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On the Energy of a Graph and its Edge–Deleted Subgraphs

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Abstract

Gutman defined the energy $\mathcal{E}(G)$ of a simple graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$ as the sum of the absolute values of eigenvalues of the adjacency matrix of G, which has been studied extensively in mathematical chemistry. In this note, we consider the relation between the energies of G and the edge-deleted subgraphs of G and prove that for any positive integer $k \leq m - 1$,

$$\binom{m-1}{k}\mathcal{E}(G) \leq \sum_{M \in \Phi_k(G)} \mathcal{E}(G - M),$$

where $\Phi_k(G) = \{M \subset E(G)||M| = k\}$. As corollary, we show that if $m \geq 2$, then

$$\mathcal{E}(G) \leq \sqrt{2}m + \frac{4 - 2\sqrt{2}}{m-1}p(G;2) = 2m - \frac{4 - 2\sqrt{2}}{m-1}\sum_{i=1}^{n}\binom{d_i}{2},$$

where $p(G;2)$ is the number of 2-matchings of G and d_i is the degree of v_i in G.

1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let $A(G)$ be the adjacency matrix of G.

Gutman [6] defined the energy of G, denoted by $\mathcal{E}(G)$, as the sum of the absolute values of eigenvalues of $A(G)$, which has been studied extensively in mathematical chemistry (see for example the book [8] and survey [7]).

Let P_s, K_s and $K_{1,s-1}$ denote the path, complete graph and star with s vertices, respectively. Denote by $p(G;i)$ the number of i-matchings of G. we use $G \cup H$ to denote the vertex disjoint union of two graphs G and H.

By using the singular-value inequality found by Fan [5], Day and So [3] proved that for any two Hermitian matrices A and B of order n, then

$$\mathcal{E}(A+B) \leq \mathcal{E}(A) + \mathcal{E}(B), \tag{1}$$

where $\mathcal{E}(A)$ is the sum of the absolute values of eigenvalues of A. By (1), they proved that for any edge e of G, then

$$\mathcal{E}(G) \leq 2 + \mathcal{E}(G-e) \tag{2}$$

with equality if and only if e is one of components of G, which results in the following upper bound of $\mathcal{E}(G)$ [10]:

$$\mathcal{E}(G) \leq 2m \tag{3}$$

with equality if and only if $G = mK_2 \cup (n-2m)K_1$, which is a previously known upper bound of $\mathcal{E}(G)$ in [2]. By using repeatedly (2) to all edges of G, except to those edges which are incident with a vertex of the maximum-degree, then the following result can be obtained [10].

$$\mathcal{E}(G) \leq 2m - 2\Delta + 2\sqrt{\Delta}$$

with equality if and only if $G = K_{1,\Delta} \cup (m-\Delta)K_2 \cup (n-2m+\Delta-1)K_1$, where Δ is the maximum degree of G.

If E_1 is a cut set of of G, Day and So [4] proved that

$$\mathcal{E}(G) \geq \mathcal{E}(G-E_1),$$

where $G - E_1$ is the graph obtained from G by deleting all edges in E_1.

On the other hand, Akbari, Ghorbani and Oboudi [1] proved that for any complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$, then

$$\mathcal{E}(K_{n_1,n_2,\ldots,n_k} - e) > \mathcal{E}(K_{n_1,n_2,\ldots,n_k})$$

for every edge $e \in E(E(K_{n_1,n_2,\ldots,n_k}))$. They also considered the change of graph energy after adding some edges and proved that for any graph G with m edges and every $t = 1, 2, \ldots, m-1$, there exist at least $\left\lceil \binom{m-1}{t-1} \frac{\mathcal{E}(G)}{2t} \right\rceil$ t-subsets S of $E(G)$ such that $\mathcal{E}(G+S) > \mathcal{E}(G)$. Particularly, for any graph G, there exist at least $\lceil \mathcal{E}(G)/2 \rceil$ edges e such that $\mathcal{E}(G+e) > \mathcal{E}(G)$.

Further results on the energy of edge-deleted subgraphs of a graph see for example [1, 9, 11, 12].

In this short note, we consider the relation between the energies of G and all k-edge-deleted subgraphs of G, and some new upper bounds of $\mathcal{E}(G)$ are obtained.

2 Main results

In this section, we prove mainly the following result.

Theorem 1. *Let G be a simple graph with edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, and let $\Phi_k(G)$ be the set of all subsets M of $E(G)$ satisfying $|M| = k$ for some positive integer $k \leq m-1$. Then*

$$\binom{m-1}{k} \mathcal{E}(G) \leq \sum_{M \in \Phi_k(G)} \mathcal{E}(G-M), \tag{4}$$

with equality if each component of G is K_2 or K_1.

Proof. Obviously, if each component of G is K_2 or K_1, then the equality holds.

Note that $A(G)$ denotes the adjacency matrix of G. We first prove that $A(G)$ and $\{A(G-M) | M \in \Phi_k(G)\}$ satisfy the following equality:

$$\binom{m-1}{k} A(G) = \sum_{M \in \Phi_k(G)} A(G-M). \tag{5}$$

Set

$$B = \binom{m-1}{k} A(G) = (b_{ij})_{n\times n}, C = \sum_{M\in\Phi_k(G)} A(G-M) = (c_{ij})_{n\times n}.$$

It is obvious that if $v_iv_j \notin E(G)$, then $b_{ij} = c_{ij} = 0$. If $v_iv_j \in E(G)$, then $b_{ij} = \binom{m-1}{k}$. On the other hand, there exist $\binom{m-1}{k-1}$ subsets M in $\Phi_k(G)$ each of which contains edge v_iv_j and hence there exist $\binom{m}{k} - \binom{m-1}{k-1} = \binom{m-1}{k}$ subgraphs $G-M$ in G each of which contains edge v_iv_j. So $c_{ij} = \binom{m-1}{k}$, and (5) holds.

The theorem is immediate from (5) and (1). ■

The following corollary is immediate from Theorem 1.

Theorem 2. *Let G be a simple graph with m edges. Then for any positive integer $k \le m-1$, then there exists an $M^* \in \Phi_k(G)$ such that*

$$\mathcal{E}(G) \le \frac{m}{m-k}\mathcal{E}(G-M^*).$$

Particularly, there exists an edge $e^* \in E(G)$ such that

$$\mathcal{E}(G) \le \frac{m}{m-1}\mathcal{E}(G-e^*).$$

Proof. If we set $\mathcal{E}(G-M^*) = \max\{\mathcal{E}(G-M)|M \in \Phi_k(G)\}$, then the corollary is immediate. ■

Remark. Set $k = m-1$ in Theorem 1. Then we can obtain (3).

Corollary 1. *Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Then*

$$\mathcal{E}(G) \le \sqrt{2}m + \frac{4-2\sqrt{2}}{m-1}p(G;2), \tag{6}$$

which is equivalent to

$$\mathcal{E}(G) \leq 2m - \frac{4-2\sqrt{2}}{m-1} \sum_{i=1}^{n} \binom{d_i}{2}, \tag{7}$$

where d_i is the degree of vertex v_i in G.

Proof. We set $k = m - 2$ in Theorem 1. Then

$$\binom{m-1}{m-2}\mathcal{E}(G) \leq \sum_{M \in \Phi_{m-2}(G)} \mathcal{E}(G-M),$$

that is,

$$(m-1)\mathcal{E}(G) \leq \sum_{M \in \Phi_{m-2}(G)} \mathcal{E}(G-M). \tag{8}$$

For any $M \in \Phi_{m-2}(G)$, since $|M| = m-2$, $G-M$ has two edges. So $G-M$ is the vertex-disjoint union of a P_3 and $n-3$ isolated vertices $(n-3)K_1$, i.e., $G-M = P_3 \cup (n-3)K_1$, or the vertex-disjoint union $2P_2$ of two 2-matchings of G and $n-4$ isolated vertices $(n-4)K_1$, i.e., $G-M = 2P_2 \cup (n-4)K_1$. It is obvious that

$$\mathcal{E}(P_3 \cup (n-3)K_1) = \mathcal{E}(P_3) = 2\sqrt{2},$$

$$\mathcal{E}(2P_2 \cup (n-2)K_1) = \mathcal{E}(2P_2) = 4.$$

Note that $\Phi_{m-2}(G)$ has $p(G;2)$ subgraphs each of which is isomorphic to $2P_2 \cup (n-4)K_1$ and $\binom{m}{m-2} - p(G;2)$ subgraphs each of which is isomorphic to $P_3 \cup (n-3)K_1$. Similarly, $\Phi_{m-2}(G)$ has $\sum_{i=1}^{n}\binom{d_i}{2}$ subgraphs each of which is isomorphic to $P_3 \cup (n-3)K_1$ and $\binom{m}{m-2} - \sum_{i=1}^{n}\binom{d_i}{2}$ subgraphs each of which is isomorphic to $2P_2 \cup (n-4)K_1$. Then

$$\begin{aligned}\sum_{M \in \Phi_{m-2}(G)} \mathcal{E}(G-M) &= p(G;2)\mathcal{E}(2P_2) + \left[\binom{m}{m-2} - p(G;2)\right]\mathcal{E}(P_3) \\ &= 4p(G;2) + [m(m-1) - 2p(G;2)]\sqrt{2} \\ &= (4-2\sqrt{2})p(G;2) + m(m-1)\sqrt{2}.\end{aligned},$$

$$\begin{aligned}\sum_{M\in\Phi_{m-2}(G)} \mathcal{E}(G-M) &= \mathcal{E}(P_3)\sum_{i=1}^{n}\tbinom{d_i}{2} + \left[\tbinom{m}{m-2} - \sum_{i=1}^{n}\tbinom{d_i}{2}\right]\mathcal{E}(2P_2) \\ &= 2\sqrt{2}\sum_{i=1}^{n}\tbinom{d_i}{2} + 4\left[\tfrac{1}{2}m(m-1) - \sum_{i=1}^{n}\tbinom{d_i}{2}\right] \qquad . \\ &= 2m(m-1) - (4-2\sqrt{2})\sum_{i=1}^{n}\tbinom{d_i}{2}.\end{aligned}$$

Then (6) and (7) are immediate from (8). The corollary has been proved.

■

By Corollary 1, the following result follows.

Corollary 2. *Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. If the minimum degree δ of G satisfies $\delta \geq 2$, then*

$$\mathcal{E}(G) \leq 2m - \frac{4-2\sqrt{2}}{m-1}n. \tag{9}$$

Remark. In the theorem above, if $k = 1$, then

$$(m-1)\mathcal{E}(G) \leq \sum_{e\in E(G)} \mathcal{E}(G-e). \tag{10}$$

Set $k = m-3$ in Theorem 1. Then we can obtain the following result.

Corollary 3. *Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Then*

$$\mathcal{E}(G) \leq \frac{4[2a + 3p(G;3) + \sqrt{5}b + \sqrt{3}\sum_{i=1}^{n}\binom{d_i}{3} + (1+\sqrt{2})c]}{(m-1)(m-2)}\,, \tag{11}$$

where $a = \Gamma_G(K_3), b = \Gamma_G(P_4)$ and $c = \Gamma_G(P_3 \cup P_2)$, and $\Gamma_G(H)$ is the number of subgraphs of G each of which is isomorphic to H. Particularly, if G is a bipartite 3-regular graph, then

$$\mathcal{E}(G) \leq \frac{4[3p(G;3) + \sqrt{5}b + \sqrt{3}n + (1+\sqrt{2})c]}{(m-1)(m-2)}\,. \tag{12}$$

References

[1] S. Akbari, E. Ghorbani, M. Oboudi, Edge addition, singular values, and energy of graphs and matrices, *Lin. Algebra Appl.* **430** (2009) 2192–2199.

[2] G. Caporossi, D. Cvetković, I. Gutman, P. Hansen, Variable neighborhood search for extremal graphs: 2. Finding graphs with extremal energy, *J. Chem. Inf. Comput. Sci.* **39** (1999) 984–996.

[3] J. Day, W. So, Singular value inequality and graph energy change, *El. J. Lin. Algebra* **16** (2007) 291–299.

[4] J. Day, W. So, Graph energy change due to edge deletion, *Lin. Algebra Appl.* **428** (2008) 2070–2078.

[5] K. Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proc. Nat. Acad. Sci. USA* **37** (1951) 760–766.

[6] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sekt. Forschungsz. Graz.* **103** (1978) 1–22.

[7] I. Gutman, B. Furtula, The total π-electron energy saga, *Croat. Chem. Acta* **90** (2017) 359–368.

[8] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[9] H. Y. Shan, C. X. He, Z. S. Yu, The energy change of the complete multipartite graph, *Lin. Algebra Appl.* **36** (2020) 309–317.

[10] W. So, M. Robbiano, N. M. M. Abreu, I. Gutman, Applications of a theorem by Ky Fan in the theory of graph energy, *Lin. Algebra Appl.* **432** (2010) 2163–2169.

[11] L. W. Tang, M. G. Lin, Q. P. Li, Graph energy change on edge deletion, *MATCH Commun. Math. Comput. Chem.* **90** (2023) 709–716.

[12] W. H. Wang, W. So, Graph energy change due to any single edge deletion, *El. J. Lin. Algebra* **29** (2015) 59–73.