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Relations between Energy and Sombor Index

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Abstract

Let G be an arbitrary simple graph. The energy of G is defined as the sum of absolute values of all eigenvalues of its adjacency matrix and denoted by $\mathcal{E}(G)$. Also, the Sombor index of G is defined as $SO(G) = \sum_{xy\in E(G)} \sqrt{d_x^2 + d_y^2}$, where d_x and d_y are the degree of vertices x and y in G, respectively. In this paper, we provide the upper and lower bounds for the Sombor index of G in terms of its energy. For every bipartite graph G, it was proved that $\mathcal{E}(G) \leq \sqrt{2/\delta^3(G)}SO(G)$, where δ is a minimum degree of G. We show that this result holds for any arbitrary graph. Also, we prove $\mathcal{E}(G) \leq SO(G)/(\sqrt{2}\delta(G))$, if $\delta(G) \geq 4$. Moreover, we show that $\sqrt{\mathcal{E}(G)} \geq SO(G)/\sqrt{m\Delta^3(G)}$, where Δ and m are maximum degree and size of G, respectively. Furthermore, we improve some of the stated inequalities between energy and degree based indices of graphs, like the first Zagreb index and the forgotten index, in the existing literature.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. By the *order* (*size*) of G, we mean the number of its vertices (edges). The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The *adjacency matrix* of G, denoted by $A(G)$, is an $n \times n$ matrix whose (i, j)-entry is 1 if v_i and v_j are adjacent and 0 otherwise. In this paper, the *energy* of a graph G, is shown by $\mathcal{E}(G)$ and is defined as the sum of the absolute values of its adjacency eigenvalues (see [11]). The *Sombor index*, the *first Zagreb index* and the *forgotten index* of G, are defined as follows, respectively:

$$SO(G) = \sum_{xy \in E(G)} \sqrt{d_x^2 + d_y^2};$$

$$M_1(G) = \sum_{xy \in E(G)} (d_x + d_y);$$

$$F(G) = \sum_{xy \in E(G)} (d_x^2 + d_y^2),$$

where d_x is the degree of vertex x. Some results for the mentioned indices can be found in [1], [3], [6], [7], [10], [12], [13], [14] and the references therein.

By [5], the energy of the vertex v_i of a graph G is given by

$$\mathcal{E}(v_i) = \sum_{i=1}^{n} |A(G)|_{ii} \qquad \text{for } i = 1, \dots, n,$$

where $|A| = (AA^*)^{1/2}$ and A is the adjacency matrix of G. In [20], the authors for an edge $e = xy \in E(G)$ defined $\mathcal{E}(e) = \dfrac{\mathcal{E}(x)}{d_x} + \dfrac{\mathcal{E}(y)}{d_y}$ and obtained the following:

$$\mathcal{E}(G) = \sum_{e \in E(G)} \mathcal{E}(e) = \sum_{xy \in E(G)} \left(\frac{\mathcal{E}(x)}{d_x} + \frac{\mathcal{E}(y)}{d_y} \right).$$

2 Preliminaries

We start this section by stating the following upper and lower bounds for the energy of a vertex which will be useful in proving the results of the article

Theorem 1. *[4, Pro. 3.2] For a graph G and a vertex $x \in V(G)$, we have $\mathcal{E}_G(x) \leq \sqrt{d_x}$ with equality if and only if the connected component containing v_x is isomorphic to the star graph $K_{1,n-1}$ and x is its center.*

Theorem 2. *[4, Pro. 3.3] Let G be a connected graph with at least one edge. Then $\mathcal{E}_G(x) \geq \frac{d_x}{\Delta(G)}$, for all $x \in V(G)$. Equality holds if and only if G is isomorphic to complete bipartite graph $K_{d,d}$.*

Theorem 3. *[4, Thm. 3.6] Let G be a graph with at least one edge. Then $\mathcal{E}_G(v_i) \geq \sqrt{\frac{d_i}{\Delta(G)}}$, for all $v_i \in V(G)$.*

The following lower bound for the energy of regular graphs, in terms of the order and the degree of regularity of graph, was stated in [9].

Theorem 4. *[9, Cor. 4] If G is a regular triangle and quadrangle-free graph of order n, then $\mathcal{E}(G) \geq \frac{n\Delta}{\sqrt{2\Delta-1}}$.*

Now, we prove the above result for quadrangle-free graphs.

Theorem 5. *Let G be a quadrangle-free graph of size m. Then*

$$\mathcal{E}(G) \geq \frac{2m}{\sqrt{2\Delta - 1}}.$$

Proof. We have $\mathcal{E}(G) \geq \frac{4m^2}{\sqrt{(2M_1(G)-2m)2m}}$, by [16, Page 2] and [15, Pro. 4]. Also $M_1(G) \leq 2m\Delta$. Thus

$$\mathcal{E}(G) \geq \frac{4m^2}{\sqrt{(4m\Delta - 2m)2m}} = \frac{4m^2}{2m\sqrt{2\Delta - 1}} = \frac{2m}{\sqrt{2\Delta - 1}}$$

and the result follows. ■

As a final result of this section, we state the following lemma which is used in the sequel.

Lemma 1. *Let $x, y \geq 4$ be two real numbers. Then $\sqrt{x^2 + y^2} \geq \sqrt{2x} + \sqrt{2y}$.*

Proof. Clearly, both $x(x-4)$ and $y(y-4)$ are non-negative. So $x^2-2x \geq 2x$ and $y^2 - 2y \geq 2y$. Thus $x^2 + y^2 - 2x - 2y \geq 2x + 2y \geq 2\sqrt{4xy}$ and consequently, $x^2 + y^2 \geq 2x + 2y + 2\sqrt{4xy} = (\sqrt{2x} + \sqrt{2y})^2$. This implies that $\sqrt{x^2 + y^2} \geq \sqrt{2x} + \sqrt{2y}$ and we are done. ■

3 Main results

The following upper bound for energy of an arbitrary graph G with $\delta(G) \geq 2$ was introduced in terms of the Sombor index of G.

Theorem 6. *[18, Pro. 3.4] Let G be a graph with $\delta(G) \geq 2$. Then we have $\mathcal{E}(G) \leq SO(G)$.*

Next, the authors in [19] prove the following result that gives an inequality between energy and Sombor index of a graph.

Theorem 7. *[19, Thm. 3.1] Let G be a connected graph with n vertices. If $n \geq 3$, then $\mathcal{E}(G) < SO(G)$.*

Later, in Theorem 3 of [2] this bound was improved as follows:

Theorem 8. *If G is a connected graph of order n which is not $P_n (n \leq 8)$, then $\mathcal{E}(G) \leq \dfrac{SO(G)}{2}$.*

As a first result, we provide the following upper bound for the energy of graphs in terms of Sombor index and minimum degree.

Theorem 9. *Let G be a graph and $\delta(G) \geq 4$. Then $\mathcal{E}(G) \leq \dfrac{SO(G)}{\sqrt{2}\delta(G)}$.*

Proof. Note that:

$$\begin{aligned}
\mathcal{E}(G) &= \sum_{xy \in E(G)} \left(\frac{\mathcal{E}(x)}{d_x} + \frac{\mathcal{E}(y)}{d_y} \right) \leq \sum_{xy \in E(G)} \left(\frac{\sqrt{d_x}}{d_x} + \frac{\sqrt{d_y}}{d_y} \right) \\
&= \sum_{xy \in E(G)} \left(\frac{1}{\sqrt{d_x}} + \frac{1}{\sqrt{d_y}} \right) = \sum_{xy \in E(G)} \left(\frac{\sqrt{d_x} + \sqrt{d_y}}{\sqrt{d_x}\sqrt{d_y}} \right). \\
&\leq \frac{1}{\sqrt{2}\delta(G)} \sum_{xy \in E(G)} \left(\sqrt{2d_x} + \sqrt{2d_y} \right).
\end{aligned}$$

Now, by Lemma 1 we have,

$$\mathcal{E}(G) \leq \frac{1}{\sqrt{2}\delta(G)} \sum_{xy \in E(G)} \sqrt{d_x^2 + d_y^2}.$$

Therefore, $\mathcal{E}(G) \leq \frac{SO(G)}{\sqrt{2}\delta(G)}$ and the result follows. ■

In [8] the authors proved the following result for bipartite graphs.

Theorem 10. *[8, Thm. 4] Let G be a bipartite graph. Then $\mathcal{E}(G) \leq \sqrt{\frac{2}{\delta^3(G)}}SO(G)$.*

Now, in the following theorem, we prove this bound for an arbitrary graph.

Theorem 11. *Let G be graph. Then $\mathcal{E}(G) \leq \sqrt{\frac{2}{\delta^3(G)}}SO(G)$. Moreover, the equality holds if and only if each connected component of G is isomorphic to K_2.*

Proof. By the inequality $\sqrt{a}+\sqrt{b} \leq \sqrt{2}\sqrt{a+b}$, we have:

$$\begin{aligned}
\mathcal{E}(G) &\leq \sum_{xy\in E(G)} \frac{\sqrt{d_x}+\sqrt{d_y}}{\sqrt{d_x d_y}} \\
&\leq \sqrt{2}\sum_{xy\in E(G)} \frac{\sqrt{d_x+d_y}}{\sqrt{d_x d_y}} \\
&\leq \frac{\sqrt{2}}{\delta(G)}\sum_{xy\in E(G)} \sqrt{\frac{d_x^2}{d_x}+\frac{d_y^2}{d_y}} \\
&\leq \frac{\sqrt{2}}{\delta(G)\sqrt{\delta(G)}}\sum_{xy\in E(G)} \sqrt{d_x^2+d_y^2} \\
&= \sqrt{\frac{2}{\delta^3(G)}}SO(G).
\end{aligned}$$

Also, if the equality holds, then clearly G is regular and consequently $SO(G) = \frac{nr^2}{\sqrt{2}}$, where r is the degree of regularity of graph G and $n = |V(G)|$. Thus, we have $\mathcal{E}(G) = \frac{\sqrt{2}}{r\sqrt{r}}.\frac{nr^2}{\sqrt{2}} = n\sqrt{r}$. By Theorem 1, this implies that every connected component of G is isomorphic to K_2, and the proof is complete. ■

In [18] the following result was proved for regular graphs.

Theorem 12. *[18, Thm. 3.2] Let G be a regular graph. Then $SO(G) \leq \mathcal{E}(G)\Delta^2(G)$.*

Here, we improve the bounds of Theorem 12 for an arbitrary graph.

Theorem 13. *Let G be a graph of size m. Then*

$$SO(G) \leq \mathcal{E}(G)\Delta^2(G) - \frac{m\sqrt{\Delta(G)}}{2}.$$

Proof. By Theorem 3 and considering the fact that $\sqrt{x}+\sqrt{y} \geq \sqrt{x+y}+\frac{1}{2}$ for $x, y \geq 1$, we have:

$$\begin{aligned}
\mathcal{E}(G) &= \sum_{xy\in E(G)} \left(\frac{\mathcal{E}(x)}{d_x} + \frac{\mathcal{E}(y)}{d_y}\right) \\
&\geq \frac{1}{\sqrt{\Delta(G)}} \sum_{xy\in E(G)} \left(\frac{1}{\sqrt{d_x}} + \frac{1}{\sqrt{d_y}}\right) \\
&\geq \frac{1}{\sqrt{\Delta(G)}} \sum_{xy\in E(G)} \frac{\sqrt{d_x}+\sqrt{d_y}}{\sqrt{d_x d_y}} \\
&\geq \frac{1}{\Delta(G)\sqrt{\Delta(G)}} \sum_{xy\in E(G)} \left(\sqrt{d_x+d_y} + \frac{1}{2}\right) \\
&\geq \frac{1}{\Delta(G)\sqrt{\Delta(G)}} \sum_{xy\in E(G)} \sqrt{\frac{d_x^2}{d_x} + \frac{d_y^2}{d_y}} + \frac{m}{2\Delta(G)\sqrt{\Delta(G)}}. \\
&\geq \frac{1}{\Delta^2(G)} SO(G) + \frac{m}{2\Delta(G)\sqrt{\Delta(G)}}.
\end{aligned}$$

Therefore,

$$SO(G) \leq \mathcal{E}(G)\Delta^2(G) - \frac{m\sqrt{\Delta(G)}}{2}$$

and we are done. ■

In [17] another upper bound for the Sombor index of regular graph was proven as follows:

Theorem 14. *[17, Thm. 10] Let G be a regular graph. Then $SO(G) \leq \frac{\Delta^2(G)\mathcal{E}(G)}{\sqrt{2}}$.*

In the following, we prove the bound of Theorem 14 for an arbitrary quadrangle-free graph.

Theorem 15. *Let G be an arbitrary quadrangle-free graph. Then*

$$SO(G) \leq \frac{\Delta(G)\sqrt{2\Delta(G)-1}}{\sqrt{2}}\mathcal{E}(G).$$

Proof. The proof is straightforward by applying $m \geq \frac{1}{\sqrt{2}\Delta}SO(G)$ to the inequality of Theorem 5. ■

In Theorems 2.1 and 2.2 of [20] the following result was proved.

Theorem 16. *Let G be a graph. Then*

$$\sqrt{\frac{\delta(G)}{\Delta^5(G)}}M_1(G) \le \mathcal{E}(G) \le \frac{\sqrt{\Delta(G)}}{\delta^2(G)}M_1(G);$$

$$\sqrt{\frac{\delta^3(G)}{\Delta^9(G)}}F(G) \le \mathcal{E}(G) \le \frac{\sqrt{\Delta^3(G)}}{\delta^4(G)}F(G).$$

Now, in the following two theorems we improve these bounds as follows:

Theorem 17. *Let G be a graph. Then*

$$\frac{M_1(G)}{\Delta^2(G)} \le \mathcal{E}(G) \le \frac{M_1(G)}{\delta(G)\sqrt{\delta(G)}}.$$

These inequalities become equalities for $G \cong tK_2$ and $G \cong tK_{\Delta,\Delta}$, respectively.

Proof. Note that

$$\begin{aligned} M_1(G) &= \sum_{x\in V(G)} d_x^2 = \sum_{x\in V(G)} d_x\sqrt{d_x}\sqrt{d_x} \\ &\ge \delta(G)\sqrt{\delta(G)} \sum_{x\in V(G)} \sqrt{d_x} \\ &\ge \delta(G)\sqrt{\delta(G)}\mathcal{E}(G). \end{aligned}$$

So, we get

$$\mathcal{E}(G) \le \frac{M_1(G)}{\delta(G)\sqrt{\delta(G)}}.$$

Also, according to the above inequalities and considering the fact that $\mathcal{E}(v_x) = \sqrt{d_x}$ if and only if $G \cong K_{1,n}$, equality holds if and only if $G \cong tK_2$, for some positive integer t.

On the other hand,

$$\begin{aligned} M_1(G) &= \sum_{x\in V(G)} d_x^2 = \sum_{x\in V(G)} \Delta(G) d_x \frac{d_x}{\Delta(G)} \\ &\le \Delta^2(G) \sum_{x\in V(G)} \frac{d_x}{\Delta(G)} \\ &\le \Delta^2(G)\mathcal{E}(G). \end{aligned}$$

Thus we obtain that

$$\mathcal{E}(G) \geq \frac{M_1(G)}{\Delta^2(G)}.$$

Finally, according to the above inequalities and considering the fact that $\mathcal{E}(v_x) = \dfrac{d_x}{\Delta(G)}$ if and only if $G \cong K_{d,d}$, equality holds if and only if $G \cong tK_{\Delta,\Delta}$, for some positive integer t and the proof is complete. ∎

Theorem 18. *Let G be a graph. Then*

$$\frac{F(G)}{\Delta^3(G)} \leq \mathcal{E}(G) \leq \frac{F(G)}{\delta^2(G)\sqrt{\delta(G)}}.$$

These inequalities become equalities for $G \cong tK_2$ and $G \cong tK_{\Delta,\Delta}$, respectively.

Proof. We have

$$\begin{aligned} F(G) &= \sum_{xy\in E(G)} (d_x^2 + d_y^2) = \sum_{x\in V(G)} d_x^3 \\ &= \sum_{x\in V(G)} d_x^2\sqrt{d_x}\sqrt{d_x} \geq \delta^2(G)\sqrt{\delta(G)} \sum_{x\in V(G)} \sqrt{d_x} \\ &\geq \delta^2(G)\sqrt{\delta(G)}\mathcal{E}(G). \end{aligned}$$

Therefore,

$$\mathcal{E}(G) \leq \frac{F(G)}{\delta^2(G)\sqrt{\delta(G)}}.$$

Also, according to the above inequalities and considering the fact that $\mathcal{E}(v_x) = \sqrt{d_x}$ if and only if $G \cong K_{1,n}$, equality holds if and only if $G \cong tK_2$, for some positive integer t.

On the other hand,

$$\begin{aligned} F(G) &= \sum_{xy\in E(G)} (d_x^2 + d_y^2) = \sum_{x\in V(G)} d_x^3 \\ &= \sum_{x\in V(G)} \Delta(G) d_x^2 \frac{d_x}{\Delta(G)} \leq \Delta^3(G) \sum_{x\in V(G)} \frac{d_x}{\Delta(G)} \\ &\leq \Delta^3(G)\mathcal{E}(G). \end{aligned}$$

So, we have,

$$\mathcal{E}(G) \geq \frac{F(G)}{\Delta^3(G)}.$$

Moreover, according to the above inequalities and considering the fact that $\mathcal{E}(v_x) = \dfrac{d_x}{\Delta(G)}$ if and only if $G \cong K_{d,d}$, equality holds if and only if $G \cong tK_{\Delta,\Delta}$ and the we are done. ∎

Now, we improve the stated upper bound for $M_1(G)$ in Theorem 17 for quadrangle-free graphs.

Theorem 19. *Let G be a quadrangle-free graph. Then*

$$M_1(G) \le \Delta(G)\sqrt{2\Delta(G)-1}\mathcal{E}(G).$$

Proof. The proof is straightforward by applying $m \ge \frac{1}{2\Delta}M_1(G)$ to the inequality of Theorem 5. ■

Theorem 20. *Let G be a graph of size m. Then*

$$\sqrt{\mathcal{E}(G)} \ge \frac{SO(G)}{\sqrt{m\Delta^3(G)}}.$$

Proof. By Cauchy-Schwartz's Inequality, we have

$$\begin{aligned} SO(G) &= \sum_{xy\in E(G)} \sqrt{d_x^2+d_y^2} \le \sqrt{m \sum_{xy\in E(G)} (d_x^2+d_y^2)} \\ &\le \sqrt{m \sum_{x\in V(G)} d_x^3} = \sqrt{m \sum_{x\in V(G)} \Delta(G) d_x^2 \frac{d_x}{\Delta(G)}} \\ &\le \sqrt{m\Delta^3(G)\mathcal{E}(G)}. \end{aligned}$$

So, we have

$$\sqrt{\mathcal{E}(G)} \ge \frac{SO(G)}{\sqrt{m\Delta^3(G)}}$$

and the result follows. ■

Theorem 21. *Let G be a graph of order n. Then*

$$\sqrt{\mathcal{E}(G)} \ge \frac{M_1(G)}{\sqrt{n}\Delta^2(G)}.$$

Proof. By Cauchy-Schwartz's Inequality, we have

$$\begin{aligned} M_1(G) &= \sum_{x\in V(G)} d_x^2 \\ &\le \sqrt{n \sum_{x\in V(G)} d_x^4} \\ &= \sqrt{n \sum_{x\in V(G)} \Delta(G) d_x^3 \frac{d_x}{\Delta(G)}} \\ &\le \sqrt{n\Delta^4(G)\mathcal{E}(G)}. \end{aligned}$$

Thus, we have

$$\sqrt{\mathcal{E}(G)} \geq \frac{M_1(G)}{\sqrt{n}\Delta^2(G)}$$

and the proof is complete. ■

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