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Unified Extremal Results for (Exponential) Bond Incident Degree Indices of Trees

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Abstract

A bond incident degree (BID) index $TI(G)$ of a connected graph G with edge-weight function $I(x,y)$ is defined as

$$TI(G) = \sum_{uv\in E(G)} I(d_G(u), d_G(v)),$$

where $I(x,y) > 0$ is a symmetric real function with $x \geq 1$ and $y \geq 1$, $d_G(u)$ is the degree of vertex u in G.

In this paper, we use a unified method to characterize the first two maximum and the first two minimum trees with respect to (exponential) BID indices, respectively. As corollaries, we deduce a number of previously established results, and state a few new. The results extend some results of Zeng et al. (2021) and Yang et al. (2023).

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1 Introduction

1.1 Background

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)| = n$ and $|E(G)| = m$ be the number of vertices and edges in G, respectively. The connected graph is called a tree if $n - m + 1 = 0$. Let $d_G(u)$ be the degree of vertex u in G. The vertex with $d_G(u) \geq 3$ is called the branching vertex of G. In this paper, all notations and terminologies used but not defined can refer to Bondy and Murty [4].

Let $I(x, y) > 0$ be a symmetric real function with $x \geq 1$ and $y \geq 1$. The bond incident degree (BID) index $TI(G)$ of a connected graph G with edge-weight function $I(x, y)$ was defined as [3, 20]

$$TI(G) = \sum_{uv \in E(G)} I(d_G(u), d_G(v)). \tag{1}$$

Let $f(x, y) > 0$ be a symmetric real function with $x \geq 1$ and $y \geq 1$. To improve the discrimination of topological indices, Rada [19] proposed the concept of exponential bond incident degree (BID) indices $e^{TI}(G)$ of G with edge-weight function $f(x, y)$, which was defined as

$$e^{TI}(G) = \sum_{uv \in E(G)} e^{f(d_u, d_v)}. \tag{2}$$

For the recent research about the exponential bond incident degree indices, one can see [5, 9, 13, 21, 26].

It is well known that in the study of topological indices or other invariants, there is basically not much difference in the research methods of many different topological indices/invariants. Therefore, it is natural and necessary to use a unified approach to study the topological indices/invariants. After doing so, we do not need to study the indices/invariants one by one. This not only saves a lot of time, but also promotes the development of related research fields. This work is meaningful. Here, we have to mention Professor Xueliang Li's outstanding contribution in this regard. In 2005, Li et al [16] used a unified method to consider different chemical indices. Li et al [15] also used a unified method to determine the extremal spectral radius of trees. Further, Zheng et al [28] consider the extremal spectral radius of trees and unicyclic graphs by a unified method. Yao et al. [23] found a unified method to identify the extremal graphs for a set of topological or spectral invariants by using the majorization theorem. It is worth mentioning that there is a review article [14] written by Li et al.

In this paper, we investigate the properties of BID indices. Gao [10] determined the trees with maximum BID indices. Cruz et al. [7] determined the trees with maximum and minimum (exponential) BID indices. Cruz et al. [6] and Gao [11] considered the extremal chemical trees with respect to BID indices. Cruz et al. [8] determined the extremal unicyclic graphs with respect to BID indices. Ali et al. [1] determined maximum values of the some BID indices among tree, unicyclic, bicyclic, tricyclic and tetracyclic graphs. Zhou et al. [29] considered the BID indices of connected graphs with fixed order and number of pendent vertices. For other related researches, one can see [2, 17, 18, 24, 25].

Let $\mathcal{T}_n$ be the set of trees with n vertices. An induced path with vertices $u_1, u_2, \cdots, u_\ell$ of G is called a pendent path of G, if $d_G(u_2) = d_G(u_3) = \cdots = d_G(u_{\ell-1}) = 2$ and $d_G(u_\ell) = 1$ (there is no requirement on $d_G(u_1)$). Denote by S_n, P_n the star and path with n vertices, respectively. Denote by $P_{n,3}$ the graph obtained from path P_{n-1} by attaching one pendent vertex to the vertex adjacent to the pendent vertex of P_{n-1}. Let $S_{a,n-a-2}$ be the graph obtained from edge uv by attaching a pendent vertices to vertex u and attaching $n-a-2$ pendent vertices to vertex v. We call it a double star tree and suppose that $1 \le a \le \lfloor \frac{n-2}{2} \rfloor$. Then $a + 2 \le n - a$.

Let $I(x, y) > 0$ be a symmetric real function with $x \ge 1$ and $y \ge 1$. If $I(x, y)$ is monotonically increasing on x (or y), $I(a - x, x)$ is monotonically decreasing on x ($x \in [1, \lfloor \frac{a}{2} \rfloor]$) for $a \ge 2$, and $I(x + c, y) - I(x, y)$ is monotonically increasing on x (or y) for $c \ge 0$, then we call $I(x, y)$ has the property P. For convenience, we say the BID index with property P if its edge-weight function $I(x, y)$ has the property P.

1.2 Main results

The main results of this paper are as follows.

Theorem 1.1. *Let $n \geq 5$ and $T \in \mathcal{T}_n$. Then for any BID index has the property P, we have*

$$TI(T) \leq (n-1)I(1, n-1),$$

with equality if and only if $T \cong S_n$.

Theorem 1.2. *Let $n \geq 5$ and $T \in \mathcal{T}_n \backslash \{S_n\}$. Then for any BID index has the property P, we have*

$$TI(T) \leq I(1,2) + I(2, n-2) + (n-3)I(1, n-2),$$

with equality if and only if $T \cong S_{1,n-3}$.

Theorem 1.3. *Let $n \geq 5$ and $T \in \mathcal{T}_n$. Then for any BID index has the property P, we have*

$$TI(T) \geq 2I(1,2) + (n-3)I(2,2),$$

with equality if and only if $T \cong P_n$.

Theorem 1.4. *Let $n \geq 5$ and $T \in \mathcal{T}_n \backslash \{P_n\}$. Then for any BID index has the property P, we have*

$$TI(T) \geq I(1,2) + 2I(1,3) + I(2,3) + (n-5)I(2,2),$$

with equality if and only if $T \cong P_{n,3}$.

1.3 Preliminaries

In this section, we give some preliminary lemmas.

Lemma 1.1. *Let $n \geq 5$ and $T \in \mathcal{T}_n$. If T has the maximum BID index with property P, then the distance between a vertex with maximum degree and a pendent vertex in T is at most two.*

Proof. On the contrary, we suppose that u is a vertex with maximum degree Δ, i.e. $d_T(u) = \Delta$. Let $uv \in E(T)$, $vw \in E(T)$, $d_T(w) = t \geq 2$, $d_T(v) = s \geq 2$ and $u \neq w$. Let $N_T(u) = \{v, u_1, u_2, \cdots, u_{\Delta-1}\}$ and

$d_T(u_i) = x_i$ for $i = 1, 2, \cdots, \Delta - 1$. Let $N_T(w) = \{v, w_1, w_2, \cdots, w_{t-1}\}$ and $d_T(w_j) = y_j$ for $j = 1, 2, \cdots, t-1$. Let $T^* = T - \{ww_j | j = 1, 2, \cdots, t-1\} + \{uw_j | j = 1, 2, \cdots, t-1\}$. Bearing in mind that the BID index has property P, then we have

$$\begin{aligned}
&TI(T^*) - TI(T) \\
&= \sum_{i=1}^{\Delta-1} (I(x_i, \Delta + t - 1) - I(x_i, \Delta)) + \sum_{j=1}^{t-1} (I(y_j, \Delta + t - 1) - I(y_j, t)) \\
&\quad + I(\Delta + t - 1, s) + I(1, s) - I(\Delta, s) - I(t, s) \\
&> (I(\Delta + t - 1, s) - I(\Delta, s)) - (I(t, s) - I(1, s)) \\
&> 0,
\end{aligned}$$

which is a contradiction with the maximality of BID index of T. Thus the distance between a vertex with maximum degree and a pendent vertex is at most two. ■

Lemma 1.2. *Let $n \geq 5$ and $T \in \mathcal{T}_n$, $uv \in E(T)$, $d_T(u) = \Delta$, $N_T(v) = \{u, v_1, v_2, \cdots, v_{t-1}\}$ $(2 \leq t \leq \Delta)$, where $d_T(v_i) = 1$ for $i = 1, 2, \cdots, t - 1$. Let $T^* = T - \{vv_i | i = 1, 2, \cdots, t - 1\} + \{uv_i | i = 1, 2, \cdots, t - 1\}$. Then for any BID index has the property P, we have $TI(T^*) > TI(T)$.*

Proof. Note that the BID index has property P. Then

$$\begin{aligned}
&TI(T^*) - TI(T) \\
&> tI(\Delta + t - 1, 1) - I(\Delta, t) - (t - 1)I(t, 1) \\
&= (t - 1)(I(\Delta + t - 1, 1) - I(t, 1)) + (I(\Delta + t - 1, 1) - I(\Delta, t)) \\
&> 0.
\end{aligned}$$

■

Lemma 1.3. *Let $n \geq 5$ and $T \in \mathcal{T}_n$. If T has minimum BID index with property P, then $\Delta \leq 2$.*

Proof. On the contrary, we suppose that $\Delta \geq 3$ in T. Let $uv \in E(T)$, $d_T(u) = t \geq 3$, $d_T(v) = s$ and $P = uu_1u_2 \cdots u_\ell$ be a pendent path, where $u_1 \neq v$ and $d_T(u_\ell) = 1$. Let $N_T(u) = \{v, u_1, z_1, z_2, \cdots, z_{t-2}\}$ and

$d_T(z_i) = x_i$ for $i = 1, 2, \cdots, t-2$. Let $T^* = T - \{uz_i | i = 1, 2, \cdots, t-2\} + \{u_\ell z_i | i = 1, 2, \cdots, t-2\}$.

Case 1. $\ell = 1$.

Bearing in mind that the BID index has property P and $t \geq 3$, we have

$$
\begin{aligned}
&TI(T^*) - TI(T) \\
&= \left(\sum_{i=1}^{t-2} I(x_i, t-1) + I(t-1, 2) + I(2, s)\right) - \left(\sum_{i=1}^{t-2} I(x_i, t) + I(t, 1) + I(t, s)\right) \\
&= \sum_{i=1}^{t-2}(I(x_i, t-1) - I(x_i, t)) + (I(t-1, 2) - I(t, 1)) + (I(2, s) - I(t, s)) \\
&< 0,
\end{aligned}
$$

which is a contradiction with the minimality of BID index of T.

Case 2. $\ell \geq 2$.

Notice that the BID index has property P and $t \geq 3$. Then

$$I(t, 1) - I(t, 2) < I(2, 1) - I(2, 2).$$

Combining with the conclusion of Case 1, we have

$$
\begin{aligned}
&TI(T^*) - TI(T) \\
&= \left(\sum_{i=1}^{t-2} I(x_i, t-1) + I(t-1, 2) + I(2, s) + I(2, 2)\right) \\
&\quad - \left(\sum_{i=1}^{t-2} I(x_i, t) + I(t, 2) + I(t, s) + I(2, 1)\right) \\
&= \sum_{i=1}^{t-2}(I(x_i, t-1) - I(x_i, t)) + (I(t-1, 2) - I(t, 1)) + (I(2, s) - I(t, s)) \\
&\quad + (I(t, 1) - I(t, 2)) - (I(2, 1) - I(2, 2)) \\
&< 0,
\end{aligned}
$$

which is a contradiction with the minimality of BID index of T.

Thus we have $\Delta \leq 2$. ∎

2 Proofs of Theorems 1.1-1.4

In this section, we give the proofs of Theorems 1.1-1.4.

Proof of Theorem 1.1. Suppose that u is the vertex with maximum degree in T. If $T \ncong S_n$, then there exists a vertex v in T such that $uv \in E(T)$ and $N_T(v) = \{u, v_1, v_2, \cdots, v_{t-1}\}$ $(2 \leq t \leq \Delta)$ where $d_T(v_i) = 1$ for $i = 1, 2, \cdots, t-1$. Let $T^* = T - \{vv_i | i = 1, 2, \cdots, t-1\} + \{uv_i | i = 1, 2, \cdots, t-1\}$. Then for any BID index has the property P, we have $TI(T^*) > TI(T)$ by Lemma 1.2. We repeatedly utilize the above transformations, and eventually we will get star S_n. Thus $TI(T) \leq (n-1)I(1, n-1)$, with equality if and only if $T \cong S_n$. This completes the proof. ■

By the conclusion of Theorem 1.1 and the proof of Lemma 1.1, we have

Lemma 2.1. *Let $n \geq 5$ and $T \in \mathcal{T}_n$. If T has the second-maximum BID index with property P, the distance between a vertex with maximum degree and a pendent vertex in T is at most two.*

Proof of Theorem 1.2. The transformation of Lemma 1.2 will decrease the number of branching vertices and BID index with property P. Since $T \in \mathcal{T}_n \backslash \{S_n\}$ $(n \geq 5)$, then eventually we will get double star $S_{a,n-a-2}$ $(1 \leq a \leq \lfloor \frac{n-2}{2} \rfloor)$ by repeatedly utilizing the transformation of Lemma 1.2.

By the fact that the BID index has property P, $a \leq n-a-2$ and the symmetry of $I(x, y)$, we have $I(n-a, 1) > I(1, a+1)$ and $I(a, n-a) > I(a+1, n-a-1)$, and then

$$
\begin{aligned}
&TI(S_{a-1,n-a-1}) - TI(S_{a,n-a-2}) \\
&= (a-1)I(1, a) + I(a, n-a) + (n-a-1)I(n-a, 1) \\
&\quad - aI(1, a+1) - I(a+1, n-a-1) - (n-a-2)I(n-a-1, 1) \\
&= (a-1)(I(1, a) - I(1, a+1)) + (n-a-2)(I(n-a, 1) - I(n-a-1, 1)) \\
&\quad + (I(n-a, 1) - I(1, a+1)) + (I(a, n-a) - I(a+1, n-a-1)) \\
&> (n-a-2)[((I(n-a, 1) - I(n-a-1, 1)) - (I(a+1, 1) - I(a, 1))] > 0.
\end{aligned}
$$

Then $\max\{TI(S_{a,n-a-2})\} = TI(S_{1,n-3})$ for $1 \leq a \leq \lfloor \frac{n-2}{2} \rfloor$. Thus $TI(T) \leq I(1, 2) + I(2, n-2) + (n-3)I(1, n-2)$, with equality if and only if $T \cong S_{1,n-3}$. This completes the proof. ■

Proof of Theorem 1.3. By Lemma 1.3, if T has minimum BID index with property P, then $\Delta \le 2$. Thus $TI(T) \ge 2I(1,2) + (n-3)I(2,2)$, with equality if and only if $T \cong P_n$. This completes the proof. ■

Proof of Theorem 1.4. On the contrary, we suppose that $T \ncong P_n$ and $T \ncong P_{n,3}$ with second minimum BID index. Let $P = u_1 u_2 \cdots u_\ell$ be the longest path in T.

Case 1. $d_T(u_2) = d_T(u_{\ell-1}) = 2$.

Since $T \ncong P_n$, there exists at one vertex, say u_i, in P such that $d_T(u_i) > 2$ for $3 \le i \le \ell - 2$. Let v be one neighbor of u_i outside P.

Subcase 1.1. $d_T(u_i) = 3$ for some $3 \le i \le \ell - 3$.

Subcase 1.1.1. $d_T(u_{\ell-2}) = 2$.

Let $T^* = T - u_i v + u_{\ell-1} v$, then $T^* \ncong P_n$. Bearing in mind that the BID index has property P, $d_T(u_{i-1}) \ge 2$ and $d_T(u_{i+1}) \ge 2$, then we have

$$
\begin{aligned}
& TI(T) - TI(T^*) \\
& = I(3, d_T(u_{i-1})) + I(3, d_T(u_{i+1})) + I(3, d_T(v)) + I(2,2) + I(1,2) \\
& \quad - I(2, d_T(u_{i-1})) - I(2, d_T(u_{i+1})) - I(3, d_T(v)) - I(2,3) - I(1,3) \\
& = (I(3, d_T(u_{i-1})) - I(2, d_T(u_{i-1}))) - (I(3,2) - I(2,2)) \\
& \quad + (I(3, d_T(u_{i+1})) - I(2, d_T(u_{i+1}))) - (I(3,1) - I(2,1)) \\
& > 0,
\end{aligned}
$$

which is a contradiction.

Subcase 1.1.2. $d_T(u_{\ell-2}) \ge 3$.

Let $T^* = T - u_i v + u_\ell v$, then $T^* \ncong P_n$. Note that the BID index has property P, $d_T(u_{i-1}) \ge 2$, $d_T(u_{i+1}) \ge 2$ and $d_T(v) \ge 1$. Then

$$
\begin{aligned}
& TI(T) - TI(T^*) = I(3, d_T(u_{i-1})) + I(3, d_T(u_{i+1})) + I(3, d_T(v)) + I(1,2) \\
& \quad - I(2, d_T(u_{i-1})) - I(2, d_T(u_{i+1})) - I(2, d_T(v)) - I(2,2) \\
& = (I(3, d_T(u_{i-1})) - I(2, d_T(u_{i-1}))) + (I(3, d_T(v)) - I(2, d_T(v))) \\
& \quad + (I(3, d_T(u_{i+1})) - I(2, d_T(u_{i+1}))) - (I(2,2) - I(2,1)) \\
& > I(3,1) - I(2,1) - I(2,2) + I(1,2) = I(3,1) - I(2,2) > 0,
\end{aligned}
$$

which is a contradiction.

Subcase 1.2. $d_T(u_i)=3$ for $i=\ell-2$.

If $d_T(u_3)=2$, we let $T^*=T-u_{\ell-2}v+u_2v$, then $T^*\ncong P_n$. Similar to Subcase 1.1.1, we have $TI(T)>TI(T^*)$, which is a contradiction.

If $d_T(u_3)\geq 3$, we let $T^*=T-u_{\ell-2}v+u_1v$, then $T^*\ncong P_n$. Similar to Subcase 1.1.2, we have $TI(T)>TI(T^*)$, which is a contradiction.

Subcase 1.3. $d_T(u_i)=k(\geq 4)$ for some $3\leq i\leq \ell-2$.

Let $N_T(u_i)=\{u_{i-1},u_{i+1},v_1,v_2,\cdots,v_{k-2}\}$ and $T^*=T-u_iv_{k-2}+u_\ell v_{k-2}$, then $T^*\ncong P_n$. Bearing in mind that the BID index has property P, $d_T(u_{i-1})\geq 2$ and $k\geq 4$, then we have

$$\begin{aligned}
&TI(T)-TI(T^*)\\
&=\sum_{i=1}^{k-3}I(k,d_T(v_i))+I(k,d_T(u_{i-1}))+I(k,d_T(u_{i+1}))+I(k,d_T(v_{k-2}))\\
&\quad+I(1,2)-\sum_{i=1}^{k-3}I(k-1,d_T(v_i))-I(k-1,d_T(u_{i-1}))\\
&\quad-I(k-1,d_T(u_{i+1}))-I(2,d_T(v_{k-2}))-I(2,2)\\
&>(I(k,d_T(u_{i-1}))-I(k-1,d_T(u_{i-1})))-(I(2,2)-I(1,2))\\
&>(I(k,2)-I(k-1,2))-(I(2,2)-I(1,2))>0,
\end{aligned}$$

which is a contradiction.

Case 2. $d_T(u_2)=3$ or $d_T(u_{\ell-1})=3$.

Let $P=u_1u_2\cdots u_\ell$ be the longest path in T. We suppose that $d_T(u_{\ell-1})=3$. Since $T\ncong P_n$ and $T\ncong P_{n,3}$, there exists at one vertex, say u_i, in P such that $d_T(u_i)>2$ for $2\leq i\leq \ell-2$. Let v be the neighbor of $u_{\ell-1}$ outside P. As $P=u_1u_2\cdots u_\ell$ is the longest path in T, then v must be a pendent vertex. Let $T^*=T-vu_{\ell-1}+vu_\ell$. Then $T^*\ncong P_n$. By the fact that the BID index has property P, $d_T(u_{\ell-2})\geq 2$, then we have

$$\begin{aligned}
&TI(T)-TI(T^*)\\
&=I(3,d_T(u_{\ell-2}))+2I(1,3)-I(2,d_T(u_{\ell-2}))-I(2,2)-I(1,2)\\
&>(I(1,3))-I(2,2)+(I(1,3)-I(1,2))>0,
\end{aligned}$$

which is a contradiction.

Case 3. $d_T(u_2) \geq 4$ or $d_T(u_{\ell-1}) \geq 4$.

Let $P = u_1u_2\cdots u_\ell$ be the longest path in T. We suppose that $d_T(u_{\ell-1}) = k \geq 4$. As $P = u_1u_2\cdots u_\ell$ is the longest path in T, then the neighbor of $u_{\ell-1}$ expect $u_{\ell-2}$ are pendent vertices. Let $N_T(u_{\ell-1}) = \{u_{\ell-2}, u_\ell, v_1, v_2, \cdots, v_{k-2}\}$.

Let $d_T(u_{\ell-2}) = t(\geq 2)$, and $T^* = T - \{u_{\ell-1}v_i | i = 1, 2, \cdots, k-2\} + \{u_\ell v_i | i = 1, 2, \cdots, k-2\}$. Then $T^* \ncong P_n$. Bearing in mind that the BID index has property P and $t \geq 2$ and $k \geq 4$, then we have

$$\begin{aligned}
&TI(T) - TI(T^*) \\
&= I(t,k) + (k-1)I(k,1) - I(t,2) - I(k-1,2) - (k-2)I(k-1,1) \\
&= (I(t,k) - I(t,2)) + (I(k,1) - I(k-1,2)) + (k-1)[I(k,1) - I(k-1,1)] \\
&> 0,
\end{aligned}$$

which is a contradiction.

Thus for $T \in \mathcal{T}_n \backslash \{P_n\}$ $(n \geq 5)$. Then for any BID index has the property P, we have $TI(T) \geq I(1,2) + 2I(1,3) + I(2,3) + (n-5)I(2,2)$, with equality if and only if $T \cong P_{n,3}$. This completes the proof. ■

3 Applications

We can easily verify that the function $x^\alpha + y^\alpha$ $(\alpha \geq 1)$ satisfies the property P. Then by Theorem 1.1 we have the following corollaries.

Corollary 3.1. *Let $n \geq 5$ and $T \in \mathcal{T}_n$. If $I(x,y) = x^\alpha + y^\alpha$ $(\alpha \geq 1)$*, we have

$$TI(T) \leq (n-1)(1 + (n-1)^\alpha),$$

with equality if and only if $T \cong S_n$.

In particular, if $\alpha = 1$, then we have $TI(T) \leq n(n-1)$ with equality if and only if $T \cong S_n$. This result is also shown in [12].

If $\alpha = 2$, then we have $TI(T) \leq (n-1)(1 + (n-1)^2)$ with equality if and only if $T \cong S_n$. This result is also shown in [10].

Corollary 3.2. [27] *Let $n \geq 5$ and $T \in \mathcal{T}_n$. If $I(x,y) = e^{x^\alpha + y^\alpha}$ $(\alpha \geq 1)$,* we have

$$TI(T) \leq (n-1)e^{(1+(n-1)^\alpha)},$$

with equality if and only if $T \cong S_n$.

By Theorem 1.2 we have the following corollaries.

Corollary 3.3. *Let $n \geq 5$ and $T \in \mathcal{T}_n \backslash \{S_n\}$. If $I(x,y) = x^\alpha + y^\alpha$ $(\alpha \geq 1)$,* we have

$$TI(T) \leq n - 2 + 2^{\alpha+1} + (n-2)^{\alpha+1},$$

with equality if and only if $T \cong S_{1,n-3}$.

Corollary 3.4. *Let $n \geq 5$ and $T \in \mathcal{T}_n \backslash \{S_n\}$. If $I(x,y) = e^{x^\alpha + y^\alpha}$ $(\alpha \geq 1)$,* we have

$$TI(T) \leq e^{n-2+2^{\alpha+1}+(n-2)^{\alpha+1}},$$

with equality if and only if $T \cong S_{1,n-3}$.

In particular, if $\alpha = 1$, then we have $TI(T) \leq e^{n+2+(n-2)^2}$, with equality if and only if $T \cong CS_{n,n-2}$. This result is also shown in [22].

By Theorem 1.3 we have the following corollaries.

Corollary 3.5. *Let $n \geq 5$ and $T \in \mathcal{T}_n$. If $I(x,y) = x^\alpha + y^\alpha$ $(\alpha \geq 1)$,* we have

$$TI(T) \geq 2(1 + 2^\alpha) + (n-3)2^{\alpha+1},$$

with equality if and only if $T \cong P_n$.

Corollary 3.6. [27] *Let $n \geq 5$ and $T \in \mathcal{T}_n$. If $I(x,y) = e^{x^\alpha + y^\alpha}$ $(\alpha \geq 1)$,* we have

$$TI(T) \geq e^{2(1+2^\alpha)+(n-3)2^{\alpha+1}},$$

with equality if and only if $T \cong P_n$.

By Theorem 1.4 we have the following corollaries.

Corollary 3.7. *Let $n \geq 5$ and $T \in \mathcal{T}_n \backslash \{P_n\}$. If $I(x,y) = x^\alpha + y^\alpha$ $(\alpha \geq 1)$,* we have

$$TI(T) \geq 3 + (2n-9)2^\alpha + 3^{\alpha+1},$$

with equality if and only if $T \cong P_{n,3}$.

Corollary 3.8. *Let* $n \geq 5$ *and* $T \in \mathcal{T}_n \backslash \{P_n\}$. *If* $I(x,y) = e^{x^\alpha + y^\alpha}$ $(\alpha \geq 1)$, we have

$$TI(T) \geq e^{3+(2n-9)2^\alpha+3^{\alpha+1}},$$

with equality if and only if $T \cong P_{n,3}$.

In particular, if $\alpha = 1$, then we have $TI(T) \geq e^{4n-6}$, with equality if and only if $T \cong P_{n,3}$. This result is also shown in [22].

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