On the Energy of a Graph and its Edge–Deleted Subgraphs

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Abstract

Gutman defined the energy $\mathcal{E}(G)$ of a simple graph $G$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$ as the sum of the absolute values of eigenvalues of the adjacency matrix of $G$, which has been studied extensively in mathematical chemistry. In this note, we consider the relation between the energies of $G$ and the edge-deleted subgraphs of $G$ and prove that for any positive integer $k \leq m - 1$,

\[
\binom{m-1}{k} \mathcal{E}(G) \leq \sum_{M \in \Phi_k(G)} \mathcal{E}(G - M),
\]

where $\Phi_k(G) = \{ M \subset E(G) ||M|| = k \}$. As corollary, we show that if $m \geq 2$, then

\[
\mathcal{E}(G) \leq \sqrt{2m} + \frac{4 - 2\sqrt{2}}{m - 1} p(G; 2) = 2m - \frac{4 - 2\sqrt{2}}{m - 1} \sum_{i=1}^{n} \left( \frac{d_i}{2} \right),
\]

where $p(G; 2)$ is the number of 2-matchings of $G$ and $d_i$ is the degree of $v_i$ in $G$.

1 Introduction

Let $G$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let $A(G)$ be the adjacency matrix of $G$. 
Gutman [6] defined the energy of $G$, denoted by $\mathcal{E}(G)$, as the sum of the absolute values of eigenvalues of $A(G)$, which has been studied extensively in mathematical chemistry (see for example the book [8] and survey [7]).

Let $P_s, K_s$ and $K_{1,s-1}$ denote the path, complete graph and star with $s$ vertices, respectively. Denote by $p(G;i)$ the number of $i$-matchings of $G$. We use $G \cup H$ to denote the vertex disjoint union of two graphs $G$ and $H$.

By using the singular-value inequality found by Fan [5], Day and So [3] proved that for any two Hermitian matrices $A$ and $B$ of order $n$, then

$$\mathcal{E}(A + B) \leq \mathcal{E}(A) + \mathcal{E}(B), \quad (1)$$

where $\mathcal{E}(A)$ is the sum of the absolute values of eigenvalues of $A$. By (1), they proved that for any edge $e$ of $G$, then

$$\mathcal{E}(G) \leq 2 + \mathcal{E}(G - e) \quad (2)$$

with equality if and only if $e$ is one of components of $G$, which results in the following upper bound of $\mathcal{E}(G)$ [10]:

$$\mathcal{E}(G) \leq 2m \quad (3)$$

with equality if and only if $G = mK_2 \cup (n - 2m)K_1$, which is a previously known upper bound of $\mathcal{E}(G)$ in [2]. By using repeatedly (2) to all edges of $G$, except to those edges which are incident with a vertex of the maximum-degree, then the following result can be obtained [10].

$$\mathcal{E}(G) \leq 2m - 2\Delta + 2\sqrt{\Delta}$$

with equality if and only if $G = K_{1,\Delta} \cup (m - \Delta)K_2 \cup (n - 2m + \Delta - 1)K_1$, where $\Delta$ is the maximum degree of $G$.

If $E_1$ is a cut set of of $G$, Day and So [4] proved that

$$\mathcal{E}(G) \geq \mathcal{E}(G - E_1),$$

where $G - E_1$ is the graph obtained from $G$ by deleting all edges in $E_1$.

On the other hand, Akbari, Ghorbani and Oboudi [1] proved that for
any complete multipartite graph $K_{n_1, n_2, \ldots, n_k}$, then

$$E(K_{n_1, n_2, \ldots, n_k} - e) > E(K_{n_1, n_2, \ldots, n_k})$$

for every edge $e \in E(E(K_{n_1, n_2, \ldots, n_k}))$. They also considered the change of graph energy after adding some edges and proved that for any graph $G$ with $m$ edges and every $t = 1, 2, \ldots, m - 1$, there exist at least $\left\lceil \frac{(m - 1)E(G)}{2t} \right\rceil$ $t$-subsets $S$ of $E(G)$ such that $E(G + S) > E(G)$. Particularly, for any graph $G$, there exist at least $\lceil E(G)/2 \rceil$ edges $e$ such that $E(G + e) > E(G)$.

Further results on the energy of edge-deleted subgraphs of a graph see for example [1, 9, 11, 12].

In this short note, we consider the relation between the energies of $G$ and all $k$-edge-deleted subgraphs of $G$, and some new upper bounds of $E(G)$ are obtained.

\section{Main results}

In this section, we prove mainly the following result.

\textbf{Theorem 1.} Let $G$ be a simple graph with edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, and let $\Phi_k(G)$ be the set of all subsets $M$ of $E(G)$ satisfying $|M| = k$ for some positive integer $k \leq m - 1$. Then

$$\left(\begin{array}{c} m - 1 \\ k \end{array}\right)E(G) \leq \sum_{M \in \Phi_k(G)} E(G - M),$$

with equality if each component of $G$ is $K_2$ or $K_1$.

\textbf{Proof.} Obviously, if each component of $G$ is $K_2$ or $K_1$, then the equality holds.

Note that $A(G)$ denotes the adjacency matrix of $G$. We first prove that $A(G)$ and $\{A(G - M) | M \in \Phi_k(G)\}$ satisfy the following equality:

$$\left(\begin{array}{c} m - 1 \\ k \end{array}\right)A(G) = \sum_{M \in \Phi_k(G)} A(G - M).$$
Set
\[ B = \left( \binom{m-1}{k} \right) A(G) = (b_{ij})_{n \times n}, C = \sum_{M \in \Phi_k(G)} A(G - M) = (c_{ij})_{n \times n}. \]

It is obvious that if \( v_iv_j \notin E(G) \), then \( b_{ij} = c_{ij} = 0 \). If \( v_iv_j \in E(G) \), then \( b_{ij} = \binom{m-1}{k} \). On the other hand, there exist \( \binom{m-1}{k-1} \) subsets \( M \) in \( \Phi_k(G) \) each of which contains edge \( v_iv_j \) and hence there exist \( \binom{m}{k} - \binom{m-1}{k-1} = \binom{m-1}{k} \) subgraphs \( G - M \) in \( G \) each of which contains edge \( v_iv_j \). So \( c_{ij} = \binom{m-1}{k} \), and (5) holds.

The theorem is immediate from (5) and (1).

The following corollary is immediate from Theorem 1.

**Theorem 2.** Let \( G \) be a simple graph with \( m \) edges. Then for any positive integer \( k \leq m - 1 \), there exists an \( M^* \in \Phi_k(G) \) such that

\[ \mathcal{E}(G) \leq \frac{m}{m-k} \mathcal{E}(G - M^*). \]

Particularly, there exists an edge \( e^* \in E(G) \) such that

\[ \mathcal{E}(G) \leq \frac{m}{m-1} \mathcal{E}(G - e^*). \]

**Proof.** If we set \( \mathcal{E}(G - M^*) = \max\{\mathcal{E}(G - M) | M \in \Phi_k(G)\} \), then the corollary is immediate.

**Remark.** Set \( k = m - 1 \) in Theorem 1. Then we can obtain (3).

**Corollary 1.** Let \( G \) be a simple graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(G) = \{e_1, e_2, \ldots, e_m\} \). Then

\[ \mathcal{E}(G) \leq \sqrt{2m} + \frac{4 - 2\sqrt{2}}{m-1} p(G; 2), \] (6)
which is equivalent to

$$
\mathcal{E}(G) \leq 2m - \frac{4 - 2\sqrt{2}}{m - 1} \sum_{i=1}^{n} \binom{d_i}{2},
$$

(7)

where \(d_i\) is the degree of vertex \(v_i\) in \(G\).

Proof. We set \(k = m - 2\) in Theorem 1. Then

$$
\binom{m-1}{m-2} \mathcal{E}(G) \leq \sum_{M \in \Phi_{m-2}(G)} \mathcal{E}(G - M),
$$

that is,

$$
(m - 1) \mathcal{E}(G) \leq \sum_{M \in \Phi_{m-2}(G)} \mathcal{E}(G - M).
$$

(8)

For any \(M \in \Phi_{m-2}(G)\), since \(|M| = m - 2\), \(G - M\) has two edges. So \(G - M\) is the vertex-disjoint union of a \(P_3\) and \(n - 3\) isolated vertices \((n - 3)K_1\), i.e., \(G - M = P_3 \cup (n - 3)K_1\), or the vertex-disjoint union \(2P_2\) of two 2-matchings of \(G\) and \(n - 4\) isolated vertices \((n - 4)K_1\), i.e., \(G - M = 2P_2 \cup (n - 4)K_1\). It is obvious that

$$
\mathcal{E}(P_3 \cup (n - 3)K_1) = \mathcal{E}(P_3) = 2\sqrt{2},
$$

$$
\mathcal{E}(2P_2 \cup (n - 2)K_1) = \mathcal{E}(2P_2) = 4.
$$

Note that \(\Phi_{m-2}(G)\) has \(p(G; 2)\) subgraphs each of which is isomorphic to \(2P_2 \cup (n - 4)K_1\) and \(\binom{m}{m-2} - p(G; 2)\) subgraphs each of which is isomorphic to \(P_3 \cup (n - 3)K_1\). Similarly, \(\Phi_{m-2}(G)\) has \(\sum_{i=1}^{n} \binom{d_i}{2}\) subgraphs each of which is isomorphic to \(P_3 \cup (n - 3)K_1\) and \(\binom{m}{m-2} - \sum_{i=1}^{n} \binom{d_i}{2}\) subgraphs each of which is isomorphic to \(2P_2 \cup (n - 4)K_1\). Then

$$
\sum_{M \in \Phi_{m-2}(G)} \mathcal{E}(G - M) = p(G; 2)\mathcal{E}(2P_2) + \left[\binom{m}{m-2} - p(G; 2)\right] \mathcal{E}(P_3)
$$

$$
= 4p(G; 2) + [m(m - 1) - 2p(G; 2)]\sqrt{2}
$$

$$
= (4 - 2\sqrt{2})p(G; 2) + m(m - 1)\sqrt{2}.
$$
\[ \sum_{M \in \Phi_{m-2}(G)} \mathcal{E}(G - M) = \mathcal{E}(P_3) \sum_{i=1}^{n} \binom{d_i}{2} + \left[ \binom{m}{m-2} - \sum_{i=1}^{n} \binom{d_i}{2} \right] \mathcal{E}(2P_2) \]

\[ = 2\sqrt{2} \sum_{i=1}^{n} \binom{d_i}{2} + 4 \left[ \frac{1}{2} m(m-1) - \sum_{i=1}^{n} \binom{d_i}{2} \right] \]

\[ = 2m(m-1) - (4 - 2\sqrt{2}) \sum_{i=1}^{n} \binom{d_i}{2}. \]

Then (6) and (7) are immediate from (8). The corollary has been proved.

By Corollary 1, the following result follows.

**Corollary 2.** Let \( G \) be a simple graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(G) = \{e_1, e_2, \ldots, e_m\} \). If the minimum degree \( \delta \) of \( G \) satisfies \( \delta \geq 2 \), then

\[ \mathcal{E}(G) \leq 2m - \frac{4 - 2\sqrt{2}}{m-1} n. \] (9)

**Remark.** In the theorem above, if \( k = 1 \), then

\[ (m-1)\mathcal{E}(G) \leq \sum_{e \in E(G)} \mathcal{E}(G - e). \] (10)

Set \( k = m - 3 \) in Theorem 1. Then we can obtain the following result.

**Corollary 3.** Let \( G \) be a simple graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(G) = \{e_1, e_2, \ldots, e_m\} \). Then

\[ \mathcal{E}(G) \leq \frac{4[2a + 3p(G; 3) + \sqrt{5}b + \sqrt{3} \sum_{i=1}^{n} \binom{d_i}{3} + (1 + \sqrt{2})c]}{(m-1)(m-2)}, \] (11)

where \( a = \Gamma_G(K_3), b = \Gamma_G(P_4) \) and \( c = \Gamma_G(P_3 \cup P_2) \), and \( \Gamma_G(H) \) is the number of subgraphs of \( G \) each of which is isomorphic to \( H \). Particularly, if \( G \) is a bipartite 3-regular graph, then

\[ \mathcal{E}(G) \leq \frac{4[3p(G; 3) + \sqrt{5}b + \sqrt{3}n + (1 + \sqrt{2})c]}{(m-1)(m-2)}. \] (12)
References


