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Incidence Energy of *k*-Uniform Hypertrees

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Abstract

For a square matrix M, its energy $E(M)$ is the sum of its singular values. Let $\mathcal{H}$ be a k-uniform hypergraph, and let $B(\mathcal{H})$ be the incidence matrix of $\mathcal{H}$. The incidence energy $BE(\mathcal{H})$ of $\mathcal{H}$ is the energy of $B(\mathcal{H})$.

Let $\mathcal{T}_{n,d}$ be the set of k-uniform hypertrees of order n and size r with diameter $3 \leq d \leq r-1$. In this article, the k-uniform hypertrees with minimum incidence energy over $\mathcal{T}_{n,d}$ are characterized. In addition, we have obtained the incidence energy of a hyperstar, and determined which hyperstar has the maximum and minimum incidence energy among all hyperstars with n vertices.

1 Introduction

In spectral graph theory, the structure of a graph is studied through the eigenvalues/eigenvectors of matrices associated with them. Many researchers around the world, motivated by this theory, have defined some matrices associated with hypergraph, aiming to develop a spectral hypergraph theory. In 2012, Cooper and Dutle [5] proposed the study of hypergraphs by means of the adjacency tensor. It is known, however, that to obtain eigenvalues of tensors has a high computational and theoretical

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cost. Perhaps for this reason, recently, some authors have renewed the interest to study the matrix representations of a hypergraph, as for example in [1,3,6,9,11,12].

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph with vertex set $V(\mathcal{H})$ and hyperedge set $E(\mathcal{H})$, where $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})}$ and $2^{V(\mathcal{H})}$ stands for the power set of $V(\mathcal{H})$. A hypergraph $\mathcal{H}$ is k-uniform if $|e| = k$ for every $e \in E(\mathcal{H})$. Especially, 2-uniform hypergraph is the ordinary graph. The distance between two vertices in a connected hypergraph is the length of the shortest walk connecting these two vertices. The diameter of a connected hypergraph is the largest distance between two of its vertices. A connected and acyclic hypergraph is called a hypertree. For a k-uniform hypertree $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ of order $n = |V(\mathcal{H})|$ and size $t = |E(\mathcal{H})|$, if there exists a vertex v satisfying $v \in e$ for any $e \in E(\mathcal{H})$, $\mathcal{H}$ is called a hyperstar with the center v, denoted it by $\mathcal{S}_{n,t}$. For convenience, let $[n] = \{1, 2, \ldots, n\}$. Let $E(v) = \{e \,|v \in e \in E(\mathcal{H})\}$, $d(v) = |E(v)|$ is the degree of v. A vertices and edges alternating sequence $\mathcal{P} = v_0 e_1 v_1 e_2 \cdots e_p v_p$ is a path if $v_{i-1}, v_i \in e_i$, and all v_i and e_i are distinct for $i \in [p]$. If $d(v_i) = 2$ for $i \in [p-1]$, and the other vertices in $V(\mathcal{P})$ are 1-degree vertices, then $\mathcal{P}$ is a loose path. An edge e is called a pendent edge if e contains exactly $k-1$ 1-degree vertices. A path $\mathcal{P} = v_0 e_1 v_1 e_2 \cdots e_p v_p$ is called pendent path if $d(v_0) \geq 3$, $d(v_i) = 2$ for $i \in [p-1]$ and the others vertices in $V(\mathcal{P})$ are 1-degree vertices. For a hypergraph $\mathcal{H}$, the subdivision graph $S(\mathcal{H})$ is obtained by adding a new vertex v_e and making it adjacent to all vertices of e for each edge of $\mathcal{H}$. Let $\mathcal{T}_{n,d}$ be the set of k-uniform hypertrees of order n and size r with diameter $2 \leq d \leq r$. Obviously, $\mathcal{T} \in \mathcal{T}_{n,2}$ is the hyperstar, and $\mathcal{T} \in \mathcal{T}_{n,r}$ is a loose path. In the following we let $3 \leq d \leq r-1$. Let $\mathcal{T}(n, d; n_1^1, \ldots, n_1^{k-1}, n_2^1, \ldots, n_2^{k-1}, \ldots, n_{d-2}^1, \ldots, n_{d-2}^{k-1}, n_{d-1}^1) \in \mathcal{T}_{n,d}$ be a hypercaterpillar obtained from a path $v_0^1 e_1 v_1^1 e_2 \cdots e_d v_d^1$ by adding n_i^j $(n_i^j \geq 0)$ pendent edges to v_i^j, where $e_i = \left\{v_{i-1}^1, v_{i-1}^2, \ldots v_{i-1}^{k-1}, v_i^1\right\}$ $(2 \leq i \leq d-1)$.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix of G, denoted by $A(G)$, is an $n \times n$ matrix (a_{ij}) in which $a_{ij} = 1$ if $v_i v_j \in E(G)$, and $a_{ij} = 0$ otherwise. The characteristic polynomial of $A(G)$, denoted by $\phi_A(G, \lambda) = |\lambda I - A(G)|$, is called the characteristic polynomial of G. The n roots of the equation $\phi_A(G,\lambda)=0$, denoted by $\lambda_1(G),\lambda_2(G),\ldots,\lambda_n(G)$, are called the eigenvalues of G. The energy $E(G)$ of G is defined [7] as

$$E(G)=\sum_{i=1}^{n}|\lambda_i(G)|.$$

If G is a bipartite graph, then its characteristic polynomial is

$$\phi(G)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k b_{2k}x^{n-2k},$$

where $b_0=1$ and $b_{2k}\geq 0$. If $G=T$ is a tree, then $b_{2k}=m(T,k)$ for all $k=1,\ldots,\left[\frac{n}{2}\right]$, where $m(T,k)$ equals to the number of k-matchings of T (see [8]). For two bipartite graphs G_1 and G_2, we define $G_1\preceq G_2$ if and only if $b_{2k}(G_1)\leq b_{2k}(G_2)$ for all $k=1,\ldots,\left[\frac{n}{2}\right]$. Moreover, if there exists a k such that $b_{2k}(G_1)<b_{2k}(G_2)$, we write $G_1\prec G_2$. The following result was proven (see [8]).

$$\begin{aligned}G_1\preceq G_2 &\Rightarrow E(G_1)\leq E(G_2),\\ G_1\prec G_2 &\Rightarrow E(G_1)<E(G_2).\end{aligned}\tag{1}$$

In 2007, Nikiforov ([10]) extended the concept of graph energy to matrices. For a square matrix M, its energy $E(M)$ is defined as the sum of its singular values. Let $B(\mathcal{H})=(b(v,e))_{|V(\mathcal{H})|\times|E(\mathcal{H})|}$ be the incidence matrix of a k-uniform hypergraph $\mathcal{H}$, where $b(v,e)=1$ if $v\in e$, and $b(v,e)=0$ otherwise. Following the definition of Nikiforov, the authors in [4] defined the energy of $B(\mathcal{H})$ as the incidence energy $BE(\mathcal{H})$ of $\mathcal{H}$, and proposed the relation

$$BE(\mathcal{H})=\frac{1}{2}E(A_s),\tag{2}$$

where A_s is the adjacent matrix of $S(\mathcal{H})$.

On this basis, the authors of [12] obtained the lower and upper bounds on $BE(\mathcal{H})$ for k-uniform hypertrees and characterized their corresponding extremal hypergraphs. Motivated by the above research, in this paper, we characterized the k-uniform hypertrees with the minimum incidence energy in $\mathcal{T}_{n,d}$. In addition, we have studied a particular class of hypergraphs and determined which hyperstar has maximum and minimum incidence energy among this class.

2 The minimum incidence energy of k-uniform hypertrees with give diameter

Lemma 1 ([4]). *If u and v are two adjacent vertices of a graph G and $e = uv$, then for $k \geq 1$,*
(i) $m(G,k) = m(G-e,k) + m(G-u-v,k-1)$;
(ii) If v is a pendent vertex, then $m(G,k) = m(G-v,k)+m(G-u-v,k-1)$.

Let $\mathcal{G}$ be a k-uniform hypertree and $e_0 = \{v_1, v_2, \dots, v_k\}$ be an edge which is not belonging to $\mathcal{G}$. Let $\mathcal{G}_1$ be the hypertree obtained by identifying v_k of e_0 and a vertex w of $\mathcal{G}$, denote the new vertex v_k. Let $\mathcal{H}_1$ be a hypertree obtained from $\mathcal{G}_1$ by attaching some pendent edges at some vertices of e_0. Let $e_{11}, \dots, e_{1t}$ be the edges attaching at v_1, and let $\mathcal{H}_2$ be the hypertree obtained from $\mathcal{H}_1$ by moving the pendent edges attaching at v_1 to v_k, as shown in Figure 1. Let $\mathcal{H}_3$ be the hypertree obtained from $\mathcal{H}_1$ by deleting the vertices in $V(\mathcal{S}_{(m-1)t+1,t}) - v_1$ and adding a loose path of length t at v_1, as shown in Figure 2.

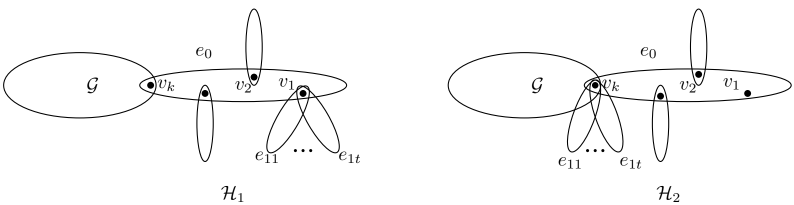

Figure 1. The hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$.

Lemma 2 ([12]). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the hypertrees as shown in Figure 2.1. Then $BE(\mathcal{H}_1) > BE(\mathcal{H}_2)$.*

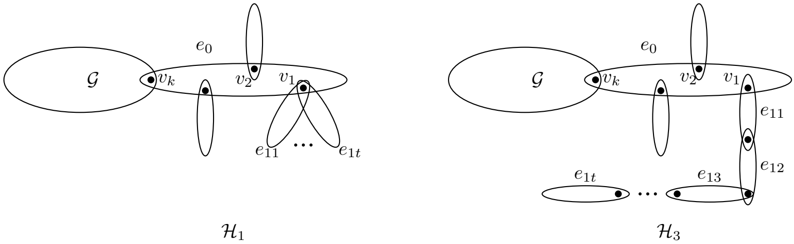

Figure 2. The hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_3$.

Lemma 3 ([12]). *Let $\mathcal{H}_1$ and $\mathcal{H}_3$ be the hypertrees as shown in Figure 2.2. Then $BE(\mathcal{H}_3) > BE(\mathcal{H}_1)$.*

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two k-uniform hypertrees and $\mathcal{P} = v_0e_1v_1e_2v_2$ be a path of length 2, which is not belonging to $\mathcal{G}_1$ and $\mathcal{G}_2$, $e_i = \{v_{i-1}, v_{i-1}^2, \ldots, v_{i-1}^{k-1}, v_i\}$. Let $\mathcal{G}_3$ be the hypertree obtained by identifying v_0 of e_1 and a vertex w of $\mathcal{G}_1$, denote the new vertex v_0, and identifying v_2 of e_2 and a vertex u of $\mathcal{G}_2$, denote the new vertex v_2. Let $\mathcal{H}_4$ be a hypertree obtained from $\mathcal{G}_3$ by attaching some pendent edges at some vertices of e_1. Let $e_{11}, \ldots, e_{1t}$ be the edges attaching at v_0^2 and $e_{21}, \ldots, e_{2s}$ be the edges attaching at v_1. Let $\mathcal{H}_5$ be the hypertree obtained from $\mathcal{H}_4$ by moving the pendent edges attaching at v_0^2 to v_1, depicted in Figure 3.

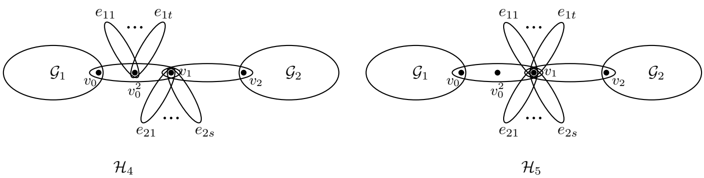

Figure 3. The hypergraphs $\mathcal{H}_4$ and $\mathcal{H}_5$.

Lemma 4. *Let $\mathcal{H}_4$ and $\mathcal{H}_5$ be the hypertrees as shown in Figure 2.3. Then $BE(\mathcal{H}_4) > BE(\mathcal{H}_5)$.*

Proof. By (1.2), $BE(\mathcal{H}_4) = \frac{1}{2}E\left(S(\mathcal{H}_4)\right)$ and $BE(\mathcal{H}_5) = \frac{1}{2}E\left(S(\mathcal{H}_5)\right)$. Now we only compare $S(\mathcal{H}_4)$ with $S(\mathcal{H}_5)$, where $S(\mathcal{H}_4)$ and $S(\mathcal{H}_5)$ are shown in Figure 4.

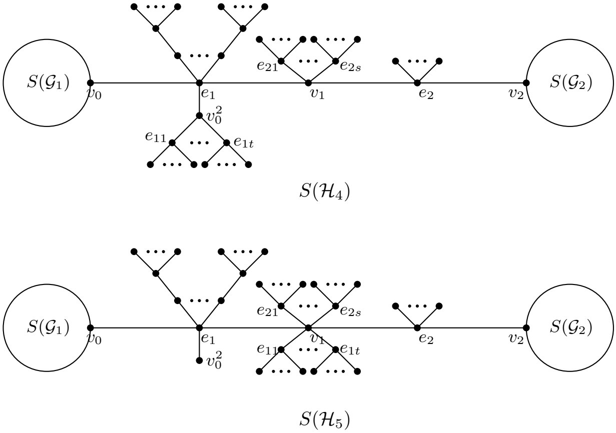

Figure 4. The graphs $S(\mathcal{H}_4)$ and $S(\mathcal{H}_5)$.

It is obvious to see that

$$S(\mathcal{H}_4)-v_0^2-e_{11}\cong S(\mathcal{H}_4)-v_0^2e_{11}-v_0^2-e_{12}\cong\cdots\cong S(\mathcal{H}_4)-\bigcup_{i=1}^{t-1}v_0^2e_{1i}-v_0^2-e_{1t},$$

$$S(\mathcal{H}_4)-\bigcup_{i=1}^{t}v_0^2e_{1i}-v_1-e_{21}\cong S(\mathcal{H}_4)-\bigcup_{i=1}^{t}v_0^2e_{1i}-v_1e_{21}-v_1-e_{22}$$

$$\cong\cdots\cong S(\mathcal{H}_4)-\bigcup_{i=1}^{t}v_0^2e_{1i}-\bigcup_{j=1}^{s-1}v_1e_{1j}-v_1-e_{2s},$$

$$S(\mathcal{H}_5)-v_1-e_{11}\cong S(\mathcal{H}_5)-v_1e_{11}-v_1-e_{12}$$

$$\cong\cdots\cong S(\mathcal{H}_5)-\bigcup_{i=1}^{t}v_1e_{1i}-\bigcup_{j=1}^{s-1}v_1e_{2j}-v_1-e_{2s},$$

$$S(\mathcal{H}_4)-\bigcup_{i=1}^{t}v_0^2e_{1i}-\bigcup_{j=1}^{s}v_1e_{1j}\cong S(\mathcal{H}_5)-\bigcup_{i=1}^{t}v_1e_{1i}-\bigcup_{j=1}^{s}v_1e_{2i},$$

$$S(\mathcal{H}_4)-\bigcup_{i=1}^{t}v_0^2e_{1i}-v_1-e_{21}\cong S(\mathcal{H}_5)-v_1-e_{11}.$$

Further by a direct calculation, for $\ell \geq 1$, we have

$$\begin{aligned} m\left(S(\mathcal{H}_4),\ell\right) &= m\left(S(\mathcal{H}_4)-v_0^2e_{11},\ell\right)+m\left(S(\mathcal{H}_4)-v_0^2-e_{11},\ell-1\right)\\ &= m\left(S(\mathcal{H}_4)-v_0^2e_{11}-v_0^2e_{12},\ell\right)\\ &\quad + m\left(S(\mathcal{H}_4)-v_0^2e_{11}-v_0^2-e_{12},\ell-1\right)\\ &\quad + m\left(S(\mathcal{H}_4)-v_0^2-e_{11},\ell-1\right)\\ &= \cdots\\ &= m\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2e_{1i},\ell\right)+tm\left(S(\mathcal{H}_4)-v_0^2-e_{11},\ell-1\right), \end{aligned}$$

and

$$\begin{aligned} & m\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2e_{1i},\ell\right)\\ =& m\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2e_{1i}-v_1e_{21},\ell\right)\\ &+ m\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2e_{1i}-v_1-e_{21},\ell-1\right)\\ =& m\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2e_{1i}-v_1e_{21}-v_1e_{22},\ell\right)\\ &+ m\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2e_{1i}-v_1e_{11}-v_1-e_{22},\ell-1\right)\\ &+ m\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2e_{1i}-v_1-e_{21},\ell-1\right)\\ =& \cdots\\ =& m\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2e_{1i}-\bigcup_{j=1}^{s} v_1e_{2j},\ell\right)\\ &+ sm\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2e_{1i}-v_1-e_{21},\ell-1\right). \end{aligned}$$

Then

$$\begin{aligned} m\left(S(\mathcal{H}_4),\ell\right) =& m\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2 e_{1i}-\bigcup_{j=1}^{s} v_1 e_{1j},\ell\right)\\ &+sm\left(S(\mathcal{H}_4)-\bigcup_{i=1}^{t} v_0^2 e_{1i}-v_1-e_{21},\ell-1\right)\\ &+tm\left(S(\mathcal{H}_4)-v_0^2-e_{11},\ell-1\right), \end{aligned}$$

and

$$\begin{aligned} m\left(S(\mathcal{H}_5),\ell\right) =& m\left(S(\mathcal{H}_5)-v_1e_{11},\ell\right)+m\left(S(\mathcal{H}_5)-v_1-e_{11},\ell-1\right)\\ =& m\left(S(\mathcal{H}_5)-v_1e_{11}-v_1e_{12},\ell\right)\\ &+m\left(S(\mathcal{H}_5)-v_1e_{11}-v_1-e_{12},\ell-1\right)\\ &+m\left(S(\mathcal{H}_5)-v_1-e_{11},\ell-1\right)\\ =&\cdots\\ =& m\left(S(\mathcal{H}_5)-\bigcup_{i=1}^{t} v_1 e_{1i}-\bigcup_{j=1}^{s} v_1 e_{2i},\ell\right)\\ &+(s+t)m\left(S(\mathcal{H}_5)-v_1-e_{11},\ell-1\right). \end{aligned}$$

If $\ell=1$, then $m\left(S(H_4),\ell\right)-m\left(S(H_5),\ell\right)=0$. If $\ell\geq 2$, since $S(\mathcal{H}_5)-v_1-e_{11}\subset S(\mathcal{H}_4)-v_0^2-e_{11}$, then

$$\begin{aligned} &m\left(S(\mathcal{H}_4),\ell\right)-m\left(S(\mathcal{H}_5),\ell\right)\\ =&t\left(m\left(S(\mathcal{H}_4)-v_0^2-e_{11},\ell-1\right)-m\left(S(\mathcal{H}_5)-v_1-e_{11},\ell-1\right)\right)>0. \end{aligned}$$

Thus $BE(\mathcal{H}_4)>BE(\mathcal{H}_5)$. The lemma holds. ■

Let $\mathcal{G}$ be a k-uniform hypertree with vertex set $V(\mathcal{G})$ and hyperedge set $E(\mathcal{G})$, and let $\mathcal{P}=v_0e_1v_1e_2\cdots e_{2p+1}v_{2p+2}$ be a path of length $2p+1$ ($p\geq 1$), which is not belonging to $\mathcal{G}$, and $e_i=\left\{v_{i-1},v_{i-1}^2,\ldots v_{i-1}^{k-1},v_i\right\}$. Let $\mathcal{G}_1$ be the hypertree obtained by identifying v_0 of e_1 and a vertex w of $\mathcal{G}$, denote the new vertex v_0. Let $\mathcal{H}_6$ be a hypertree obtained from $\mathcal{G}_1$ by attaching $n_i\geq 0$ pendent edges at v_i ($1\leq i\leq p+1$). There exists

$k \in \{1, 2, \ldots, p\}$ such that $n_k \neq 0$ if $E(\mathcal{G}) = \emptyset$. Let $e_{11}, \ldots, e_{1s}$ $(s \geq 0)$ be the edges attaching at v_p, and let $e_{21}, \ldots, e_{2t}$ $(t \geq 1)$ be the edges attaching at v_{p+1}. Let $\mathcal{H}_7$ be the hypertree obtained from $\mathcal{H}_6$ by moving the pendent edges attaching at v_{p+1} to v_p, depicted in Figure 5.

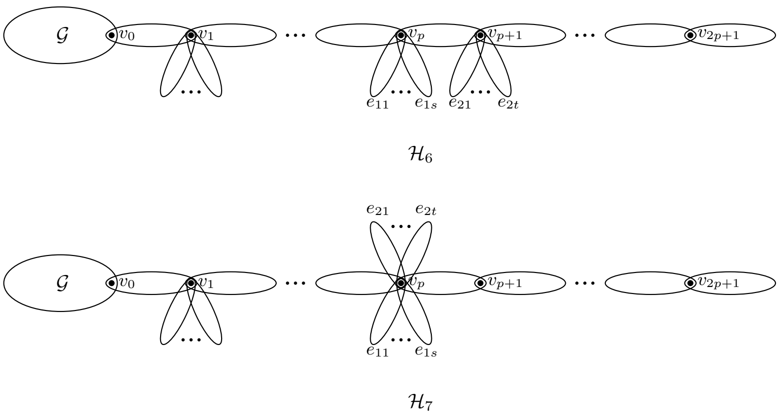

Figure 5. The hypergraphs $\mathcal{H}_6$ and $\mathcal{H}_7$.

Lemma 5. *Let $\mathcal{H}_6$ and $\mathcal{H}_7$ be the hypertrees as shown in Figure 2.5. Then $BE(\mathcal{H}_6) > BE(\mathcal{H}_7)$.*

Proof. By (1.2), $BE(\mathcal{H}_6) = \frac{1}{2}E(S(\mathcal{H}_6))$ and $BE(\mathcal{H}_7) = \frac{1}{2}E(S(\mathcal{H}_7))$. Now we only compare $S(\mathcal{H}_6)$ with $S(\mathcal{H}_7)$, where $S(\mathcal{H}_6)$ and $S(\mathcal{H}_7)$ are shown in Figure 6.

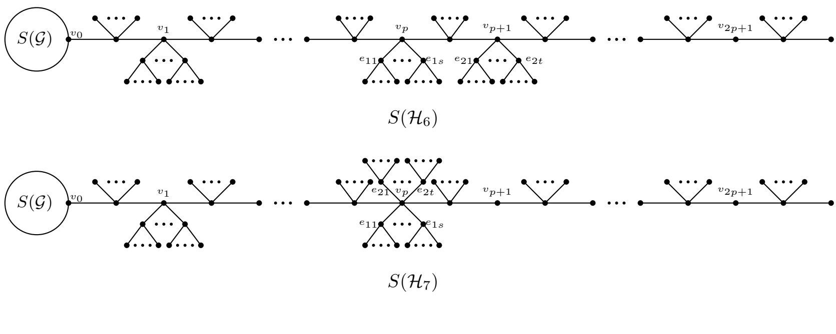

Figure 6. The graphs $S(\mathcal{H}_6)$ and $S(\mathcal{H}_7)$.

It is obvious to see that

$$S(\mathcal{H}_6) - v_{p+1} - e_{21} \cong S(\mathcal{H}_6) - v_{p+1}e_{21} - v_{p+1} - e_{22}$$

$$\cong \cdots \cong S(\mathcal{H}_6) - \bigcup_{i=1}^{t-1} v_{p+1}e_{2i} - v_{p+1} - e_{2t},$$

$$S(\mathcal{H}_6) - \bigcup_{i=1}^{t} v_{p+1}e_{2i} - v_p - e_{11} \cong S(\mathcal{H}_6) - \bigcup_{i=1}^{t} v_{p+1}e_{2i} - v_p e_{11} - v_p - e_{22}$$

$$\cong \cdots \cong S(\mathcal{H}_6) - \bigcup_{i=1}^{t} v_{p+1}e_{2i} - \bigcup_{j=1}^{s-1} v_p e_{1j} - v_p - e_{1s},$$

$$S(\mathcal{H}_7) - v_p - e_{11} \cong S(\mathcal{H}_7) - v_p e_{11} - v_p - e_{12}$$

$$\cong \cdots \cong S(\mathcal{H}_7) - \bigcup_{i=1}^{s} v_p e_{1i} - \bigcup_{j=1}^{t-1} v_p e_{2j} - v_p - e_{2t},$$

$$S(\mathcal{H}_6) - \bigcup_{i=1}^{t} v_{p+1}e_{2i} - \bigcup_{j=1}^{s} v_p e_{1j} \cong S(\mathcal{H}_7) - \bigcup_{i=1}^{t} v_p e_{2i} - \bigcup_{j=1}^{s} v_p e_{1j},$$

$$S(\mathcal{H}_6) - \bigcup_{i=1}^{t} v_{p+1}e_{2i} - v_p - e_{11} \cong S(\mathcal{H}_7) - v_p - e_{11}.$$

By a direct calculation, for $\ell \geq 1$, we have

$$\begin{aligned} m\left(S(\mathcal{H}_6), \ell\right) &= m\left(S(\mathcal{H}_6) - v_{p+1}e_{21}, \ell\right) + m\left(S(\mathcal{H}_6) - v_{p+1} - e_{21}, \ell - 1\right) \\ &= m\left(S(\mathcal{H}_6) - v_{p+1}e_{21} - v_{p+1}e_{22}, \ell\right) \\ &\quad + m\left(S(\mathcal{H}_6) - v_{p+1}e_{21} - v_{p+1} - e_{22}, \ell - 1\right) \\ &\quad + m\left(S(\mathcal{H}_6) - v_{p+1} - e_{21}, \ell - 1\right) \\ &= \cdots \\ &= m\left(S(\mathcal{H}_6) - \bigcup_{i=1}^{t} v_{p+1}e_{2i}, \ell\right) \\ &\quad + tm\left(S(\mathcal{H}_6) - v_{p+1} - e_{21}, \ell - 1\right), \end{aligned}$$

and

$$
\begin{aligned}
& m\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i},\ell\right) \\
=& m\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i}-v_pe_{11},\ell\right) \\
& + m\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i}-v_p-e_{11},\ell-1\right) \\
=& m\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i}-v_pe_{11}-v_pe_{12},\ell\right) \\
& + m\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i}-v_pe_{11}-v_p-e_{12},\ell-1\right) \\
& + m\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i}-v_p-e_{11},\ell-1\right) \\
=& \cdots \\
=& m\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i}-\bigcup_{j=1}^{s} v_pe_{1j},\ell\right) \\
& + sm\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i}-v_p-e_{11},\ell-1\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
m\left(S(\mathcal{H}_6),\ell\right) =& m\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i}-\bigcup_{j=1}^{s} v_pe_{1j},\ell\right) \\
& + sm\left(S(\mathcal{H}_6)-\bigcup_{i=1}^{t} v_{p+1}e_{2i}-v_p-e_{11},\ell-1\right) \\
& + tm\left(S(\mathcal{H}_6)-v_{p+1}-e_{21},\ell-1\right),
\end{aligned}
$$

and

$$
m\left(S(\mathcal{H}_7),\ell\right) = m\left(S(\mathcal{H}_7)-v_pe_{11},\ell\right) + m\left(S(\mathcal{H}_7)-v_p-e_{11},\ell-1\right)
$$

$$
\begin{aligned}
&= m\left(S(\mathcal{H}_7) - v_p e_{11} - v_p e_{12}, \ell\right)\\
&\quad + m\left(S(\mathcal{H}_7) - v_p e_{11} - v_p - e_{12}, \ell - 1\right)\\
&\quad + m\left(S(\mathcal{H}_7) - v_p - e_{11}, \ell - 1\right)\\
&= \cdots\\
&= m\left(S(\mathcal{H}_7) - \bigcup_{i=1}^{t} v_p e_{2i} - \bigcup_{j=1}^{s} v_p e_{1j}, \ell\right)\\
&\quad + (s+t) m\left(S(\mathcal{H}_7) - v_p - e_{11}, \ell - 1\right).
\end{aligned}
$$

So

$$
\begin{aligned}
&m\left(S(\mathcal{H}_6), \ell\right) - m\left(S(\mathcal{H}_7), \ell\right)\\
=&t\left(m\left(S(\mathcal{H}_6) - v_{p+1} - e_{21}, \ell - 1\right) - m\left(S(\mathcal{H}_7) - v_p - e_{11}, \ell - 1\right)\right).
\end{aligned}
$$

We consider two cases.

Case 1. $E(\mathcal{G}) \neq \emptyset$.

Let A be the graph as shown in Figure 7. By repeatedly utilizing the Lemma 1 (i), it can be concluded that

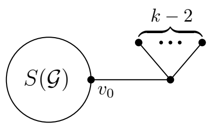

Figure 7. The graph A.

$$
\begin{aligned}
&m\left(S(\mathcal{H}_6) - v_{p+1} - e_{21}, \ell - 1\right) - m\left(S(\mathcal{H}_7) - v_p - e_{11}, \ell - 1\right)\\
\geq& m\left(S(G) \cup \left(\sum_{i=1}^{p+1} n_i\right) S_k \cup S_{k-1}, \ell - 2p\right)\\
&- m\left(A \cup \left(\left(\sum_{i=1}^{p+1} n_i\right) - 1\right) S_k \cup S_{k-1}, \ell - 2p\right).
\end{aligned}
$$

Note that $m\left(S(\mathcal{H}_6), \ell\right) - m\left(S(\mathcal{H}_7), \ell\right) \geq 0$ for any ℓ, and there exists

$\ell = 2p+2$ such that $m\left(S(\mathcal{H}_6), \ell\right) - m\left(S(\mathcal{H}_7), \ell\right) > 0$. Then $BE(\mathcal{H}_6) > BE(\mathcal{H}_7)$.

Case 2. $E\left(\mathcal{G}\right) = \emptyset$ and there exists $k \in \{1, 2, \ldots, p\}$ such that $n_k \neq 0$. By repeatedly utilizing the Lemma 1(i), it can be concluded that

$$\begin{aligned} & m\left(S(\mathcal{H}_6) - v_{p+1} - e_{21}, \ell - 1\right) - m\left(S(\mathcal{H}_7) - v_p - e_{11}, \ell - 1\right) \\ > & m\left(\left(\sum_{i=1}^{p+1} n_i\right) S_k \cup S_{k-1}, \ell - 2p\right) - m\left(\left(\sum_{i=1}^{p+1} n_i\right) S_k \cup S_{k-1}, \ell - 2p\right) \\ = & 0. \end{aligned}$$

Note that $m\left(S(\mathcal{H}_6), \ell\right) - m\left(S(\mathcal{H}_7), \ell\right) \geq 0$ for any ℓ, and when $\ell = 2p+1$, $m\left(S(\mathcal{H}_6), \ell\right) - m\left(S(\mathcal{H}_7), \ell\right) > 0$. Then $BE(\mathcal{H}_6) > BE(\mathcal{H}_7)$. The lemma holds. ■

Theorem 1. *Let $\mathcal{T} \in \mathcal{T}_{n,d}$ with $3 \leq d \leq r-1$.*

(i) If d is even, then

$$BE(\mathcal{T}(n, d; \underbrace{0, \ldots, 0}_{\frac{(d-2)(k-1)}{2}}, r-d, \underbrace{0, \ldots, 0}_{\frac{(d-2)(k-1)}{2}})) \leq BE(\mathcal{T}),$$

with the equation holds if and only if $\mathcal{T} \cong \mathcal{T}(n, d; \underbrace{0, \ldots, 0}_{\frac{(d-2)(k-1)}{2}}, r-d, \underbrace{0, \ldots, 0}_{\frac{(d-2)(k-1)}{2}})$.

(ii) If d is odd, then

$$BE(\mathcal{T}(n, d; \underbrace{0, \ldots, 0}_{\frac{(d-3)(k-1)}{2}}, r-d, \underbrace{0, \ldots, 0}_{\frac{(d-1)(k-1)}{2}})) \leq BE(\mathcal{T}),$$

with the equation holds if and only if $\mathcal{T} \cong \mathcal{T}(n, d; \underbrace{0, \ldots, 0}_{\frac{(d-3)(k-1)}{2}}, r-d, \underbrace{0, \ldots, 0}_{\frac{(d-1)(k-1)}{2}})$.

Proof. Let $\mathcal{T} \in \mathcal{T}_{n,d}$ with $3 \leq d \leq r-1$. If $\mathcal{T}$ is not a hypercaterpillar, then by Lemmas 2 and 3, there is a hypercaterpillar $\mathcal{T}' \in \mathcal{T}_{n,d}$ such that $BE(\mathcal{T}') > BE(\mathcal{T})$. By Lemmas 4 and 5, the theorem holds. ■

3 The incidence energy of a hyperstar

In this section, we will obtain the incidence energy of a hyperstar, and determine which hyperstar has the maximum and minimum incidence energy among all hyperstars with n vertices. The definition of a power graph was introduced in [3] as follows:

Definition 1. Let $G = (V, E)$ be a graph and let $k \geq 2$ be an integer. We define the power graph $\mathcal{G}^k$ as the k-graph with the following vertex set and edge set

$$V\left(\mathcal{G}^k\right) = V\left(G\right) \cup \left(\bigcup_{e \in E(G)} \varsigma_e\right) \text{ and } E\left(\mathcal{G}^k\right) = \{e \cup \varsigma_e : e \in E(G)\},$$

where $\varsigma_e = \left\{v_1^e, \ldots, v_{k-2}^e\right\}$ for each edge $e \in E(G)$.

We define a hyperstar as a power graph of a star. A generalization of the join operation was introduced in [2] as follows:

Definition 2. Consider a family of k-graphs, $\mathcal{F} = \{G_1, \ldots, G_k\}$, where each graph G_i has order n_i for $i = 1, \ldots, k$, and H is a graph with $V(H) = \{v_1, \ldots, v_k\}$. Each vertex $v_i \in V(H)$ is assigned to the graph $G_i \in \mathcal{F}$. The H-join of $G_1, \ldots, G_k$ is the graph $G = H[G_1, \ldots, G_k]$ such that $V(G) = \bigcup_{i=1}^{k} V(G_i)$ and edge set:

$$E\left(G\right) = \left(\bigcup_{i=1}^{k} E\left(G_i\right)\right) \cup \left(\bigcup_{uw \in E(H)} \{ij : i \in V(G_u), j \in V(G_w)\}\right).$$

The spectrum of the H-join of regular graphs was characterized in [2]. Let H be a graph with k vertices without isolated vertices. Let $G_1, \ldots, G_k$ be a sequence of k disjoint arbitrary p_j-regular graphs of orders n_j, $j = 1, \ldots, k$. Let $G = H[G_1, \ldots, G_k]$. For $j = 1, \ldots, k$, we use A_j to denote the adjacency matrices of G_j. Let $A(H) = (\delta_{ij})$ be the adjacency matrix of H. Define

$$\hat{G} = \begin{pmatrix} p_1 & \delta_{12}\sqrt{n_1 n_2} & \cdots & \delta_{1k}\sqrt{n_1 n_k} \\ \delta_{12}\sqrt{n_1 n_2} & p_2 & \cdots & \delta_{2k}\sqrt{n_2 n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1k}\sqrt{n_1 n_k} & \cdots & \delta_{k-1,k}\sqrt{n_{k-1} n_k} & p_k \end{pmatrix}. \quad (3)$$

Theorem 2 ([2]). *For $j = 1, \ldots, k$, let G_j be a p_j-regular graph of order n_j with spectrum σ_{G_i}. If $G = H[G_1, \ldots, G_k]$, and $\hat{G}$ is as defined in (3.1), then*

$$\sigma(G) = \sigma_{\hat{G}} \cup \left(\bigcup_{j=1}^{k} \left(\sigma_{G_j} \setminus \{p_j\}\right) \right).$$

Theorem 3. *Let S_n be the star on n vertices. If $k \geq 2$ is an integer, then*

$$BE\left((\mathcal{S}_n)^k \right) = \sqrt{k+n-2} + (n-2)\sqrt{k-1}.$$

Proof. By (1.2), $BE\left((\mathcal{S}_n)^k \right) = \frac{1}{2} E\left(S\left((\mathcal{S}_n)^k \right) \right)$. Now we only calculate $E\left(S\left((\mathcal{S}_n)^k \right) \right)$. Let H_0 be the tree formed by attaching a vertex to all pendent vertex of the star S_n. The adjacency matrix of A_{H_0}, takes the form

$$A_{H_0} = \begin{pmatrix} A_n & B_{n\times(n-1)} \\ {B_{n\times(n-1)}}^T & O \end{pmatrix},$$

where

$$A_n = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \ B_{n\times(n-1)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Useing the above notation,

$$S\left((\mathcal{S}_n)^k \right) = H_0 \left[\underbrace{K_1, \ldots, K_1}_{n}, \underbrace{\bar{K}_{k-1}, \ldots, \bar{K}_{k-1}}_{n-1} \right]. \quad (4)$$

By the identification in (3.2), the cardinality is

$$n_i = \begin{cases} 1, & \text{if } i = 1, \ldots, n, \\ k-1, & \text{if } i = n+1, \ldots, 2n-1, \end{cases}$$

and the regularity p_i is equal to zero for $1 \leq i \leq 2n-1$. Hence by applying Theorem 2 to $S\left((\mathcal{S}_n)^k\right)$, we obtain

$$\sigma_{A\left(S\left((\mathcal{S}_n)^k\right)\right)} = \left\{0^{[(n-1)(k-2)]}\right\} \cup \sigma_{C_{2n-1}},$$

where $C_{2n-1} = \begin{pmatrix} A_n & \sqrt{k-1}B_{n\times(n-1)} \\ \sqrt{k-1}B_{n\times(n-1)}{}^T & O \end{pmatrix}$.

Further by a direct calculation, we have

$$\sigma_{C_{2n-1}} = \left\{0, \sqrt{k+n-2}, -\sqrt{k+n-2}, \sqrt{k-1}^{[n-2]}, \left(-\sqrt{k-1}\right)^{[n-2]}\right\}.$$

Thus $BE\left((\mathcal{S}_n)^k\right) = \frac{1}{2}E\left(S\left((\mathcal{S}_n)^k\right)\right) = \sqrt{k+n-2} + (n-2)\sqrt{k-1}$. ■

Corollary. *If $\mathcal{S}$ is a hyperstar with $t \geq 2$ vertices, then*

$$\sqrt{t} = BE\left((\mathcal{S}_2)^t\right) \leq BE(\mathcal{S}) \leq BE(\mathcal{S}_t) = t + \sqrt{t} - 2.$$

Proof. Let $\mathcal{S}$ be a hyperstar with $t \geq 2$ vertices. Then there are $2 \leq n \leq t$ and $2 \leq k \leq t$ such that $\mathcal{S} = (\mathcal{S}_n)^k$. In this way, we have that $t = (n-1)(k-1)+1$, and so $n = \frac{t-1}{k-1} + 1$. Therefore

$$BE(\mathcal{S}) = \sqrt{\frac{t-1}{k-1} + k - 1} + \frac{t-1}{\sqrt{k-1}} - \sqrt{k-1}.$$

Consider the function $f : [2, t] \to \mathbb{R}$, defined by

$$f(x) = \sqrt{\frac{t-1}{x-1} + x - 1} + \frac{t-1}{\sqrt{x-1}} - \sqrt{x-1}.$$

Computing its derivatives, we obtain

$$f'(x) = -\frac{(x-1)(x+t-2)\sqrt{\frac{x^2-2x+t}{x-1}} + (-x^2+2x+t-2)\sqrt{x-1}}{2(x-1)^{\frac{5}{2}}\sqrt{\frac{x^2-2x+t}{x-1}}}$$
$$\leq \frac{(t-3)\sqrt{x-1}}{2(x-1)^{\frac{5}{2}}\sqrt{\frac{x^2-2x+t}{x-1}}}.$$

If $t > 4$, then $f'(x) < 0$ for all $x \in [2, t]$, and $f(x)$ is an increasing function. If $t = 3$ or $t = 2$, then $\mathcal{S} = (\mathcal{S}_2)^t$. Hence the result follows. ■

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