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On Orderenergetic Graphs

Rakshith B. R.[a], **Kinkar Chandra Das**[b,*]

[a]*Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal 576 104, India*

[b]*Department of Mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea*

E-mail: ranmsc08@yahoo.co.in, rakshith.br@manipal.edu, kinkardas2003@gmail.com

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Abstract

Let G be a simple graph of order n. The eigenvalue of a graph G is the eigenvalue of its adjacency matrix. The energy $\mathcal{E}(G)$ of G is the sum of absolute values of its eigenvalues. A graph G of order n is orderenergetic if $\mathcal{E}(G) = n$. The algebraic multiplicity of the number zero in the spectrum of G is referred to as its nullity, and is denoted by η. In this paper, we show that if the cycle C_4 is not an induced subgraph of a graph G with nullity $\eta = 3$, then G is not orderenergetic. We also obtain some results connecting orderenergetic graphs and minimum degree. Finally, we show that there is a connected orderenergetic graph on $10k + 8$ vertices for all $k \geq 0$.

1 Introduction

All graphs considered here are simple and undirected. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| =$

*Corresponding author

n and $|E(G)| = m$. The degree of a vertex v_i is denoted by d_i and the minimum degree of G is denoted by δ. The adjacency matrix of the graph G with vertex set $V(G)$ is the $n \times n$ symmetric matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i v_j \in E(G)$, and $a_{ij} = 0$, otherwise. The spectrum of $A(G)$ is called the spectrum of the graph G. The algebraic multiplicity of the number zero in the spectrum of a graph is referred to as its nullity, and is denoted by η. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the adjacency matrix $A(G)$ of graph G. The energy of graph G is denoted by $\mathcal{E}(G)$ and is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_1|.$$

The definition of graph energy is motivated from chemistry and was first introduced by Gutman in 1978, see [16]. Studies on graph energy have been intensified in last few years. For properties and studies on the energy of graphs we refer to the book entitled "Graph Energy" by Li, Shi and Gutman [20], and especially the most recent works [1,8–11]. A graph G is borderenergtic if $\mathcal{E}(G) = 2(n-1)$. The definition was first put forwarded in [14]. Plently of research work on borderenergetic graphs have been presented in [12–14, 19, 24]. As usual, we denote by C_n, nG, $K_{n_1,n_2,\dots,n_k}$ the cycle of length n and n copies of G, and the complete multipartite graph of order $n_1 + n_2 + \cdots + n_k$. The complement graph of G is denoted by $\overline{G}$.

A graph is said to be orderenergetic if its energy is equal to its order, that is, if $\mathcal{E}(G) = n$. This concept was introduced in [2] for the first time. It is shown in [2] that there are infinitely many connected orderenergetic graphs. If $n = 4$, then the only connected orderenergetic graph is the cycle C_4 and the complete bipartite graph $K_{p,p}$ is orderenergetic for all $p \geq 1$. It is well-known that the energy of graph is never an odd integer, see [6]. Hence orderenergetic graphs must have even number of vertices. In [2], using a computer-aided search all orderenergetic connected graphs up to 10 vertices is presented. Among all non-singular connected graphs only P_2 is orderenergetic and also, there is no orderenergetic graph with nullity $\eta = 1$, see [2]. Several open problems and conjectures are pointed out in [2]. For example, we have the following:

Conjecture 1. [2] *For a given non-negative integer k, there are finitely many connected orderenergetic graphs with nullity k.*

Problem 1. [2] *Find a method for constructing connected orderenergetic graphs, not using the direct product.*

In [2], it was shown that Conjecture 1 is true for $k = 1$ and in [4] it was proved for $k = 2$. Motivated by this, in this paper, we show that if the cycle C_4 is not an induced subgraph of a graph G with nullity $\eta = 3$, then G is not orderenergetic. Again motivated by Problem 1, we show that there are connected orderenergetic graphs on 10k+8 vertices for all $k \geq 0$. Some basic results on connected orderenergetic graphs of order $2n$ with minimum degree $\delta = n$ or $n - 1$ are obtained.

2 Main results

The following lemma is due to Gutman [15].

Lemma 1. [15] *Let G be a graph with n vertices and m edges, possessing q quadrangles, and let d_1, d_2,...,d_n be its vertex degrees. Then*

$$\mathcal{E}(G) \geq 2m\sqrt{\frac{2m}{2\sum_{i=1}^{n} d_i^2 - 2m + 8q}}.$$

The following lemma gives a lower bound for the energy of a graph in terms of energies of vertex disjoint induced subgraphs.

Lemma 2. [3] *Let G be a graph and $H_1, H_2, \ldots, H_k$ be its k vertex disjoint induced subgraphs. Then*

$$\mathcal{E}(G) \geq \sum_{i=1}^{k} \mathcal{E}(H_i).$$

Bapat and pati [6] proved the following result.

Lemma 3. [6] *There are no connected orderenergetic graphs of odd order.*

A $\{1,2\}$-factor of a graph G is a spanning subgraph of G whose each component is an edge or a cycle.

Lemma 4. [2] *Let G be graph of order n. If G has a $\{1,2\}$-factor, then $\mathcal{E}(G) \geq n$. Equality holds if and only if G is the disjoint union of balanced complete bipartite graphs.*

The following result is obtained from [2] (see, Lemma 10 and the proof of Lemma 11).

Lemma 5. [5] *For any odd integer $n \geq 3$, $\mathcal{E}(C_n) \geq n+1$ with equality if and only if $n = 3$.*

The adjacency polynomial $P(G, x)$ of G is defined as

$$P(G,\, x) = \det(xI_n - A(G)) = \sum_{i=0}^{n} a_i\, x^{n-i}. \tag{1}$$

Lemma 6. [7,17] *Let G be a graph with order n and adjacency polynomial $P(G,x) = \sum\limits_{i=0}^{n} a_i\, x^{n-i}$. Then*

$$a_i = \sum_{S \in L_i} (-1)^{p(S)}\, 2^{c(S)}, \tag{2}$$

where L_i denotes the set of Sachs graphs of G with i vertices, that is, the graphs in which every component is either a K_2 or a cycle, $p(S)$ is the number of components of S and $c(S)$ is the number of cycles contained in S. In addition $a_0 = 1$.

Lemma 7. *Let G be a graph such that the cycle C_4 is not an induced subgraph of G. Let x be a vertex adjacent to at least two vertices in the induced cycle $C : v_1v_2 \dots v_{2p+1}$ of odd length $2p+1$ $(p \geq 2)$. Then $C' \cup \mathcal{M}$ is a Sachs subgraph of order $2p+1$ such that $(V(C) \cup \{v\}) \setminus V(C' \cup \mathcal{M}) = \{v_k\}$ for some $1 \leq k \leq 2p+1$ and v_k is adjacent to an edge component of $\mathcal{M}$, where $\mathcal{M}$ is a maximum matching of the induced path $C \setminus V(C')$ and C' is an induced cycle of odd length.*

Proof. Without loss of generality, we can assume that vertex x is adjacent to s (≥ 2) vertices in cycle C of odd length $2p+1$, where $V(C) =$

$\{v_1, v_2, \ldots, v_{2p+1}\}$. Let $Q = \{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\} \subseteq V(C)$ be the set of vertices adjacent to vertex x and $1 \leq i_1 < i_2 < \cdots < i_s \leq 2p+1$.
Case 1: Suppose $v_{i_j}v_{i_{j+1}} \in E(C)$ for some $1 \leq j \leq s-1$ or $v_{i_s}v_{i_1} \in E(C)$. Then $C' : xv_{i_j}v_{i_{j+1}}x$ $(1 \leq j \leq s-1)$ or $C' : xv_{i_1}v_{i_s}x$ is an odd cycle of length 3. Since $|V(C)| = 2p+1 \geq 5$, the induced graph $C\backslash V(C')$ is a path of even length. Let $\mathcal{M}$ be a maximum matching of the induced path $C\backslash V(C')$. Then the graph $C' \cup \mathcal{M}$ is a Sachs subgraph of order $2p+1$ such that the vertex $v_k \in V(C)\,(v_k \notin V(C' \cup \mathcal{M}))$ is adjacent to an edge component of $\mathcal{M}$.
Case 2: Suppose $v_{i_j}v_{i_{j+1}} \notin E(C)$ for all j, $1 \leq j \leq s-1$ and $v_{i_s}v_{i_1} \notin E(C)$. Then there exists an integer j $(1 \leq j \leq s-1)$ such that the induced cycle $C' : xv_{i_j} \ldots v_{i_{j+1}}x$ or $C' : xv_{i_1} \ldots v_{i_s}x$ is of odd length. (Otherwise, $2p+1 = \sum\limits_{j=1}^{s-1} d(v_{i_j}, v_{i_{j+1}}) + d(v_{i_s}, v_{i_1})$ = even number as each term is even, a contradiction). Now, the induced graph $C\backslash V(C')$ is either an isolated vertex or a path of even length. Suppose the induced graph $C\backslash V(C')$ is an isolated vertex v_ℓ, (say). Then $xv_{\ell-1}v_\ell v_{\ell+1}x$ is the cycle C_4 in G, a contradiction as C_4 is not an induced subgraph of G. Thus $C\backslash V(C')$ is a path of even length. Let $\mathcal{M}$ be a maximum matching of the induced path $C\backslash V(C')$. Then the graph $C' \cup \mathcal{M}$ is a Sachs subgraph of order $2p+1$ such that the vertex $v_k \in V(C)\,(v_k \notin V(C' \cup \mathcal{M}))$ is adjacent to an edge component of $\mathcal{M}$. ■

Let G be a graph with induced cycle C. Also let v be a vertex in G such that $v \notin V(C)$. We define

$$\Omega(C, v) = |\{v_j \in V(C) : vv_j \in E(G)\}|.$$

Theorem 2. *If the cycle C_4 is not an induced subgraph of a connected graph G with nullity $\eta = 3$, then G is not orderenergetic.*

Proof. We prove this result by contradiction. For this we assume that there exists an orderenergetic connected graph G of order n with nullity $\eta = 3$. Since G is connected and $\eta = 3$, we have $n \geq 5$. By Lemma 3, n must be even and hence $n \geq 6$. Since $\eta = 3$, by (1), the coefficient a_{n-3}

of $P(G, x)$ is non-zero. Therefore, by (2), we obtain

$$0 \neq a_{n-3} = \sum_{S \in L_{n-3}} (-1)^{p(S)}\, 2^{c(S)}.$$

Then there exists a Sachs subgraph of G of order $n-3$. Without loss of generality, we can assume that H (subgraph of G) is a Sachs graph of order $n-3$ such that all the components of H are edges and induced odd cycles of G (as $p\,K_2$ is a subgraph of cycle C_{2p} of order $2p$). Since n is even, $n-3$ is odd and so H contains at least one odd cycle C^1, (say). Since $|V(H)| = n-3$, there are exactly three vertices x, y and z of G which are not in the vertex set of H. Let $F = \{x, y, z\}$ and $G[F]$ denotes the subgraph of G induced by F.

Case 1. $E(G[F]) \neq \emptyset$. First we assume that $|E(G[F])| \geq 2$. In this case $G[F] \cong P_3$ or C_3. Then $\mathcal{E}(G[F]) > 2$. Therefore by Lemma 2, we obtain

$$\mathcal{E}(G) \geq \mathcal{E}(H\backslash C^1) + \mathcal{E}(C^1) + \mathcal{E}(G[F]) > |V(H\backslash C^1)| + (|V(C^1)|+1) + 2 = n,$$

a contradiction.

Next we assume that $|E(G[F])| = 1$. Without loss of generality, we can assume that $xy \in E(G)$. Let $R = \{x, y\}$ and we have $\mathcal{E}(G[R]) = 2$. Since G is connected, vertex z is adjacent to at least one vertex in H. If z is adjacent with some edge component $e = uw$ of H, then it is easy to check that $\mathcal{E}(G[K]) > 2$, where $K = \{u, w, z\}$. By Lemma 5, $\mathcal{E}(C^1) \geq |V(C^1)| + 1$. Thus from Lemma 2, we get

$$\begin{aligned}
\mathcal{E}(G) &\geq \mathcal{E}(H\backslash\{V(C^1) \cup \{u, w\}\}) + \mathcal{E}(C^1) + \mathcal{E}(G[K]) + \mathcal{E}(G[R]) \\
&> |V(H\backslash\{V(C^1) \cup \{u, w\}\})| + |V(C^1)| + 1 + 2 + 2 = n,
\end{aligned}$$

a contradiction. Otherwise, $zv_j \in E(G)$, where $v_j \in V(C^1)$. Since the cycle C^1 is of odd length, the path graph $C^1\backslash\{z, v_j\}$ is of even order and so it has a perfect matching $\mathcal{M}$, (say). We define a graph H^1 as follows:

$$V(H^1) = V(H) \cup \{x, y, z\},\ \ E(H^1) = E(H\backslash C^1) \cup \{x, y\} \cup \mathcal{M} \cup \{z, v_j\}.$$

Thus the graph H^1 is a Sachs subgraph of G of order n. Therefore by Lemma 2, we obtain

$$\mathcal{E}(G) \geq \mathcal{E}(H^1) \geq |V(H \backslash C^1)| + 2 + |\mathcal{M}| + 2 = n.$$

By Lemma 4, $\mathcal{E}(G) = n$ if and only if G is a balanced complete bipartite graph (that is, $\eta = n - 2$ and n is even), a contradiction to the hypothesis that $\eta = 3$ and $n \geq 6$. Hence $\mathcal{E}(G) > n$, a contradiction.

Case 2. $E(G[F]) = \emptyset$. Let t be the number of odd cycles in H. First we assume that $t > 1$. Since $|V(H)|\,(= n - 3)$ is odd, $t \geq 3$. If any one cycle length is at least 5 or $t > 3$, then by Lemmas 2 and 5, we obtain

$$\mathcal{E}(G) \geq \mathcal{E}(H) > n,$$

a contradiction. Otherwise, $t = 3$ and each cycle length is 3. Since G is connected, each vertex in F is adjacent to some vertex in H. For vertex x is adjacent to the edge component in H or adjacent to a vertex in some cycle of H. For both cases, again by Lemma 2, one can easily see that

$$\mathcal{E}(G) > n,$$

a contradiction as $\mathcal{E}(P_3) > 2.82$, $\mathcal{E}(\overline{P_3 \cup K_1}) > 4$, $\mathcal{E}(\overline{K_2 \cup 2\,K_1}) > 4$ and $\mathcal{E}(K_4) > 4$.

Next we assume that $t = 1$. Then $H \cong C^1 \cup p\,K_2$, where C^1 is a induced cycle of odd length of graph G and $2p = n - 3 - |V(C^1)|$. Moreover, we have $|V(G)| = |V(H)| + |F|$. Without loss of generality, we can assume that $\Omega(C^1, x) \geq \Omega(C^1, y) \geq \Omega(C^1, z)$. We consider the following cases:

Case 2.1. $\Omega(C^1, x) \leq 1$. We divided the following cases:

Case 2.1.1. $\Omega(C^1, x) = \Omega(C^1, y) = \Omega(C^1, z) = 0$. In this case the vertices x, y and z must be adjacent with the vertices of some of the edge components of H as shown in Fig. 1. Let L be any one of the graphs G_1, G_2, G_3, G_4, or G_5 as depicted in Fig. 1. Also let $L^\star$ be the graph obtained from L by considering the induced subgraph of each components of L in G. By Sage [23], one can easily check that $\mathcal{E}(L^\star) \geq |V(L^*)| - 1$ with equality if and only if $L^\star \cong K_{1,4}$. By Lemma 5, we obtain $\mathcal{E}(C^1) \geq |V(C^1)| + 1$ with equality if and only if $C^1 \cong C_3$.

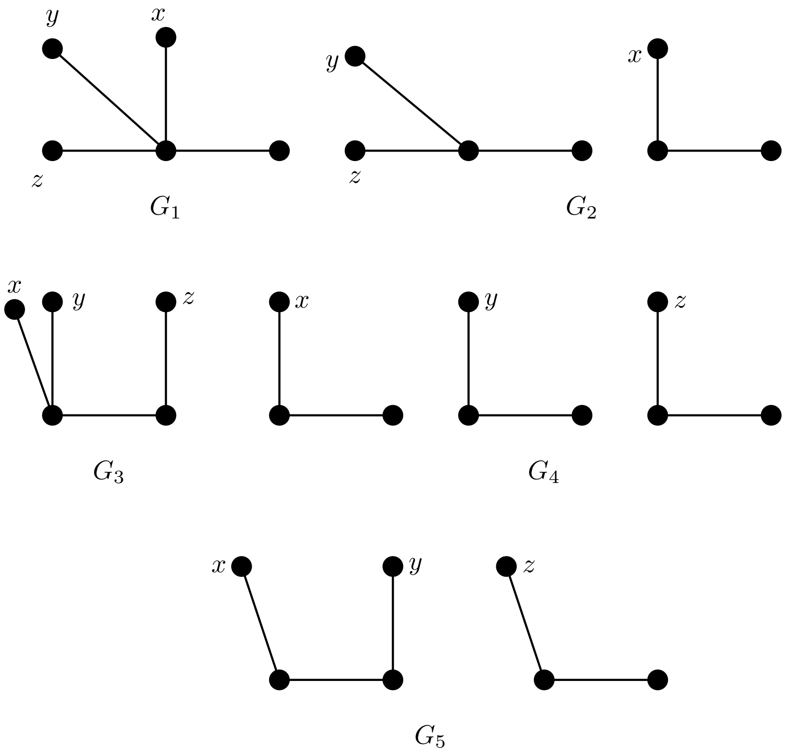

Figure 1. Graphs G_1, G_2, G_3, G_4 and G_5.

First we assume that $C^1 \ncong C_3$. Then by using the above results with Lemma 2, we obtain

$$\begin{aligned}\mathcal{E}(G) &\geq \mathcal{E}(H\backslash \left\{C^1 \cup L^\star \backslash \{x,y,z\}\right\}) + \mathcal{E}(C^1) + \mathcal{E}(L^\star)\\ &> |V(H\backslash \left\{C^1 \cup L^\star \backslash \{x,y,z\}\right\})| + |V(C^1)| + 1 + |V(L^*)| - 1 = n,\end{aligned}$$

a contradiction.

Next we assume that $C^1 \cong C_3$. If $L^\star \ncong K_{1,4}$, then by using the above results with Lemma 2, we obtain

$$
\begin{aligned}
\mathcal{E}(G) &\geq \mathcal{E}(H\backslash \left\{C^1 \cup L^\star \backslash \{x,y,z\}\right\}) + \mathcal{E}(C^1) + \mathcal{E}(L^\star) \\
&> |V(H\backslash \left\{C^1 \cup L^\star \backslash \{x,y,z\}\right\})| + |V(C^1)| + 1 + |V(L^*)| - 1 = n,
\end{aligned}
$$

a contradiction. Otherwise, $L^\star \cong K_{1,4}$. Since G is connected, a vertex of the odd cycle C^1 is either adjacent with an end vertex of an edge component of H or it is adjacent to a vertex of $L^\star$. Let H_1 be the subgraph induced by the vertices of C^1 and an edge component e of H. Then it is not difficult to see that $\mathcal{E}(H_1) > |V(H_1)| + 1$. Thus by Lemma 2, we have

$$
\begin{aligned}
\mathcal{E}(G) &\geq \mathcal{E}(H\backslash \{H_1 \cup L^\star \backslash \{x,y,z\}\}) + \mathcal{E}(H_1) + \mathcal{E}(L^\star) \\
&> |V(H\backslash \{H_1 \cup L^\star \backslash \{x,y,z\}\})| + |V(H_1)| + 1 + |V(L^\star)| - 1 = n,
\end{aligned}
$$

a contradiction. Let H_2 be the subgraph induced by the vertices of C^1 and $L^\star$. By Sage [23], one can easily check that $\mathcal{E}(H_2) > |V(H_2)|$. Thus by Lemma 2, we have

$$
\begin{aligned}
\mathcal{E}(G) &\geq \mathcal{E}(H\backslash H_2) + \mathcal{E}(H_2) \\
&> |V(H\backslash H_2)| + |V(H_2)| = n,
\end{aligned}
$$

a contradiction.

Case 2.1.2. $\Omega(C^1, x) = \Omega(C^1, y) = \Omega(C^1, z) = 1$. In this case the vertices x, y and z must be adjacent with the vertices of C^1 as shown in Fig. 2. Let $C^\star$ be any one of the graphs G_6, G_7 or G_8 as depicted in Fig. 2.

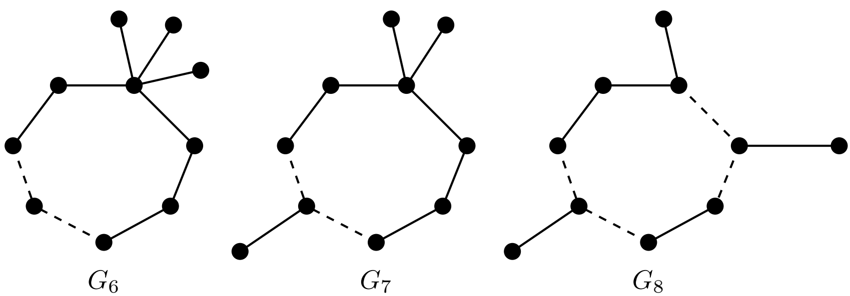

Figure 2. Graphs G_6, G_7 and G_8.

If $C^\star$ is the graph G_6 as depicted in Fig. 2, then by Lemma 1, it is easy to see that $\mathcal{E}(C^\star) > |V(C^\star)|$ for $|V(C^1)| \geq 11$. For $3 \leq |V(C^1)| \leq 9$, using Sage [23], we see that $\mathcal{E}(C^\star) > |V(C^\star)|$. Thus by Lemma 2, we obtain

$$\mathcal{E}(G) \geq \mathcal{E}(H\backslash C^1) + \mathcal{E}(C^\star) > |V(H\backslash C^1)| + |V(C^\star)| = n,$$

a contradiction. Similarly, if $C^\star$ is isomorphic to the graph G_7 or G_8 as depicted in Fig. 2, then by using Sage [23] and Lemma 2, it can be shown that $\mathcal{E}(G) > n$, a contradiction.

Case 2.1.3. $\Omega(C^1, x) = \Omega(C^1, y) = 1$, $\Omega(C^1, z) = 0$. In this case two vertices x and y are adjacent to the vertices of the cycle C^1 and the vertex z is adjacent to an edge component e of H. Let $C^\star$ be the subgraph induced by the vertices of the cycle C^1 and the vertices x and y of G. Then $C^\star \cong G_9$ or $C^\star \cong G_{10}$ (G_9 and G_{10} are shown in Fig 3).

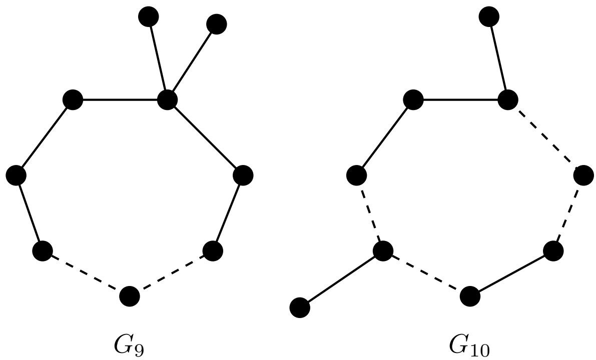

Figure 3. Graphs G_9 and G_{10}.

First we assume that $C^\star \cong G_9$. If $|V(C^\star)| < 8$, then by Sage [23], we obtain $\mathcal{E}(C^\star) > |V(C^\star)| + 0.2$. Otherwise, by Lemma 1, we have $\mathcal{E}(C^\star) > |V(C^\star)| + 0.2$. Thus if $L^\star$ is the subgraph induced by the vertices of e and the vertex z, then $\mathcal{E}(L^\star) > |V(L^\star)| - 0.2$ and hence by Lemma 2, we obtain

$$\begin{aligned}\mathcal{E}(G) &\geq \mathcal{E}(H\backslash\{C^1 \cup e\}) + \mathcal{E}(C^\star) + \mathcal{E}(L^\star)\\ &> |V(H\backslash C^1)| + |V(C^\star)| + 0.2 + |V(L^\star)| - 0.2 = n,\end{aligned}$$

a contradiction. Next we assume that $C^\star \cong G_{10}$. Similarly, it can be shown that $\mathcal{E}(G) > n$.

Case 2.1.4. $\Omega(C^1, x) = 1$, $\Omega(C^1, y) = \Omega(C^1, z) = 0$. In this case only one vertex x is adjacent to the exactly one vertex of C^1 and vertices y & z are adjacent to edge components of H. Let $C^\star$ be the subgraph induced by the vertices of the cycle C^1 and the vertex x of graph G. If $|V(C^\star)| < 7$, then by Sage [23], we obtain $\mathcal{E}(C^\star) > |V(C^\star)| + 0.6$. Otherwise, by Lemma 1, we get $\mathcal{E}(C^\star) > |V(C^\star)| + 0.6$. Now, we assume that y and z are adjacent to the vertices of the edge components of H. Let L be any one of the graphs G_{11}, G_{12} or G_{13} as depicted in Fig. 4.

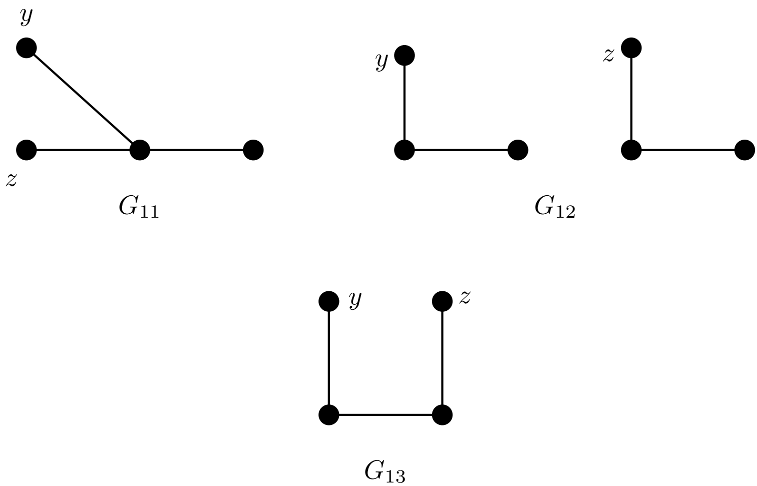

Figure 4. Graphs mentioned in the proof of Theorem 2.

Let $L^\star$ be the subgraph induced by the vertices of L of graph G. By Sage [23], we get $\mathcal{E}(L^\star) > |V(L^\star)| - 0.6$. Then by Lemma 2, we obtain

$$\begin{aligned}\mathcal{E}(G) &\geq \mathcal{E}(H \backslash \{C^1 \cup L^\star \backslash \{y,z\}\}) + \mathcal{E}(C^\star) + \mathcal{E}(L^\star)\\ &> |V(H \backslash \{C^1 \cup L^\star \backslash \{y,z\}\})| + |V(C^\star)| + 0.6 + |V(L^\star)| - 0.6 = n,\end{aligned}$$

a contradiction.

Case 2.2. $\Omega(C^1, x) \geq 2$. In this case vertex x is adjacent to at least two vertices in the induced cycle C^1 of graph G. Then by Lemma 7, we have $H^1 \cong (H \backslash C^1) \cup C^2 \cup \mathcal{M}_1$ is a Sachs subgraph of order $n-3$ of graph G such that $v_k \notin V(H^1)$ ($v_k \in V(C^1)$) is adjacent to an edge component of H^1, where $\mathcal{M}_1$ is a maximum matching of the induced path $C^1 \backslash V(C^2)$ and C^2 is an induced cycle of odd length of graph G.

Without loss of generality, we can assume that $\Omega(C^2, y) \geq \Omega(C^2, z)$. We consider the following two cases:

Case 2.2.1. $\Omega(C^2, y) \leq 1$. In this case we have the following three cases:

Case 2.2.1.1. $\Omega(C^2, y) = \Omega(C^2, z) = 0$. The proof of this case is similar to the **Case 2.1.1**.

Case 2.2.1.2. $\Omega(C^2, y) = \Omega(C^2, z) = 1$. The proof of this case is similar to the **Case 2.1.3**.

Case 2.2.1.3. $\Omega(C^2, y) = 1, \Omega(C^2, z) = 0$. The proof of this case is similar to the **Case 2.1.4**.

Case 2.2.2. $\Omega(C^2, y) \geq 2$. In this case vertex y is adjacent to at least two vertices in the induced cycle C^2 of graph G. Then by Lemma 7, we have $H^2 \cong (H^1 \backslash C^2) \cup C^3 \cup \mathcal{M}_2$ is a Sachs subgraph of order $n-3$ of graph G such that $v_\ell \notin V(H^2)$ ($v_\ell \in V(C^2)$) is adjacent to an edge component of H^2, where $\mathcal{M}_2$ is a maximum matching of the induced path $C^2 \backslash V(C^3)$ and C^3 is an induced cycle of odd length of graph G. We consider the following cases:

Case 2.2.2.1. $\Omega(C^3, z) = 0$. The proof of this case is similar to the **Case 2.1.1**.

Case 2.2.2.2. $\Omega(C^3, z) = 1$. The proof of this case is similar to the **Case 2.1.4**.

Case 2.2.2.3. $\Omega(C^3, z) \geq 2$. In this case vertex z is adjacent to at least two vertices in the induced cycle C^3 of graph G. Then by Lemma 7, we have $H^3 \cong (H^2 \backslash C^3) \cup C^4 \cup \mathcal{M}_3$ is a Sachs subgraph of order $n-3$ of graph G such that $v_s \notin V(H^3)$ ($v_s \in V(C^3)$) is adjacent to an edge component of H^3, where $\mathcal{M}_3$ is a maximum matching of the induced path $C^3 \backslash V(C^4)$ and C^4 is an induced cycle of odd length of graph G. Then the proof of this case is similar to the **Case 2.1.1**. This completes the proof of the theorem. ■

Lemma 8. [22] *A graph has exactly one positive eigenvalue if and only if its non isolated vertices form a complete multipartite graph.*

Lemma 9. [21] *Let G be a connected graph with minimum degree $\delta(G)$. Then $\mathcal{E}(G) \geq 2\delta(G)$ and the equality holds if and only if G is a complete multipartite graph with equal size of parts.*

Lemma 10. [7] *Let G be a connected graph of order n and size m. Then $\lambda_1 \geq 2m/n$. Equality holds if and only if G is a regular graph.*

Lemma 11. [18] *Let G be an r-regular graph ($r > 0$) of order n. Then $\mathcal{E}(G) \geq n$. Equality is attained if and only if every component of G is isomorphic to the complete bipartite graph $K_{r,r}$.*

Theorem 3. *If a positive integer is an eigenvalue of a connected graph G of order $2n$ and minimum degree $\delta = n - 1$, then G is not orderenergetic.*

Proof. Suppose G is orderenergetic. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_0}$ be the positive eigenvalues of $A(G)$. Then it is well-known that

$$\mathcal{E}(G) = 2(\lambda_1 + \lambda_2 + \cdots + \lambda_{n_0}). \tag{3}$$

If G is regular, then by Lemma 11, $G \cong K_{n-1,n-1}$, a contradiction to the order of the graph G. Otherwise, G is a non-regular graph. By Lemma 10, $\lambda_1 \geq n - 1$ and the equality holds if and only if G is regular. Thus $\lambda_1 > n - 1$. From (3), we obtain

$$\mathcal{E}(G) > 2(n-1) + 2(\lambda_2 + \lambda_3 + \ldots + \lambda_{n_0}).$$

Since a positive integer is an eigenvalue of G, we must have $\lambda_1 = n$, $\lambda_2 = \lambda_3 = \ldots = \lambda_{n_0} = 0$. Therefore G has only one positive eigenvalue. Hence by Lemma 8, G must be a complete multipartite graph. Since the minimum degree of G is $n-1$, $K_{n+1,n-1}$ is a subgraph of G and so $|E(G)| \geq n^2 - 1$. If $|E(G)| \geq n^2$, then by Lemma 10, $\lambda_1 > n$, a contradiction. Otherwise,

$|E(G)| \leq n^2 - 1$ and hence $|E(G)| = n^2 - 1$. Thus $G \cong K_{n-1,n+1}$. Therefore, $\mathcal{E}(G) = 2\sqrt{n^2 - 1}$. Hence G is not orderenergetic. This completes the proof. ∎

The following corollary is immediate from the above theorem.

Corollary. *There is no orderenergetic connected integral graph G of order $2n$ and minimum degree $n - 1$.*

Theorem 4. *Let G be an orderenergetic connected graph of order $2n$ and minimum degree n. Then $G \cong K_{n,n}$.*

Proof. Let G be an orderenergetic connected graph of order $2n$ and minimum degree n. Then $\mathcal{E}(G) = 2n$ and hence by Lemma 9, G must be a regular complete multipartite graph. Let $G \cong K_{t,\,t,\dots,\,t}$. Then $\lambda_1 = 2n - t$ and so $\mathcal{E}(G) = 2\lambda_1 = 2(2n - t)$. Since $\mathcal{E}(G) = 2n$, we get $2(2n - t) = 2n$. Thus $n = t$. Therefore $G \cong K_{n,n}$. ∎

The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$ is obtained by joining every vertex of G with the vertices of H.

Lemma 12. [7] *If G_1 is an r_1 regular with n_1 vertices and G_2 is r_2 regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \vee G_2$ is given by*

$$P(G_1 \vee G_2, x) = \frac{P(G_1, x)\, P(G_2, x)}{(x - r_1)\,(x - r_2)} \Big((x - r_1)(x - r_2) - n_1 n_2\Big).$$

Theorem 5. *The graph $aK_{p,p} \vee \overline{K}_{2p(4a-1)}$ is orderenergetic.*

Proof. From Lemma 12, it follows that the spectrum of $aK_{p,p} \vee \overline{K}_{2p(4a-1)}$ is

$$\Big\{4ap, \underbrace{p,\ p, \dots,\ p}_{a-1}, \underbrace{-p,\ -p, \dots,\ -p}_{a}, \underbrace{0,\ 0, \dots,\ 0}_{2p(4a-1)+2a(p-1)-1}, -4ap + p\Big\}.$$

Thus $\mathcal{E}(K_{a,a} \vee \overline{K_{6a}}) = 2p(5a - 1)$. Hence $aK_{p,p} \vee \overline{K}_{2p(4a-2)}$ is orderenergetic. ∎

Letting $p = 1$ in Theorem 5, we obtain the following corollary.

Corollary. *There exists orderenergetic graphs of order* $10k+8$ *for all* $k \geq 0$

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