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Graph Energy Change Due to Vertex Deletion

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Abstract

Let G be a graph with adjacency matrix A. The energy of G is denoted by $\mathcal{E}(G)$ and defined as the sum of the absolute values of the eigenvalues of A. In this paper we study the problem of variation of the energy when a vertex is deleted. Concretely, we show that if G is a graph and $G^{(j)}$ is the graph obtained from G by deleting the vertex v_j of G, then

$$\mathcal{E}(G) - \mathcal{E}(G^{(j)}) \leq 2\sqrt{d_j},$$

where d_j is the degree of v_j. Moreover, equality occurs if and only if the connected component of G containing v_j is isomorphic to a star tree and v_j is its center. Afterwards, we introduce a new approach to the local energy of a vertex and initiate the study of its basic properties.

1 Introduction

Let G be a graph with adjacency matrix A. The energy of G is denoted by $\mathcal{E}(G)$ and defined as the sum of the absolute values of the eigenvalues of A. We refer the reader to [9,12] for further information on graph energy and [1,2,10] for recent results.

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The problem of how the energy of a graph changes when some of its edges are deleted was first studied by Day and So [5,6], and more recently in [13]. It is our main concern in this paper to study the problem of the variation of the energy when a vertex is deleted. Concretely, based on Ky Fan's triangular inequality theorem for the trace norm, we show that if G is a graph and $G^{(j)}$ is the graph obtained from G by deleting the vertex v_j of G, then

$$\mathcal{E}(G) - \mathcal{E}(G^{(j)}) \leq 2\sqrt{d_j},$$

where d_j is the degree of v_j. Moreover, equality occurs if and only if the connected component of G containing v_j is isomorphic to a star tree and v_j is its center.

Intuitively, $\mathcal{E}(G) - \mathcal{E}(G^{(j)})$ measures the contribution of the vertex v_j to the energy of G. So in Section 3, we define naturally the local concept of energy of G at vertex v_j and initiate the study of its basic properties. This new approach should be compared to the interesting concept of energy of a vertex introduced in [3], defined as the diagonal elements of the matrix $(AA^*)^{\frac{1}{2}}$. Finally, we introduce a new energy defined as the sum of the local energies, and show that it is upper bounded by the regular energy of a graph.

2 Graph energy change due to vertex deletion

Recall that a real symmetric matrix M is positive semidefinite whenever it satisfies $x^\top Mx \geq 0$ for all $x \in \mathbb{R}^n$. This is well-known to be equivalent to the fact that all eigenvalues of M are nonnegative [11]. We will use in our arguments below the property that if the diagonal element $[M]_{kk}$ of a positive semidefinite matrix M is equal to zero, then all elements in the row k of M and all elements in the column k of M are equal to zero.

Recall that the trace norm of the matrix M, denoted by $\|M\|_*$, is the sum of its singular values. When M is real and symmetric, $\|M\|_*$ is precisely the sum of the absolute values of its eigenvalues. The following result will be crucial in our study of the variation of the energy when a vertex is deleted.

Theorem 1. *[5], [8]. Let X, Y be square matrices of the same size. Then*

$$\|X+Y\|_* \leq \|X\|_* + \|Y\|_*.$$

Equality holds if and only if there exists an orthogonal matrix P, such that PX and PY are both positive semidefinite.

Let G be a graph with set of vertices $V(G)=\{v_1,\ldots,v_n\}$ and $G^{(j)}$ the graph obtained from G by deleting the vertex v_j. Let A be the adjacency matrix of G and $A^{(j)}$ the matrix obtained from A by deleting the row and the column j. Clearly, $A^{(j)}$ is the adjacency matrix of $G^{(j)}$. Let $B^{(j)}$ be the matrix obtained from A by substituting the row and column j of A by zeroes.

Theorem 2. *Let v_j be a vertex of a graph G of degree d_j. Then*

$$0 \leq \mathcal{E}(G) - \mathcal{E}(G^{(j)}) \leq 2\sqrt{d_j}.$$

Equality in the left occurs if and only if v_j is an isolated vertex. Equality in the right occurs if and only if the connected component of G containing v_j is isomorphic to a star tree and v_j is its center.

Proof. The left inequality and equality condition is well known [12, Theorem 4.19].

Next we prove the right inequality. Note that

$$\mathcal{E}(G)-\mathcal{E}(G^{(j)}) = \|A\|_* - \|A^{(j)}\|_* = \|A\|_* - \|B^{(j)}\|_* \leq \|A-B^{(j)}\|_* = 2\sqrt{d_j}. \tag{1}$$

Suppose that $\mathcal{E}(G)-\mathcal{E}(G^{(j)}) = 2\sqrt{d_j}$. Without loosing generality, we may assume that $j=1$. By (1), $\|A\|_* - \|B^{(1)}\|_* = \|A-B^{(1)}\|_*$ which implies

$$\|(A-B^{(1)}) + B^{(1)}\|_* = \|A\|_* = \|A-B^{(1)}\|_* + \|B^{(1)}\|_*.$$

It follows from Theorem 1 that there exists an orthogonal matrix $P=(p_{ij})$

such that $P(A - B^{(1)})$ and $PB^{(1)}$ are positive semidefinite. Note that $P(A - B^{(1)})$ and $(A - B^{(1)})P$ are similar matrices since $P^\top = P^{-1}$ and

$$(A - B^{(1)})P = P^\top P(A - B^{(1)})P.$$

In particular, $(A - B^{(1)})P$ is also positive semidefinite.

Let $v_2, \ldots, v_s$ be the vertices of G that are adjacent to v_1. Then $A - B^{(1)}$ is the 2×2 block matrix

$$A - B^{(1)} = \begin{pmatrix} C & O \\ O & O \end{pmatrix},$$

where C is the $s \times s$ matrix

$$C = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and the rest are zero matrices of the adequate size. It easily follows that

$$(A - B^{(1)})P = \begin{pmatrix} X & Y \\ O & O \end{pmatrix},$$

where X is the $s \times s$ matrix

$$X = \begin{pmatrix} p_{21} + \cdots + p_{s1} & p_{22} + \cdots + p_{s2} & \cdots & p_{2s} + \cdots + p_{ss} \\ p_{11} & p_{12} & \cdots & p_{1s} \\ p_{11} & p_{12} & \cdots & p_{1s} \\ \vdots & \vdots & \vdots & \vdots \\ p_{11} & p_{12} & \cdots & p_{1s} \end{pmatrix},$$

and Y is the $s \times (n - s)$ matrix

$$Y = \begin{pmatrix} p_{2,s+1}+\cdots+p_{s,s+1} & p_{2,s+2}+\cdots+p_{s,s+2} & \cdots & p_{2,n}+\cdots+p_{s,n} \\ p_{1,s+1} & p_{1,s+2} & \cdots & p_{1,n} \\ p_{1,s+1} & p_{1,s+2} & \cdots & p_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{1,s+1} & p_{1,s+2} & \cdots & p_{1,n} \end{pmatrix}.$$

Since $(A - B^{(1)})P$ is positive semidefinite and the diagonal elements

$$\left[(A - B^{(1)})P\right]_{kk} = 0$$

for all $s+1 \le k \le n$, we deduce that $Y = 0$. Moreover, the diagonal elements $p_{12}, p_{13}, \ldots, p_{1s}$ of X are all strictly positive, otherwise we would have a zero row in the orthogonal matrix P, a contradiction.

On the other hand,

$$\left[PB^{(1)}\right]_{11} = \sum_{k=1}^{n} p_{1k}\left[B^{(1)}\right]_{k1} = 0.$$

Since $PB^{(1)}$ is positive semidefinite and $p_{1k} = 0$ for all $s+1 \le k \le n$, it follows that

$$0 = \left[PB^{(1)}\right]_{1j} = \sum_{k=1}^{n} p_{1k}\left[B^{(1)}\right]_{kj} = p_{12}a_{2j} + p_{13}a_{3j} + \cdots + p_{1s}a_{sj}$$

for all $2 \le j \le n$. But $p_{1k} > 0$ for all $2 \le k \le s$ implies that $a_{kj} = 0$ for all $2 \le k \le s$ and $2 \le j \le n$. In other words,

$$A = \begin{pmatrix} C & O \\ O & * \end{pmatrix}.$$

This clearly implies that the connected component containing v_1 is a star tree and v_1 is its center.

Conversely, assume that G is isomorphic to the direct sum of a star tree S with center vertex v_1 and a graph L. Then clearly

$$\mathcal{E}(G) - \mathcal{E}(G^{(1)}) = \mathcal{E}(S) = 2\sqrt{d_1}. \qquad \blacksquare$$

3 Local energy of a graph at a vertex

We keep the notation introduced in the previous section. Intuitively, $\mathcal{E}(G) - \mathcal{E}(G^{(j)})$ measures the contribution of the vertex v_j to the energy of G.

Definition 1. Let G be a graph with set of vertices $\{v_1, \ldots, v_n\}$. We define the local energy of G at vertex v_j as

$$\mathcal{E}_G(v_j) = \mathcal{E}(G) - \mathcal{E}(G^{(j)}). \tag{2}$$

Directly from Theorem 2 we deduce the following result.

Corollary 1. *Let G be a graph and $v \in V(G)$ with degree d_v. Then*

$$0 \leq \mathcal{E}_G(v) \leq 2\sqrt{d_v}.$$

Moreover, equality in the left inequality occurs if and only if v is an isolated vertex. Equality in the right inequality occurs if and only if v is the center of a star.

Example 1. Let us compute the local energy for some special graphs.

1. Let $G = K_n$, the complete graph with n vertices. We know that $\mathcal{E}(K_n) = 2\,(n-1)$ and $\mathcal{E}(K_n^{(j)}) = 2\,(n-2)$, for all vertex v_j of K_n. Hence
$$\mathcal{E}_{K_n}(v_j) = 2\,(n-1) - 2\,(n-2) = 2,$$
for all j.

2. Let $G = C_n$ be the cycle on n vertices. Then

$$\mathcal{E}(C_n) = \begin{cases} 4\cot\frac{\pi}{n} & if\ n \equiv 0 \pmod 4 \\ 4\csc\frac{\pi}{n} & if\ n \equiv 2 \pmod 4 \\ 2\csc\frac{\pi}{2n} & if\ n \equiv 1 \pmod 2 \end{cases},$$

and for each vertex v_j in C_n

$$\mathcal{E}(C_n^{(j)}) = \mathcal{E}(P_{n-1}) = \begin{cases} 2\cot\frac{\pi}{2n} - 2 & if\ n \equiv 0 \pmod 2 \\ 2\csc\frac{\pi}{2n} - 2 & if\ n \equiv 1 \pmod 2 \end{cases}.$$

Hence,

$$\begin{aligned} \mathcal{E}_{C_n}(v_j) &= \mathcal{E}(C_n) - \mathcal{E}(C_n^{(j)}) \\ &= \begin{cases} 2 - 2\cot\frac{\pi}{2n} + 4\cot\frac{\pi}{n} & if\ n \equiv 0 \pmod 4 \\ 2 - 2\cot\frac{\pi}{2n} + 4\csc\frac{\pi}{n} & if\ n \equiv 2 \pmod 4 \\ 2 & if\ n \equiv 1 \pmod 2 \end{cases}. \end{aligned}$$

Note that the complete graph and the cycle are regular graphs for which the local energy is constant for every vertex. This is not always the case for general regular graphs.

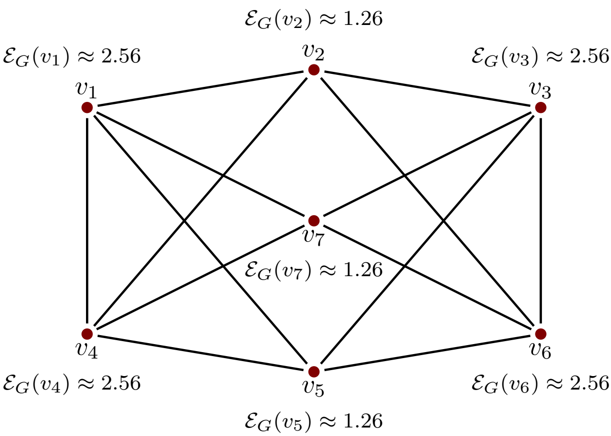

Figure 1. A regular graph with different local energies.

Example 2. Consider the graph G depicted in Figure 1. G is a 4-regular graph, however, the local energy is not constant at every vertex.

The following problem naturally arises.

Problem 1. *Assume that the local energy of a graph is constant at each vertex of the graph. Is the graph regular?*

Next we consider the local energy of the complete bipartite graph $K_{p,q}$.

Example 3. Let $G = K_{p,q}$, the complete bipartite graph with $p + q$ vertices. Assume that P is the set of p vertices and Q is the set of q vertices of G. Then $\mathcal{E}(K_{p,q}) = 2\sqrt{pq}$ and for $v_i \in P$ and $v_j \in Q$,

$$\mathcal{E}(K_{p,q}^{(i)}) = 2\sqrt{(p-1)q},$$

and

$$\mathcal{E}(K_{p,q}^{(j)}) = 2\sqrt{p(q-1)}.$$

Consequently,

$$\mathcal{E}_{K_{p,q}}(v_i) = 2\sqrt{pq} - 2\sqrt{(p-1)q},$$

and

$$\mathcal{E}_{K_{p,q}}(v_j) = 2\sqrt{pq} - 2\sqrt{p(q-1)}.$$

In particular, in the star tree S_n, the local energy at the center vertex v_i is

$$\mathcal{E}_{S_n}(v_i) = 2\sqrt{n-1},$$

and for any leaf v_j

$$\mathcal{E}_{S_n}(v_j) = 2\sqrt{n-1} - 2\sqrt{n-2}.$$

From the fact that the function $g\left(x\right) = \frac{x}{x-1}$ is nonincreasing in the interval $[2, +\infty]$, we easily deduce that if $q \geq p \geq 1$ then

$$\mathcal{E}_{K_{p,q}}(v_i) \geq \mathcal{E}_{K_{p,q}}(v_j),$$

for all $v_i \in P$ and $v_j \in Q$. In other words, if $u, v \in K_{p,q}$ and $d_u \geq d_v$, then $\mathcal{E}_{K_{p,q}}(u) \geq \mathcal{E}_{K_{p,q}}(v)$. This is not always the case for general graphs, as we can see in our next example.

Example 4. Consider the graph G depicted in Figure 2. Although $d_{v_1} < d_{v_2}$, we have that $\mathcal{E}_G(v_1) \approx 2.02 > 1.34 \approx \mathcal{E}_G(v_2)$.

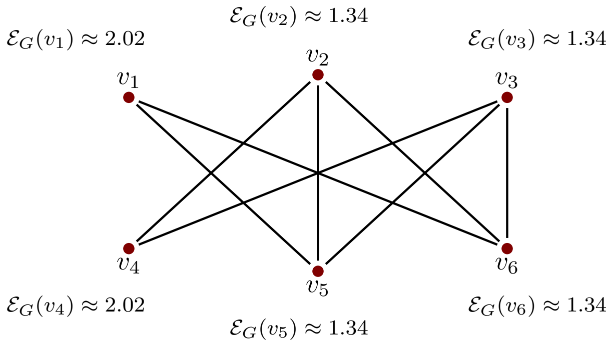

Figure 2. Local energy is not increasing with respect to the degree of the vertices.

Recall that if G is a bipartite graph with n vertices, then

$$\phi_G(x) = \sum_{k \geq 0} (-1)^k m(G,k) x^{n-2k}, \tag{3}$$

where $m(G,k)$ is the number of k-matchings of G (that is, the number of selection of k independent edges of G). If G_1 and G_2 are bipartite graphs such that $m(G_1,k) \geq m(G_2,k)$ holds for all $k \geq 0$, then we write $G_1 \succeq G_2$. If $m(G_1,k) > m(G_2,k)$ for at least one k, then we write $G_1 \succ G_2$. It is well known [9] that

$$G_1 \succ G_2 \Rightarrow \mathcal{E}(G_1) > \mathcal{E}(G_2). \tag{4}$$

Proposition 3. *Let v_i and v_j be vertices of the bipartite graph G. If $G^{(i)} \succ G^{(j)}$ then $\mathcal{E}_G(v_j) > \mathcal{E}_G(v_i)$.*

Proof. By (4), if $G^{(i)} \succ G^{(j)}$ then $\mathcal{E}(G^{(i)}) > \mathcal{E}(G^{(j)})$. Now by Definition

1,

$$\mathcal{E}_G(v_j) - \mathcal{E}_G(v_i) = \mathcal{E}(G^{(i)}) - \mathcal{E}(G^{(j)}) > 0.$$

Hence, $\mathcal{E}_G(v_j) > \mathcal{E}_G(v_i)$. ■

Example 5. Let P_n be the path on n vertices. For all $i = 1, \ldots, n$,

$$P_n^{(i)} = P_{i-1} \cup P_{n-i},$$

where P_0 is the empty graph. Let $n = 4k + s$, where $s \in \{0, 1, 2, 3\}$. By [12, Lemma 4.6],

$$P_n^{(1)} \succ P_n^{(3)} \succ \cdots \succ P_n^{(2k+1)} \succ P_n^{(2k)} \succ P_n^{(2k-2)} \succ \cdots \succ P_n^{(4)} \succ P_n^{(2)}.$$

It follows from Proposition 3 that

$$\begin{aligned}\mathcal{E}_{P_n}(v_1) < \mathcal{E}_{P_n}(v_3) < \cdots < \mathcal{E}_{P_n}(v_{2k+1}) < \mathcal{E}_{P_n}(v_{2k}) \\ < \mathcal{E}_{P_n}(v_{2k-2}) < \cdots < \mathcal{E}_{P_n}(v_4) < \mathcal{E}_{P_n}(v_2).\end{aligned}$$

We end this section with the following natural question: if H is a subgraph of the graph G and v is a vertex of H, is it true that $\mathcal{E}_H(v) \leq \mathcal{E}_G(v)$?

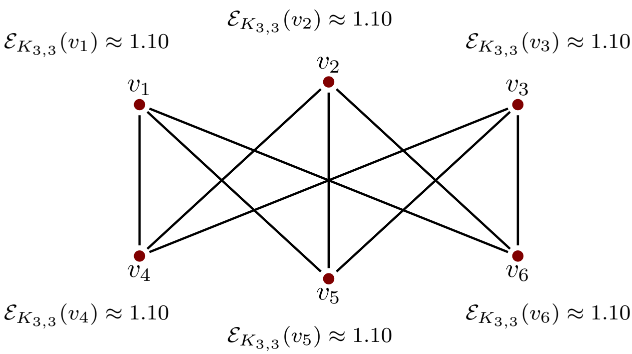

Figure 3. Complete bipartite graph $K_{3,3}$.

Example 6. Let H be the graph depicted in Figure 2. Note that this is a generator subgraph of the complete bipartite graph $K_{3,3}$ (shown in Figure

3), since it is obtained by adding the edge v_1v_4. However, $\mathcal{E}_H(v_1) \approx 2.02 > 1.10 \approx \mathcal{E}_{K_{3,3}}(v_1)$. Consequently, adding an edge to a vertex does not necessarily increase the local energy at the vertex.

On the other hand, consider the graph G depicted in Figure 4 and the cycle C_4. Clearly, C_4 is an induced subgraph of G. However, $\mathcal{E}_{C_4}(v_2) \approx 1.18 > 1.12 \approx \mathcal{E}_G(v_2)$.

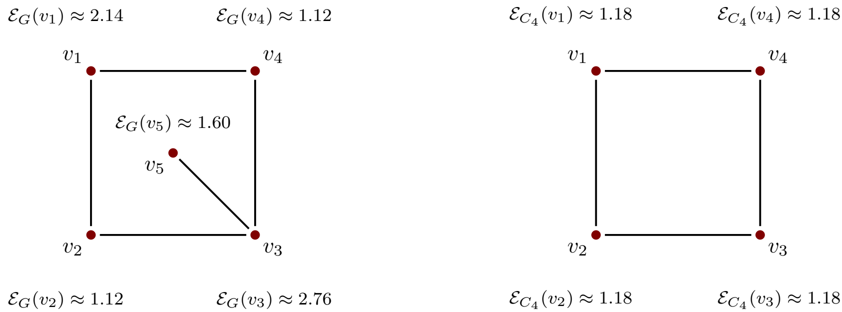

Figure 4. The cycle C_4 as an induced subgraph of G.

4 Local energy of a graph

We introduce now a new type of graph energy.

Definition 2. Let G be a graph. The local energy of G is defined as

$$e(G) = \sum_{v \in V(G)} \mathcal{E}_G(v).$$

Example 7. Let us compute the local energy of some special graphs.

1. Let K_n be the complete graph on n vertices. By Example 1 item 1,

$$e(K_n) = \sum_{v \in V(K_n)} \mathcal{E}_{K_n}(v) = 2n.$$

2. Let C_n be the cycle on n vertices. By Example 1 item 2,

$$e(C_n) = \sum_{v \in V(C_n)} \mathcal{E}_{C_n}(v)$$
$$= \begin{cases} 2n - 2n\cot\frac{\pi}{2n} + 4n\cot\frac{\pi}{n} & if\ n \equiv 0 \pmod 4 \\ 2n - 2n\cot\frac{\pi}{2n} + 4n\csc\frac{\pi}{n} & if\ n \equiv 2 \pmod 4 \\ 2n & if\ n \equiv 1 \pmod 2 \end{cases}.$$

3. Let $K_{p,q}$ be the complete bipartite graph. It follows from Example 3,

$$\begin{aligned} e(K_{p,q}) &= \sum_{v \in V(K_{p,q})} \mathcal{E}_{K_{p,q}}(v) \\ &= \sum_{v \in P} \mathcal{E}_{K_{p,q}}(v) + \sum_{v \in Q} \mathcal{E}_{K_{p,q}}(v) \\ &= 2p\left(\sqrt{pq} - \sqrt{(p-1)q}\right) + 2q\left(\sqrt{pq} - \sqrt{p(q-1)}\right) \\ &= 2(p+q)\sqrt{pq} - 2p\sqrt{(p-1)q} - 2q\sqrt{p(q-1)}. \end{aligned}$$

In particular,

$$e(S_n) = 2n\sqrt{n-1} - 2(n-1)\sqrt{n-2}.$$

We have the following bounds on the local energy of a graph.

Theorem 4. *Let G be a graph. Then,*

$$0 \leq e(G) \leq 2\sum_{v \in V(G)} \sqrt{d_v}. \tag{5}$$

Equality in the left inequality occurs if and only if G is totally disconnected. Equality in the right inequality occurs if and only if G is a direct sum of copies of K_2 plus some isolated vertices.

Proof. The left inequality (and equality condition) follows directly from

Corollary 1. To see the right inequality, note that again by Corollary 1,

$$e(G) = \sum_{v \in V(G)} \mathcal{E}_G(v) \leq \sum_{v \in V(G)} 2\sqrt{d_v}. \tag{6}$$

If

$$e(G) = \sum_{v \in V(G)} 2\sqrt{d_v},$$

then by the inequality (6) we deduce that

$$\sum_{v \in V(G)} \mathcal{E}_G(v) = \sum_{v \in V(G)} 2\sqrt{d_v},$$

and so by Corollary 1, $\mathcal{E}_G(v) = 2\sqrt{d_v}$ for all $v \in V(G)$, which implies that every vertex in G is the center of a star tree. This is equivalent to say that all connected components of G are K_2 or isolated vertices.

Conversely, if G is a direct sum of r copies of K_2 plus s isolated vertices, where r, s are nonnegative integers, then it is clear that

$$e(G) = 4r = \sum_{v \in V(G)} 2\sqrt{d_v}. \qquad \blacksquare$$

We can improve the upper bound given in Theorem 4 for the local energy of a graph using the regular energy of a graph.

Theorem 5. *Let G be a graph. Then*

$$e(G) \leq 2\,\mathcal{E}(G) \leq 2 \sum_{v \in V(G)} \sqrt{d_v}. \tag{7}$$

Proof. We first show the left inequality in (7). Let A be the adjacency matrix of a graph G on n vertices and $A^{(j)}$ the matrix obtained from A by deleting the row and the column j. Let $B^{(j)}$ be the matrix obtained from A by substituting the row and column j of A by zeroes. Observe that

$$(n-2)\,A = \sum_{j=1}^{n} B^{(j)}, \tag{8}$$

and $\|A^{(j)}\|_* = \|B^{(j)}\|_*$, for all $j = 1, \dots, n$. Hence,

$$(n-2)\,\|A\|_* = \|(n-2)\,A\|_* = \left\|\sum_{j=1}^{n} B^{(j)}\right\|_* \leq \sum_{j=1}^{n} \left\|B^{(j)}\right\|_* = \sum_{j=1}^{n} \left\|A^{(j)}\right\|_* . \tag{9}$$

In other words,

$$(n-2)\,\mathcal{E}(G) \leq \sum_{j=1}^{n} \mathcal{E}(G^{(j)}),$$

or equivalently,

$$e(G) = \sum_{j=1}^{n} \Big(\mathcal{E}(G) - \mathcal{E}(G^{(j)})\Big) \leq 2\,\mathcal{E}(G).$$

The right inequality in (7) was shown in [3, Theorem 3.1]. ■

Remark. The equality holds in the right inequality of (7) if and only if G is a direct sum of copies of K_2 plus some isolated vertices. This is a consequence of [3, Proposition 3.2], since $\mathcal{E}(G) = \sum_{v \in V(G)} \sqrt{d_v}$ if and only if every vertex of G is the center of a star.

On the other hand, it is clear that if G is a direct sum of r copies of K_2 plus s isolated vertices, then $e(G) = 4r = 2\,\mathcal{E}(G)$. So for these graphs, equality holds in the left inequality of (7).

An interesting problem to solve is the following.

Problem 2. *Characterize graphs G such that $e(G) = 2\,\mathcal{E}(G)$.*

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