\left(2m - (2\cos(\frac{\pi}{n+1}))^2\right)}$, *where* G *is a connected graph, and the equality holds if and only if* $G \cong P_2$.

xii) [40] $\mathcal{E}(G) \leq 2\beta\sqrt{\Delta}$ *and the equality holds if and only if* $G \cong \beta K_{1,\Delta} \cup (n-\beta)K_1$.

xiii) [6] $\mathcal{E}(G) \leq \sqrt{2m(n-\delta) + 4\sqrt{m^3(1 - \frac{1}{\omega})}}$.

Corollary 3. *Let* G *be a graph with* $m \geq 1$. *Then*

i) $AS(G) \geq \frac{2\sqrt{2mn}}{n}$ *and the equality holds if* $G \cong \frac{n}{2}K_2$.

ii) $AS(G) \geq 4\,m\left(2\,\frac{m}{n}+\sqrt{(n-1)(2\,m-4\,\frac{m^2}{n^2})}\right)^{-1}$ *for* $2m \geq n$, *and the equality holds if and only if* G *is isomorphic to* $\frac{n}{2}K_2$ *or* K_n *or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value* $\sqrt{(2m-(\frac{2m}{n})^2)/(n-1)}$.

iii) $AS(G) \geq 4m\left(\frac{m}{R}+\sqrt{(n-1)(2m-(\frac{m}{R})^2)}\right)^{-1}$ *and the equality holds if and only if* $G \cong K_n$.

iv) $AS(G) \geq 4m\left((\chi-1)+\sqrt{(n-1)(2m-(\chi-1)^2)}\right)^{-1}$ *and the equality holds if and only if* $G \cong K_n$.

v) $AS(G) \geq 4m\left(\sqrt[D]{n-1}+\sqrt{(n-1)(2m-(\sqrt[D]{n-1})^2)}\right)^{-1}$ *and the equality holds if and only if* $G \cong K_n$.

vi) $AS(G) \geq 4m\left(2m-\frac{2m}{n}(\frac{2m}{n}-1)-\ln(\frac{n|\det A|}{2m})\right)^{-1}$, *where* A *is a non-singular matrix, and the equality holds if and only if* $G \cong K_n$.

vii) $AS(G) \geq 4m\left(2\alpha\sqrt{2\Delta_e+1}\right)^{-1}$, *where* Δ_e *is an even number, and the equality holds if and only if each component of* G *is isomorphic to* K_1 *or* K_2.

viii) $AS(G) \geq 4m\left(\alpha\left(\sqrt{2\nu+2\sqrt{2\nu}}+\sqrt{2\nu-2\sqrt{2\nu}}\right)\right)^{-1}$, *where* $\nu = \Delta_e + 1$ *is an even number, and the equality holds if and only if each component of* G *is isomorphic to* K_1 *or* P_3.

ix) $AS(G) \geq 4m\left(\sqrt{\frac{M_1}{n}}+\sqrt{(n-1)(2m-\frac{M_1}{n})}\right)^{-1}$, *and the equality holds if and only if* G *is isomorphic to* $\frac{n}{2}K_2$ *or* K_n *or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value* $\sqrt{(2m-(\frac{2m}{n})^2)/(n-1)}$.

x) $AS(G) \geq 4m\left(\sqrt{\Delta}+\sqrt{(n-1)(2m-\Delta)}\right)^{-1}$ *and the equality holds if and only if* $G \cong \frac{n}{2}K_2$.

xi) $AS(G) \geq 4m\Big(2\cos(\frac{\pi}{n+1})+\sqrt{(n-1)\big(2m-(2\cos(\frac{\pi}{n+1}))^2\big)}\Big)^{-1}$, *where* G *is a connected graph, and the equality holds if and only if* $G \cong P_2$.

xii) $AS(G) \geq 4m\left(2\beta\sqrt{\Delta}\right)^{-1}$ *and the equality holds if and only if* $G \cong \beta K_{1,\Delta} \cup (n-\beta)K_1$.

xiii) $AS(G) \geq 4m\left(\sqrt{2m(n-\delta)+4\sqrt{m^3(1-\frac{1}{\omega})}}\right)^{-1}$.

Proof. The proof follows from Theorems 3(i) and 6. ∎

Theorem 7. *If G is a graph with $m \geq 1$, then*

i) [15] $AS(G) \leq 2\sqrt{m}$ *and the equality holds if and only if G is the union of a complete bipartite subgraph and some trivial subgraphs.*

ii) [15] *For* $m > \lfloor \frac{n^2}{4} \rfloor$, $AS(G) \leq \frac{2m}{n-1} + \sqrt{2m\left(\frac{n-2}{n-1}\right)\left(1-\frac{2m}{n(n-1)}\right)}$ *and the equality holds if and only if* $G \cong K_n$.

iii) [15] *For* $m > \lfloor \frac{n^2}{4} \rfloor$, $AS(G) \leq \frac{2m}{n} + \sqrt{2m - \left(\frac{2m}{n}\right)^2 - \omega + 2}$ *and the equality holds if and only if* $G \cong K_n$.

iv) [29] $AS(G) \leq 2\sqrt[4]{M_1 - m + 4f}$, *where f is the number of all 4-cycles in G, and the equality holds if and only if G is the union of a complete bipartite subgraph and some trivial subgraphs.*

v) [27] $AS(G) \leq n$, *where G is a regular graph, and the equality holds if* $G \cong K_n$ *or* $K_{\frac{n}{2},\frac{n}{2}}$.

vi) [28] $AS(G) \leq \sqrt{2kn}$, *where G is a connected k-regular graph, and the equality holds if* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

vii) [28] $AS(G) \leq 2\sqrt{2(n-2)}$, *where $n \geq 3$ and $\chi = 2$, and the equality holds if* $G \cong K_{2,n-2}$.

Corollary 4. *If G is a graph with $m \geq 1$, then*

i) $\mathcal{E}(G) \geq 2\sqrt{m}$ *and the equality holds if and only if G is the union of a complete bipartite subgraph and some trivial subgraphs.*

ii) *For* $m > \lfloor \frac{n^2}{4} \rfloor$, $\mathcal{E}(G) \geq 4m\left(\frac{2m}{n-1} + \sqrt{2m\left(\frac{n-2}{n-1}\right)\left(1-\frac{2m}{n(n-1)}\right)}\right)^{-1}$ *and the equality holds if and only if* $G \cong K_n$.

iii) *For* $m > \lfloor \frac{n^2}{4} \rfloor$, $\mathcal{E}(G) \geq 4m \left(\frac{2m}{n} + \sqrt{2m - \left(\frac{2m}{n}\right)^2 - \omega + 2} \right)^{-1}$ *and the equality holds if and only if* $G \cong K_n$.

iv) $\mathcal{E}(G) \geq 2m \left(\sqrt[4]{M_1 - m + 4f} \right)^{-1}$, *where* f *is the number of all 4-cycles in* G*, and the equality holds if and only if* G *is the union of a complete bipartite subgraph and some trivial subgraphs.*

v) $\mathcal{E}(G) \geq \frac{4m}{n}$, *where* G *is a regular graph, and the equality holds if* $G \cong K_n$ *or* $K_{\frac{n}{2}, \frac{n}{2}}$.

vi) $\mathcal{E}(G) \geq \frac{4m}{\sqrt{2kn}}$, *where* G *is a connected* k*-regular graph, and the equality holds if* $G \cong K_{\frac{n}{2}, \frac{n}{2}}$.

vii) $\mathcal{E}(G) \geq \frac{2m}{\sqrt{2(n-2)}}$, *where* $n \geq 3$ *and* $\chi = 2$*, and the equality holds if* $G \cong K_{2,n-2}$.

Proof. The proof follows from Theorems 3(i) and 7. ∎

Theorem 8. *Let* G *be a graph with* $m \geq 1$*. Then*

i) [17] $L\mathcal{E}(G) \leq \sqrt{nM_1 + 2m(n-2m)}$ *and the equality holds if and only if there exist* $\alpha \geq 1$ *and* $k \geq 2$ *such that* $G \cong \alpha K_k \cup (k-2)\alpha K_1$.

ii) [17] $L\mathcal{E}(G) \leq \frac{2m}{n}\rho + \sqrt{(n-\rho)\left(\frac{nM_1+2m(n-2m)}{n} - \rho\left(\frac{2m}{n}\right)^2\right)}$, *and for* $\rho = 1$ *the equality holds if and only if* G *is a complete graph or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value* $\sqrt{(2m - (\frac{2m}{n})^2)/(n-1)}$*; Otherwise, the equality holds if there exist* $\alpha \geq 1$ *and* $k \geq 2$ *such that* $G \cong \alpha K_k \cup (k-2)\alpha K_1$ *and* $(k-1)\alpha = \rho$.

iii) [17] G *has no isolated vertex. Then* $L\mathcal{E}(G) \leq \frac{nM_1+2m(n-2m)}{n}$ *the equality holds if and only if* $G \cong \frac{n}{2}K_2$.

iv) [43] $L\mathcal{E}(G) \leq \sqrt{\frac{nM_1+2m(n-2m)}{n}(n-1) + nd^{\frac{2}{n}}}$, *where* $d = \left| \det\left(L(G) - \frac{2m}{n}I\right)\right|$.

Corollary 5. *Let* G *be a graph with* $m \geq 1$*. Then*

i) $\alpha_1 \geq (2nM_1 + 4m(n-2m))\left(n\sqrt{nM_1 + 2m(n-2m)}\right)^{-1}$ *and the equality holds if and only if there exist* $\alpha \geq 1$ *and* $k \geq 2$ *such that* $G \cong \alpha K_k \cup (k-2)\alpha K_1$.

ii) $\alpha_1 \geq \left(2m\rho + n\sqrt{(n-\rho)\left(\frac{nM_1+2m(n-2m)}{n} - \rho\left(\frac{2m}{n}\right)^2\right)}\right)^{-1} \big(2nM_1 + 4m(n - 2m)\big)$, *and for* $\rho = 1$ *the equality holds if and only if* G *is a complete graph or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value* $\sqrt{\frac{2m-(\frac{2m}{n})^2}{n-1}}$*; Otherwise, the equality holds if there exist* $\alpha \geq 1$ *and* $k \geq 2$ *such that* $G \cong \alpha K_k \cup (k-2)\alpha K_1$ *and* $(k-1)\alpha = \rho$.

iii) *For every graph* G *with no isolated vertex,* $\alpha_1 \geq 2$ *and the equality holds if and only if* $G \cong \frac{n}{2}K_2$.

iv) $\alpha_1 \geq \big(2nM_1 + 4m(n-2m)\big)\left(n\sqrt{\frac{nM_1+2m(n-2m)}{n}(n-1) + nd^{\frac{2}{n}}}\right)^{-1}$, *where* $d = \left|\det\left(L(G) - \frac{2m}{n}I\right)\right|$.

Proof. The proof follows from Theorems 3(ii) and 8. ∎

Corollary 6. *Let* G *be a graph with* $m \geq 1$*. Then* $\alpha_1 \geq \frac{2\sqrt{mn}}{n}$ *and the equality holds if and only if* $G \cong \frac{n}{2}K_2$.

Proof. One can observe that the function $f(x) = \frac{(2nx+4m(n-2m))}{\left(n\sqrt{nx+2m(n-2m)}\right)}$ is increasing on $(-\infty, +\infty)$. Thus, the proof follows from Corollary 5(i) and Theorem 5. ∎

Theorem 9. *Let* G *be a graph with* $m \geq 1$*. Then*

i) [2] $\alpha_1 \leq \Delta_e + 2$ *and the equality holds if and only if* G *is a bipartite regular or a bipartite semi-regular, where* G *is a connected graph.*

ii) [20] $\alpha_1 \leq n$ *with equality holds if and only if the complement* G^c *of* G *is disconnected.*

iii) [39] $\alpha_1 \leq \sqrt{2\Delta^2 + 4m - 2\delta(n-1) + 2\Delta(\delta-1)}$ *and the equality holds if and only if* G *is a bipartite regular, where* G *is a connected graph.*

iv) [39] *For every nonregular connected graph* G*,* $\alpha_1 < 2\Delta - \frac{2}{2n^2-n}$ *and* $\alpha_1 < 2\Delta - \frac{2}{(2D+1)n}$.

v) [23] $\alpha_1 \leq \frac{2m+\sqrt{m(n-2)(n(n-1)-2m)}}{n-1}$ *and the equality holds if and only if* $G \cong K_n$ *or* $K_{1,n-1}$, *where* G *is a connected graph.*

vi) [41] $\alpha_1 \leq \Delta + \sqrt{2m + \Delta(\delta - 1) - \delta(n-1)}$ *and the equality holds if and only if* G *is a bipartite regular, where* G *is a connected graph.*

vii) [31] $\alpha_1 \leq \Delta + \sqrt{2m\left(1 - \frac{1}{\omega}\right)}$.

viii) [25] *For every triangle-free graph,* $\alpha_1 \leq \Delta + \sqrt{m}$ *and* $\alpha_1 \leq \Delta + \frac{n}{2}$ *with either equality if* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Corollary 7. *Let* G *be a graph with* $m \geq 1$. *Then*

i) $L\mathcal{E}(G) \geq (2nM_1 + 4m(n-2m))\,(n(\Delta_e + 2))^{-1}$ *with equality if* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

ii) $L\mathcal{E}(G) \geq \frac{2nM_1+4m(n-2m)}{n^2}$ *with equality if* $G \cong K_n$.

iii) $L\mathcal{E}(G) \geq \left(n\sqrt{2\Delta^2 + 4m - 2\delta(n-1) + 2\Delta(\delta-1)}\right)^{-1} (2nM_1 + 4m(n - 2m))$ *with equality if* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

iv) *For every nonregular connected graph* G,

$$L\mathcal{E}(G) > (2nM_1 + 4m(n-2m))\left(n(2\Delta - \frac{2}{2n^2 - n})\right)^{-1},$$

$$L\mathcal{E}(G) > (2nM_1 + 4m(n-2m))\left(n(2\Delta - \frac{2}{(2D+1)n})\right)^{-1}.$$

v) $L\mathcal{E}(G) \geq (2nM_1 + 4m(n-2m))\left(n\frac{2m+\sqrt{m(n-2)\left(n(n-1)-2m\right)}}{n-1}\right)^{-1}$ *and the equality holds if and only if* $G \cong K_n$, *where* G *is a connected graph.*

vi) $L\mathcal{E}(G) \geq \left(n(\Delta + \sqrt{2m + \Delta(\delta-1) - \delta(n-1)})\right)^{-1} (2nM_1 + 4m(n - 2m))$ *with equality if* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

vii) $L\mathcal{E}(G) \geq \left(n(\Delta + \sqrt{2m\left(1 - \frac{1}{\omega}\right)})\right)^{-1} (2nM_1 + 4m(n - 2m)$.

viii) *For every triangle-free graph,*

$$L\mathcal{E}(G) \geq (2nM_1 + 4m(n-2m))\left(n(\Delta + \sqrt{m})\right)^{-1},$$
$$L\mathcal{E}(G) \geq (2nM_1 + 4m(n-2m))\left(n(\Delta + \frac{n}{2})\right)^{-1}$$

with either equality if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof. The proof follows from Theorems 3(ii) and 9. ■

Corollary 8. *Let G be a graph with $m \geq 1$. Then*

i) $L\mathcal{E}(G) \geq \frac{4m}{\Delta_e+2}$ *with equality if* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

ii) $L\mathcal{E}(G) \geq \frac{4m}{n}$ *with equality if* $G \cong K_n$.

iii) $L\mathcal{E}(G) \geq \frac{4m}{\sqrt{2\Delta^2+4m-2\delta(n-1)+2\Delta(\delta-1)}}$ *with equality if* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

iv) *For every nonregular connected graph* G, $L\mathcal{E}(G) > 4m\big(2\Delta - \frac{2}{2n^2-n}\big)^{-1}$ *and* $L\mathcal{E}(G) > 4m\big(2\Delta - \frac{2}{(2D+1)n}\big)^{-1}$.

v) $L\mathcal{E}(G) \geq 4m\left(\frac{2m+\sqrt{m(n-2)(n(n-1)-2m)}}{n-1}\right)^{-1}$ *and the equality holds if and only if* $G \cong K_n$, *where* G *is a connected graph.*

vi) $L\mathcal{E}(G) \geq 4m\left(\Delta + \sqrt{2m + \Delta(\delta-1) - \delta(n-1)}\right)^{-1}$ *with equality if* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

vii) $L\mathcal{E}(G) \geq 4m\left(\Delta + \sqrt{2m\left(1-\frac{1}{\omega}\right)}\right)^{-1}$.

viii) *For every triangle-free graph,* $L\mathcal{E}(G) \geq 4m\left(\Delta + \sqrt{m}\right)^{-1}$ *and* $L\mathcal{E}(G) \geq 4m\left(\Delta + \frac{n}{2}\right)^{-1}$ *with either equality if* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof. The proof follows from Corollary 7 and Theorem 5. ■

Theorem 10. *Let G be a graph with $m \geq 1$. Then*

i) [1] $Q\mathcal{E}(G) \leq 4m(1-\frac{1}{n})$ *and the equality holds if and only if* $G \cong K_2 \cup (n-2)K_1$.

ii) [14] $Q\mathcal{E}(G) \leq 2(2m+1-\Delta-\frac{2m}{n})$ *and the equality holds if and only if* $G \cong K_{1,n-1}$.

iii) [22] $Q\mathcal{E}(G) \leq \frac{2m}{n-1} + n - 2 + \sqrt{(n-2)\left(\frac{2m^2}{n-1} + \frac{8m\Delta - 4m^2}{n} + mn - 4\right)}$ *and the equality holds if and only if* $G \cong K_2$.

Corollary 9. *Let* G *be a graph with* $m \geq 1$. *Then*

i) $SQ(G) \geq (2nM_1 + 4m(n-2m))\left(4nm(1-\frac{1}{n})\right)^{-1}$ *and the equality holds if and only if* $G \cong K_2 \cup (n-2)K_1$.

ii) *Every connected graph* G *satisfies*

$$SQ(G) > (nM_1 + 2m(n-2m))\left(n(2m+1-\Delta-\frac{2m}{n})\right)^{-1}.$$

iii) $SQ(G) \geq \frac{2nM_1 + 4m(n-2m)}{n\left(\frac{2m}{n-1} + n - 2 + \sqrt{(n-2)\left(\frac{2m^2}{n-1} + \frac{8m\Delta - 4m^2}{n} + mn - 4\right)}\right)}$ *and the equality holds if and only if* $G \cong K_2$.

Proof. The proof follows from Theorems 3(iii) and 10. ∎

Corollary 10. *Let* G *be a graph with* $m \geq 1$. *Then*

i) $SQ(G) \geq \frac{n}{n-1}$ *and the equality holds if and only if* $G \cong K_2$.

ii) *Every connected graph* G *satisfies* $SQ(G) > 4m\left(2m+1-\Delta-\frac{2m}{n}\right)^{-1}$.

iii) $SQ(G) \geq \frac{4m}{\frac{2m}{n-1} + n - 2 + \sqrt{(n-2)\left(\frac{2m^2}{n-1} + \frac{8m\Delta - 4m^2}{n} + mn - 4\right)}}$ *and the equality holds if and only if* $G \cong K_2$.

Proof. The proof follows from Corollary 9 and Theorem 5. ∎

Theorem 11. *Let* G *be a graph with* $m \geq 1$. *Then*

i) [30] $SQ(G) \leq \max\{d_G(v) + m(v) : v \in V(G)\}$ *where*

$$m(v) = \sum_{uv \in E(G)} \frac{d_G(u)}{d_G(v)}.$$

The equality holds if and only if G *is a bipartite regular or a bipartite semi-regular.*

ii) [36] *For every connected graph G with $\Delta \leq n-2$, $SQ(G) \leq 2n-4$ with equality if and only if $G \cong C_4$.*

iii) [36] *For $n \geq 5$, $SQ(G) \leq 2n-4$ and the equality holds if and only if $G \cong K_{n-1} \cup K_1$.*

iv) [3] $SQ(G) \leq \sqrt{2M_1 + 4m - \frac{8m^2}{n}}$ *and* $SQ(G) \leq \Big(2m\Big(\frac{2m}{n-1} + \frac{n-2}{n-1}\Delta + (\Delta-\delta)(1-\frac{\Delta}{n-1})\Big) + 4m - \frac{8m^2}{n}\Big)^{\frac{1}{2}}$ *with either equality if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

v) [3] *For every k-regular graph G, $SQ(G) \leq \sqrt{2nk}$ and the equality holds if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

vi) [34] *For every regular graph G, $SQ(G) \leq n$ with equality holds if and only if the complement G^c of G is disconnected.*

Corollary 11. *Let G be a graph with $m \geq 1$. Then*

i) *For every connected graph G, $Q\mathcal{E}(G) \geq (2nM_1 + 4m(n-2m))(n\max\{d_G(v) + m(v) : v \in V(G)\})^{-1}$, where $m(v) = \sum_{uv\in E(G)} \frac{d_G(u)}{d_G(v)}$ with equality if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

ii) *For every connected graph G with $\Delta \leq n-2$,*

$$Q\mathcal{E}(G) \geq (nM_1 + 2m(n-2m))\,(n(n-2))^{-1}.$$

Also the equality holds if and only if $G \cong C_4$.

iii) *For $n \geq 5$, $Q\mathcal{E}(G) > (nM_1 + 2m(n-2m))\,(n(n-2))^{-1}$.*

iv) $Q\mathcal{E}(G) \geq (2nM_1 + 4m(n-2m))\left(n\sqrt{2M_1 + 4m - \frac{8m^2}{n}}\right)^{-1}$ *and*

$$Q\mathcal{E}(G) \geq \frac{2nM_1 + 4m(n-2m)}{n\sqrt{2m\left(\frac{2m}{n-1} + \frac{n-2}{n-1}\Delta + (\Delta-\delta)(1-\frac{\Delta}{n-1})\right) + 4m - \frac{8m^2}{n}}}$$

with either equality if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

v) *For every k-regular graph G, $Q\mathcal{E}(G) \geq (2nM_1 + 4m(n-2m))(n\sqrt{2nk})^{-1}$ and the equality holds if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

vi) *For every regular graph G, $Q\mathcal{E}(G) \geq \frac{2nM_1+4m(n-2m)}{n^2}$ with equality if $G \cong K_n$ or $K_{\frac{n}{2},\frac{n}{2}}$.*

Proof. The proof follows from Theorems 3(iii) and 11. ■

Corollary 12. *Let G be a graph with $m \geq 1$. Then*

i) *For every connected graph G, $Q\mathcal{E}(G) \geq 4m(\max\{d_G(v) + m(v) : v \in V(G)\})^{-1}$ where $m(v) = \sum_{uv\in E(G)} \frac{d_G(u)}{d_G(v)}$. The equality holds if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

ii) *For every connected graph G with $\Delta \leq n-2$, $Q\mathcal{E}(G) \geq \frac{2m}{n-2}$ with equality if and only if $G \cong C_4$.*

iii) *For $n \geq 5$, $Q\mathcal{E}(G) > \frac{2m}{n-2}$.*

iv) $Q\mathcal{E}(G) \geq 2\sqrt{m}$ *and* $Q\mathcal{E}(G) \geq \frac{4m}{\sqrt{2m\left(\frac{2m}{n-1}+\frac{n-2}{n-1}\Delta+(\Delta-\delta)(1-\frac{\Delta}{n-1})\right)+4m-\frac{8m^2}{n}}}$ *with either equality if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

v) *For every k-regular graph G, $Q\mathcal{E}(G) \geq \sqrt{2nk}$ and the equality holds if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

vi) *For every regular graph G, $Q\mathcal{E}(G) \geq \frac{4m}{n}$ with equality if $G \cong K_n$ or $K_{\frac{n}{2},\frac{n}{2}}$.*

Proof. The proof follows from Corollary 11 and Theorem 5. ■

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