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Relations between p-Sombor and Other Degree–Based Indices

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Abstract

The Sombor index (SO) and its extended version p-Sombor index (SO_p) are vertex degree-based topological indices that have potential applications in mathematical chemistry. In this article, we obtain several new relations for these indices. Precisely, we find relations between SO and SO_p and characterize the graphs where equality occurs. Also, we present relations for SO and SO_p involving other topological indices, such as the first Zagreb, Randić, reciprocal Randić, inverse sum indeg, and Albertson indices. Furthermore, we set up relations between the p-Sombor indices for different values of p.

1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The neighborhood of a vertex v, denoted by $N_G(v)$, is the set of all vertices which are adjacent to v. The degree d_v of v in G is the cardinality of $N_G(v)$. We denote $\Delta = \Delta(G)$ and $\delta = \delta(G)$ for the maximum degree and minimum degree of the graph G, respectively.

Degree-based topological indices have potential applications in mathematical chemistry. At present time, a study on mathematical properties

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and chemical applications of degree-based topological indices is a burning area of research [1, 3, 4, 6–12, 15]. On the basis of Euclidean metric, Gutman [7] introduced the Sombor index, which is defined by

$$SO(G) = \sum_{uv \in E(G)} \sqrt{(d_u^2 + d_v^2)}.$$

Based on the p-norm, Rêti et al. [17] extended the concept of the Sombor index to p-Symbor index ($p \neq 0$), which was defined as

$$SO_p(G) = \sum_{uv \in E(G)} (d_u^p + d_v^p)^{\frac{1}{p}}.$$

They also observed that the first Zagreb index [11, 12] defined by

$$M_1(G) \quad = \quad \sum_{uv \in E(G)} d_u^2 = \sum_{uv \in E(G)} (d_u + d_v)$$

is equals to $SO_1(G)$ and the Sombor index is equals to $SO_2(G)$. The Inverse sum indeg index is defined as

$$ISI(G) = \sum_{uv \in E(G)} \frac{2d_u d_v}{d_u + d_v} = 2SO_{-1}(G).$$

The Randić index [16] and reciprocal Randić index [13] are defined as

$$R(G) \quad = \quad \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$$

$$RR(G) \quad = \quad \sum_{uv \in E(G)} \sqrt{d_u d_v}.$$

The Albertston irregularity index [2] is defined as

$$Alb(G) = \sum_{uv \in E(G)} |d_u - d_v|.$$

In this article, we obtain several new relations for $SO_p(G)$ and $SO(G)$. Precisely, we find relations between SO and SO_p and characterize the graphs where equality occurs. Also, we present relations for SO and SO_p involving other topological indices, such as the first Zagreb, Randić, reciprocal Randić, inverse sum indeg, and Albertson indices. Furthermore, we set up relations between the p-Sombor indices for different values of p.

2 Relations between p-Sombor, first Zagreb and reciprocal Randić indices

Lemma 1. *Let G be a simple graph such that $d_u = d_v$ for all $uv \in E(G)$. Then each component of G is regular. In particular, if G is connected and $d_u = d_v$ for all $uv \in E(G)$, then G is regular.*

Proof. Let C be a component of G. Then C is a connected graph. Since $d_u = d_v$ for all $uv \in E(G)$ and so any two adjacent vertices in C are of the same degree. Let u be a vertex in C and $\deg(u) = m$, a finite number. Since C is connected, then for any arbitrary vertex $v \in V(C)$, there exists a $u - v$ path in C. Then by condition $\deg(u) = \deg(v) = m$. Therefore, the degree of every vertex v is equal to the degree of u. Hence C is regular. Thus proof is complete. ■

Recall that $SO_p(G) = M_1(G)$ for $p = 1$. In the following, we represent bounds for $SO_p(G)$ in terms of $M_1(G)$ and characterize the graphs achieving the bounds.

Theorem 1. *For a simple graph G with minimum degree $\delta > 0$ and maximum degree Δ,*

(a) $\frac{1}{2^{\frac{p-1}{p}}} M_1(G) \leq SO_p(G) \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta} M_1(G)$ *if* $p > 1$.

(b) $\frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta} M_1(G) \leq SO_p(G) \leq \frac{1}{2^{\frac{p-1}{p}}} M_1(G)$ *if* $p < 1$.

Moreover, the left-side equality in (a) or reight-side equality in (b) will occur if and only if each component of G is regular, and the right-side equality in (a) or the left-side equality in (b) occurs only when G is a bi-regular.

Proof. Let's consider the following ratio

$$\frac{(d_u^p + d_v^p)^{\frac{1}{p}}}{d_u + d_v} = \frac{\left\{1 + \left(\frac{d_u}{d_v}\right)^p\right\}^{\frac{1}{p}}}{1 + \frac{d_u}{d_v}}, \quad d_u, d_v \in [\delta, \Delta]. \tag{1}$$

Since $d_u, d_v \in [\delta, \Delta]$, so $\frac{\delta}{\Delta} \leq \frac{d_u}{d_v} \leq \frac{\Delta}{\delta}$. In view of expression (1), we take the following function h defined by

$$h(t) = \frac{(1 + t^p)^{\frac{1}{p}}}{1 + t}, \quad t \in \left[\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right].$$

Then

$$\begin{aligned} h'(t) &= \frac{t^{p-1}(1 + t^p)^{\frac{1}{p}-1} - (1 + t^p)^{\frac{1}{p}}}{(1 + t)^2} \\ &= \frac{(1 + t^p)^{\frac{1}{p}-1}\{(1 + t)t^{p-1} - (1 + t^p)\}}{(1 + t)^2} \\ &= \frac{(1 + t^p)^{\frac{1}{p}-1}(t^{p-1} - 1)}{(1 + t)^2}. \end{aligned} \tag{2}$$

Case-1: $p > 1$**.** Since $p > 1$, from (2) we get that h is monotonically decreasing on $\left[\frac{\delta}{\Delta}, 1\right]$ and monotonically increasing on $\left[1, \frac{\Delta}{\delta}\right]$ and so

$$\frac{1}{2^{\frac{p-1}{p}}} \leq h(t) \leq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{\delta + \Delta}$$

$$i.e., \quad \frac{1}{2^{\frac{p-1}{p}}} \leq \frac{(1 + t^p)^{\frac{1}{p}}}{1 + t} \leq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{\delta + \Delta}, \quad t \in \left[\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right]. \tag{3}$$

From the expression (1) and inequality (3), we have

$$\frac{1}{2^{\frac{p-1}{p}}} \leq \frac{\left\{1+\left(\frac{d_u}{d_v}\right)\right\}^{\frac{1}{p}}}{1+\frac{d_u}{d_v}} \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta} \tag{4}$$

Then

$$\begin{aligned}\frac{1}{2^{\frac{p-1}{p}}} &\leq \frac{(d_u^p+d_v^p)^{\frac{1}{p}}}{d_u+d_v} \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta}\\ \frac{1}{2^{\frac{p-1}{p}}}(d_u+d_v) &\leq (d_u^p+d_v^p)^{\frac{1}{p}} \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta}(d_u+d_v)\end{aligned}$$

Taking summation over $uv \in E(G)$, we have

$$\frac{1}{2^{\frac{p-1}{p}}}\sum_{uv\in E(G)}(d_u+d_v) \leq (d_u^p+d_v^p)^{\frac{1}{p}} \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta}\sum_{uv\in E(G)}(d_u+d_v)$$

Therefore,

$$\frac{1}{2^{\frac{p-1}{p}}}M_1(G) \leq SO_p(G) \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta}M_1(G) \text{ when } p>1. \tag{5}$$

The left-side equality in (3) will occurs only when $t=1$ and so left-side equality of (5) will occurs when $\frac{d_u}{d_v}=1$, i.e., $d_u=d_v$. Thus left-side equality in (5) will occurs only when $d_u=d_v$ for all $uv \in E(G)$, i.e., when each component of G is regular due to Lemma 1. Again right-side equality in (3) will occurs if and only if $t \in \left\{\frac{\Delta}{\delta}, \frac{\delta}{\Delta}\right\}$ and so right-side equality of (4) will occurs only when $\frac{d_u}{d_v} \in \left\{\frac{\Delta}{\delta}, \frac{\delta}{\Delta}\right\}$, i.e., only when $d_u, d_v \in \{\delta, \Delta\}$. Thus right equality in (5) will occurs only when $d_u, d_v \in \{\delta, \Delta\}$ for all $uv \in E(G)$, i.e., only when G is a bi-regular graph.

Case-2: $p<1$**.** Since $p<1$, from (2) we get that h is monotonically increasing on $\left[\frac{\delta}{\Delta}, 1\right]$ and monotonically decreasing on $\left[1, \frac{\Delta}{\delta}\right]$ and so

$$\frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta} \le h(t) \le \frac{1}{2^{\frac{p-1}{p}}}$$

$$\text{i.e.,} \qquad \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta} \le \frac{(1+t^p)^{\frac{1}{p}}}{1+t} \le \frac{1}{2^{\frac{p-1}{p}}}, \quad t \in \left[\tfrac{\delta}{\Delta}, \tfrac{\Delta}{\delta}\right]. \tag{6}$$

Notice that (6) is the reverse of the inequality (3) and so we get the following reverse inequality of (4)

$$\frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta} \le \frac{(d_u^p+d_v^p)^{\frac{1}{p}}}{d_u+d_v} \le \frac{1}{2^{\frac{p-1}{p}}}.$$

By employing similar type calculations as of Case-1, we get

$$\frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\delta+\Delta} M_1(G) \le SO_p(G) \le \frac{1}{2^{\frac{p-1}{p}}} M_1(G). \tag{7}$$

By a similar explanation of Case-1, the right-side equality of (5) occurs only when G is regular, and the left-side equality occurs only when G is bi-regular. ■

Remark. Applying Theorem 1 for an r-regular graph G, $SO_p(G) = \frac{1}{2^{\frac{p-1}{p}}} 2r|E(G)|$.

The following result is a consequence of Theorem 1 for $p = 2$.

Corollary. *([5]) For a simple graph* $SO_2(G) \ge \frac{1}{\sqrt{2}} M_1(G)$.

It is well known that for a simple connected graph with n vertices and m edges occurs $M_1 \ge \frac{4m^2}{n}$. Using this inequality in Theorem 1, we get the following result.

Corollary. *For a simple connected graph G with n vertices and m edges,* $SO_p(G) \ge \frac{4}{2^{\frac{p-1}{p}}} \frac{m^2}{n}$ *if* $p > 1$.

A consequence of Corollary 2 and of Theorem 1 is the following one.

Corollary. *([5]) For a simple connected graph G with n vertices and m edges, $SO_2(G) \geq \frac{2\sqrt{2}}{n}m^2$.*

Corollary. *([5], [14], [15]) For a simple graph G with minimum degree δ and maximum degree Δ, $\frac{1}{\sqrt{2}}M_1(G) \leq SO_2(G) \leq \frac{(\delta^2+\Delta^2)^{\frac{1}{2}}}{\delta+\Delta}M_1(G)$.*

Now we give the bound of $SO_p(G)$ in terms of $RR(G)$.

Theorem 2. *For a simple graph G with minimum degree $\delta > 0$ and maximum degree Δ,*

$$(a)\ 2^{\frac{2}{p}}\, RR(G) \leq SO_p(G) \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\sqrt{\delta\Delta}}\, RR(G) \text{ when } p > 0.$$

$$(b)\ \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\sqrt{\delta\Delta}} RR(G) \leq SO_p(G) \leq 2^{\frac{1}{p}}\, RR(G) \text{ when } p < 0.$$

Also, left-side equality in (a) or right-side equality in (b) are held if and only if each component of G is regular and the right-side equality in (a) or lefts-side equality in (b) hold if and only if G is a bi-regular graph.

Proof. Since $d_u, d_v \in [\delta, \Delta]$, so $\frac{\delta}{\Delta} \leq \frac{d_u}{d_v} \leq \frac{\Delta}{\delta}$. We consider the ratio,

$$\frac{(d_u^p+d_v^p)^{\frac{2}{p}}}{d_u d_v} = \frac{\left(1+\left(\frac{d_u}{d_v}\right)^p\right)^{\frac{2}{p}}}{\frac{d_u}{d_v}}$$

and an equivalent function corresponding to this expression can be taken as

$$f(t) = \frac{(1+t^p)^{\frac{2}{p}}}{t}, \quad t \in \left[\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right].$$

Then

$$\begin{aligned} f'(t) &= \frac{2t^p(1+t^p)^{\frac{2}{p}-1} - (1+t^p)^{\frac{2}{p}}}{t^2} \\ &= \frac{(1+t^p)^{\frac{2}{p}-1}(2t^p - (1+t^p)}{t^2} \\ &= \frac{(1+t^p)^{\frac{2}{p}-1}(t^p-1)}{t^2}. \end{aligned}$$

Case-1: $p > 0$. Since $p > 0$, f is monotonically decreasing on $\left[\frac{\delta}{\Delta}, 1\right]$ and monotonically increasing on $\left[1, \frac{\Delta}{\delta}\right]$ and so

$$2^{\frac{2}{p}} \quad \leq \quad f(t) \leq \frac{(\delta^p + \Delta^p)^{\frac{2}{p}}}{\delta\Delta}. \tag{8}$$

Therefore,

$$2^{\frac{2}{p}} \quad \leq \quad \frac{\left(1 + \left(\frac{d_u}{d_v}\right)^p\right)^{\frac{2}{p}}}{\frac{d_u}{d_v}} \leq \frac{(\delta^p + \Delta^p)^{\frac{2}{p}}}{\delta\Delta} \tag{9}$$

$$2^{\frac{1}{p}}\sqrt{d_u d_v} \quad \leq \quad (d_u + d_v)^{\frac{1}{p}} \leq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{\sqrt{\delta\Delta}}\sqrt{d_u d_v}.$$

Taking summation over $uv \in E(G)$, and using $SO_p(G) = \sum\limits_{uv\in E(G)} (d_u + d_v)^{\frac{2}{p}}$ and $RR(G) = \sum\limits_{uv\in E(G)} \sqrt{d_u d_v}$, we get

$$2^{\frac{1}{p}} RR(G) \quad \leq \quad (SO_p(G) \leq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{\sqrt{\delta\Delta}} RR(G). \tag{10}$$

The left side equality in (7) holds if and only if $t = 1$ and so left side equality of (9) holds if and only if $\frac{d_u}{d_v} = 1$, $d_u = d_v$ for all $uv \in E(G)$, i.e., if and only if each component of G is regular due to Lemma 1. Again right side equality in (7) hold if and only if $t \in \left\{\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right\}$ and so right side equality of (9) is hold if and only if $\frac{d_u}{d_v} \in \left\{\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right\}$, i.e., if and only if $d_u, d_v \in \{\delta, \Delta\}$. Thus the right-side equality in (10) is hold if and only if $d_u, d_v \in \{\delta, \Delta\}$ for all $uv \in E(G)$, i.e., if and only if G is bi-regular.

Case-2: $p < 0$. Since $p < 0$, f is monotonically increasing on $\left[\frac{\delta}{\Delta}, 1\right]$ and monotonically decreasing on $\left[1, \frac{\Delta}{\delta}\right]$ and so

$$\frac{(\delta^p + \Delta^p)^{\frac{2}{p}}}{\delta\Delta} \quad \leq \quad f(t) \leq 2^{\frac{2}{p}}; \tag{11}$$

which is a reverse inequality of (7).

Therefore,

$$\frac{(\delta^p+\Delta^p)^{\frac{2}{p}}}{\delta\Delta} \leq \frac{\left(1+\left(\frac{d_u}{d_v}\right)^p\right)^{\frac{2}{p}}}{\frac{d_u}{d_v}} \leq 2^{\frac{2}{p}} \tag{12}$$

$$i.e., \quad \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\sqrt{\delta\Delta}}\sqrt{d_u d_v} \leq (d_u+d_v)^{\frac{1}{p}} \leq 2^{\frac{1}{p}}\sqrt{d_u d_v}.$$

Proceeding similar way as in Case-1, we get the result. ■

Corollary. *([19]) For a simple graph G with minimum degree $\delta > 0$ and maximum degree Δ,*

$$\sqrt{2}\,RR(G) \leq SO_2(G) \leq \sqrt{\frac{\delta}{\Delta}+\frac{\Delta}{\delta}}\,RR(G).$$

Moreover, the left-side equality holds if and only if each component of G is regular and the right-side equality holds if and only if G is a bi-regular graph.

Corollary. *For a simple graph G with minimum degree $\delta > 0$ and maximum degree Δ,*

$$\frac{2\sqrt{\delta\,\Delta}}{\delta+\Delta} \leq ISI(G) \leq 2^{\frac{p+1}{p}}\,RR(G).$$

Moreover, the right-side equality holds if and only if each component of G is regular and the left-side equality holds if and only if G is a bi-regular graph.

Proof. From definition $ISI(G) = 2SO_{-1}(G)$ and so putting $p = -1$ in Theorem 2, we get

$$\frac{(\delta^{-1}+\Delta^{-1})^{-1}}{\sqrt{\delta\Delta}}\,RR(G) \leq SO_{-1}(G) \leq 2^{\frac{1}{p}}\,RR(G)$$

$$i.e., \quad \frac{\sqrt{\delta\,\Delta}}{\delta+\Delta} \leq \frac{1}{2}\,ISI(G) \leq 2^{\frac{1}{p}}\,RR(G),$$

which completes the proof. ■

3 Relations between *p*-Sombor, Sombor, Randić and Albertson indices

Theorem 3. *For a simple graph G with minimum degree $\delta > 0$ and maximum degree Δ,*

(a) $\frac{1}{2^{\frac{p-2}{2p}}} SO(G) \leq SO_p(G) \leq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{\sqrt{\delta^2+\Delta^2}} SO(G)$ *if* $p > 2$*;*

(b) $\frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{\sqrt{\delta^2+\Delta^2}} SO(G) \leq SO_p(G) \leq \frac{1}{2^{\frac{p-2}{2p}}} SO(G)$ *if* $p < 2$.

Also, left-side equality in (a) or right-side equality in (b) are held if and only if each component of G is regular and the right-side equality in (a) or lefts-side equality in (b) hold if and only if G is a bi-regular graph.

Proof. We know that $SO_p(G) = \sum_{uv \in E(G)} (d_u^p + d_v^p)^{\frac{1}{p}}$ and $SO(G) = \sum_{uv \in E(G)} (d_u^2 + d_v^2)^{\frac{1}{2}}$. In view of the ratio $\frac{(\delta^p+\Delta^p)^{\frac{2}{p}}}{\delta^2+\Delta^2}$ we take the following function f defined as

$$f(t) = \frac{(1+t^p)^{\frac{2}{p}}}{1+t^2}, \quad t \in \left[\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right].$$

Then

$$\begin{aligned} f'(t) &= \frac{(1+t^2)2t^{p-1}(1+t^p)^{\frac{2}{p}-1} - (1+t^p)^{\frac{2}{p}}2t}{(1+t^2)^2} \\ &= \frac{2t(1+t^p)^{\frac{2}{p}-1}\{(1+t^2)t^{p-2} - (1+t^p)\}}{(1+t^2)^2} \\ &= \frac{2t(1+t^p)^{\frac{2}{p}-1}(t^{p-2}-1)}{(1+t^2)^2}. \end{aligned} \tag{13}$$

Case 1 : $p > 2$. In this case f is monotonically decreasing on $\left[\frac{\delta}{\Delta}, 1\right]$ and monotonically increasing on $\left[1, \frac{\Delta}{\delta}\right]$ and so,

$$\frac{2^{\frac{2}{p}}}{2} \leq f(t) \leq \frac{(\delta^p+\Delta^p)^{\frac{2}{p}}}{\delta^2+\Delta^2}. \tag{14}$$

Then,

$$\frac{1}{2^{\frac{p-2}{p}}} \leq \frac{\left(1+\left(\frac{d_u}{d_v}\right)^p\right)^{\frac{2}{p}}}{1+\left(\frac{d_u}{d_v}\right)^2} \leq \frac{(\delta^p+\Delta^p)^{\frac{2}{p}}}{\delta^2+\Delta^2}$$

$$i.e., \qquad \frac{1}{2^{\frac{p-2}{2p}}}\left(d_u^2+d_v^2\right)^{\frac{1}{2}} \leq (d_u^p+d_v^p)^{\frac{1}{p}} \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\sqrt{\delta^2+\Delta^2}}\left(d_u^2+d_v^2\right)^{\frac{1}{2}}.$$

Taking summation over $uv \in E(G)$, we have

$$\frac{1}{2^{\frac{p-2}{2p}}} SO(G) \leq SO_p(G) \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\sqrt{\delta^2+\Delta^2}} SO(G). \tag{15}$$

By a similar argument as described in Theorem 1, the left side equality will occur if and only if G is regular and the right side equality will occur if and only if G is bi-regular.

Case 2 : $p < 2$. In this case f is a monotonically increasing on $\left[\frac{\delta}{\Delta}, 1\right]$ and monotonically decreasing on $\left[1, \frac{\Delta}{\delta}\right]$ and so

$$\frac{(\delta^p+\Delta^p)^{\frac{2}{p}}}{\delta^2+\Delta^2} \leq f(t) \leq \frac{1}{2^{\frac{p-2}{p}}};$$

which is a reverse inequality of (14) and proceeding by similar arguments of Case 1 we get the following inequality, which is a reverse of the inequality (15)

$$\frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{\sqrt{\delta^2+\Delta^2}} SO(G) \leq SO_p(G) \leq \frac{1}{2^{\frac{p-2}{2p}}} SO(G).$$

■

Theorem 4. *For graph G be a graph with maximum degree Δ and minimum degree $\delta > 0$,*

(a) $\delta^2 R(G)\, 2^{\frac{1}{p}} \leq SO_p(G) \leq \Delta^2 R(G)\, 2^{\frac{1}{p}}$ *for* $p > 0$;

(b) $\Delta^2 R(G)\, 2^{\frac{1}{p}} \leq SO_p(G) \leq \delta^2 R(G)\, 2^{\frac{1}{p}}$ *for* $p < 0$.

The bounds are attained if and only if G is a regular graph.

Proof. First we assume that $p > 0$. Since $\delta \leq d_u \leq \Delta$, so for $p > 0$, we get

$$\delta \leq \sqrt{d_u\, d_v} \leq \Delta$$

$$\delta^p \leq d_u^p \leq \Delta^p$$

$$2\delta^p \leq d_u^p + d_v^p \leq 2\Delta^p$$

$$2^{\frac{1}{p}}\,\delta \leq (d_u^p + d_v^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p}}\,\Delta.$$

Therefore $\delta^2\, 2^{\frac{1}{p}} \leq \sqrt{d_u\, d_v}\,(d_u^p + d_v^p)^{\frac{1}{p}} \leq \Delta^2\, 2^{\frac{1}{p}}$, which gives

$$\delta^2\, 2^{\frac{1}{p}}\, \frac{1}{\sqrt{d_u\, d_v}} \leq (d_u^p + d_v^p)^{\frac{1}{p}} \leq \Delta^2\, 2^{\frac{1}{p}}\, \frac{1}{\sqrt{d_u\, d_v}}.$$

Taking summation over $uv \in E(G)$, we get the required inequality for $p > 0$. Similarly, we can prove the result for $p < 0$. ■

Putting $p = 2$, in the above theorem we get the following result.

Corollary. *([19]) For graph G be a graph with maximum degree Δ and minimum degree $\delta > 0$,*

$$\delta^2\, R(G)\sqrt{2} \leq SO_2(G) \leq \Delta^2\, R(G)\sqrt{2}.$$

Theorem 5. *For a simple graph G with minimum degree $\delta > 0$ and maximum degree $\Delta > \delta$,*

$$SO_p(G) \geq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{\Delta - \delta} Alb(G).$$

Also, equality occurs if and only if G is a bi-regular graph.

Proof. In view of the ratio $\frac{|d_u-d_v|}{(d_u^p+d_v^p)^{\frac{1}{p}}}$, let us consider the following function

$$\begin{aligned} f(t) &= \frac{|1-t|}{(1+t^p)^{\frac{1}{p}}}, \quad t \in \left[\frac{\delta}{\Delta}, \ \frac{\Delta}{\delta}\right]; \\ &= \begin{cases} \frac{1-t}{(1+t^p)^{\frac{1}{p}}}, & \text{if } \frac{\delta}{\Delta} \le t \le 1; \\ \frac{t-1}{(1+t^p)^{\frac{1}{p}}}, & \text{if } 1 \le t \le \frac{\Delta}{\delta}. \end{cases} \end{aligned}$$

For $t = 1$, $f(t) = 0$ and for others values of t, $f'(t)$ are as follows.

For $t < 1$,

$$\begin{aligned} f'(t) &= \frac{(1+t^p)^{\frac{1}{p}}(-1) - t^{p-1}(1+t^p)^{\frac{1}{p}-1}}{(1+t^p)^{\frac{2}{p}}} \\ &= -\frac{(1+t^p)^{\frac{1}{p}-1}(1+t^p+t^{p-1})}{(1+t^p)^{\frac{2}{p}}} < 0 \end{aligned}$$

and for $t > 1$,

$$\begin{aligned} f'(t) &= \frac{(1+t^p)^{\frac{1}{p}} \cdot 1 - t^{p-1}(1+t^p)^{\frac{1}{p}-1}}{(1+t^p)^{\frac{2}{p}}} \\ &= \frac{(1+t^p)^{\frac{1}{p}-1}\{1+t^{p-1}(t-1)\}}{(1+t^p)^{\frac{2}{p}}} > 0. \end{aligned}$$

Therefore, f is monotonically decreasing on $\left[\frac{\delta}{\Delta}, 1\right]$ and monotonically increasing on $\left[1, \frac{\Delta}{\delta}\right]$. Thus the maximum of f attains at $t = \frac{\delta}{\Delta}$ or $t = \frac{\Delta}{\delta}$ or at both the points. Clearly, $f\left(\frac{\delta}{\Delta}\right) = \frac{|\delta-\Delta|}{(\delta^p+\Delta^p)^{\frac{1}{p}}} = f\left(\frac{\Delta}{\delta}\right)$.

Therefore,

$$f(t) \le \frac{|\delta-\Delta|}{(\delta^p+\Delta^p)^{\frac{1}{p}}} \quad \text{for all } t \in \left[\tfrac{\delta}{\Delta}, \ \tfrac{\Delta}{\delta}\right]; \tag{16}$$

$$\frac{\left|1-\frac{d_v}{d_u}\right|}{\left(1+\frac{d_v}{d_u}\right)^{\frac{1}{p}}} \le \frac{|\delta-\Delta|}{(\delta^p+\Delta^p)^{\frac{1}{p}}} \quad \text{for all } \tfrac{d_v}{d_u} \in \left[\tfrac{\delta}{\Delta}, \ \tfrac{\Delta}{\delta}\right]. \tag{17}$$

By simple calculation and taking summation over $uv \in E(G)$, we get

$$Alb(G) \leq \frac{\Delta - \delta}{(\delta^p + \Delta^p)^{\frac{1}{p}}} SO_p(G);$$

$$i.e., \qquad \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{\Delta - \delta} Alb(G) \leq SO_p(G). \tag{18}$$

Equality in (16) hold if and only if $t \in \left\{\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right\}$ and so equality of (17) is hold if and only if $\frac{d_u}{d_v} \in \left\{\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right\}$, i.e., if and only if $d_u, d_v \in \{\delta, \Delta\}$. Thus the equality in (18) is hold if and only if $d_u, d_v \in \{\delta, \Delta\}$ for all $uv \in E(G)$, i.e., if and only if G is bi-regular. ■

4 Relations between *p*-Sombor indices with different values of *p*

Theorem 6. *For $p > 0$ and a simple graph G with minimum degree $\delta > 0$ and maximum degree Δ,*

$$2^{\frac{2}{p}} SO_{-p}(G) \leq SO_p(G) \leq \frac{(\delta^p + \Delta^p)^{\frac{2}{p}}}{\delta\Delta} SO_{-p}(G).$$

Also, the left-side equality holds if and only if G is regular, and the right-side equality holds if and only if G is a bi-regular graph.

Proof. In view of the ratio $\frac{(d_u^p + d_v^p)^{\frac{1}{p}}}{\left(d_u^{-p} + d_v^{-p}\right)^{\frac{-1}{p}}} = \frac{\left(1 + \left(\frac{d_v}{d_u}\right)^p\right)^{\frac{1}{p}}}{\left(1 + \left(\frac{d_v}{d_u}\right)^{-p}\right)^{\frac{-1}{p}}}$ of the terms in $SO_p(G)$ and $SO_{-p}(G)$, we consider a function f defined by

$$\begin{aligned} f(t) &= \frac{(1 + t^p)^{\frac{1}{p}}}{(1 + t^{-p})^{-\frac{1}{p}}}, \quad t \in \left[\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right] \\ &= (1 + t^p)^{\frac{1}{p}} \cdot (1 + \frac{1}{t^p})^{\frac{1}{p}} \\ &= \frac{(1 + t^p)^{\frac{2}{p}}}{t}, \quad t \in \left[\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right]. \end{aligned}$$

Then

$$f'(t) = \frac{t\cdot 2t^{p-1}(1+t^p)^{\frac{2}{p}-1} - (1+t^p)^{\frac{2}{p}}\cdot 1}{t^2}$$
$$= \frac{(1+t^p)^{\frac{2}{p}-1}\{2t^p-(1+t^p)\}}{t^2} = \frac{(1+t^p)^{\frac{2}{p}-1}(t^p-1)}{t^2}.$$

Since $p>0$, f is monotonically decreasing on $\left[\frac{\delta}{\Delta},1\right]$ and monotonically increasing on $\left[1,\frac{\Delta}{\delta}\right]$ and so,

$$2^{\frac{2}{p}} \leq f(t) \leq \frac{(\delta^2+\Delta^2)^{\frac{2}{p}}}{\delta\Delta}, \quad t\in\left[\frac{\delta}{\Delta},\frac{\Delta}{\delta}\right].$$

Consequently, we have

$$2^{\frac{2}{p}} \leq \frac{\left(1+\left(\frac{d_v}{d_u}\right)^p\right)^{\frac{1}{p}}}{\left(1+\left(\frac{d_v}{d_u}\right)^{-p}\right)^{\frac{-1}{p}}} \leq \frac{(\delta^2+\Delta^2)^{\frac{2}{p}}}{\delta\Delta}, \quad t\in\left[\frac{\delta}{\Delta},\frac{\Delta}{\delta}\right],$$

which is equivalent to

$$2^{\frac{2}{p}} \leq \frac{(d_u^p+d_v^p)^{\frac{1}{p}}}{(d_u^{-p}+d_v^{-p})^{\frac{-1}{p}}} \leq \frac{(\delta^2+\Delta^2)^{\frac{2}{p}}}{\delta\Delta}, \quad t\in\left[\frac{\delta}{\Delta},\frac{\Delta}{\delta}\right].$$

Taking summation over $uv\in E(G)$, we get

$$2^{\frac{2}{p}}SO_{-p}(G) \leq SO_p(G) \leq \frac{(\delta^2+\Delta^2)^{\frac{2}{p}}}{\delta\Delta}SO_{-p}(G).$$

■

Theorem 7. *For a simple graph G with minimum degree $\delta>0$ and maximum degree Δ,*

$$\frac{1}{2^{\frac{p-q}{pq}}}SO_q(G) \leq SO_p(G) \leq \frac{(\delta^p+\Delta^p)^{\frac{1}{p}}}{(\delta^q+\Delta^q)^{\frac{1}{q}}}SO_q(G), \quad \textit{provided } p>q.$$

Also, the left-side equality holds if and only if G is regular, and the right-

side equality holds if and only if G is a bi-regular graph.

Proof. Consider the following function $f(t) = \frac{(1+t^p)^{\frac{1}{p}}}{(1+t^q)^{\frac{1}{q}}}$, $t \in \left[\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right]$. Then,

$$\begin{aligned} f'(t) &= \frac{(1+t^q)^{\frac{1}{q}} \cdot t^{p-1} \cdot (1+t^p)^{\frac{1}{p}-1} - (1+t^p)^{\frac{1}{p}} \cdot t^{q-1} \cdot (1+t^q)^{\frac{1}{q}-1}}{(1+t^q)^{\frac{2}{q}}} \\ &= \frac{(1+t^q)^{\frac{1}{q}-1}(1+t^p)^{\frac{1}{p}-1} \left\{t^{p-1}(1+t^q) - t^{q-1}(1+t^p)\right\}}{(1+t^q)^{\frac{2}{q}}} \\ &= \frac{(1+t^q)^{\frac{1}{q}-1}(1+t^p)^{\frac{1}{p}-1}(t^{p-1} - t^{q-1})}{(1+t^q)^{\frac{2}{q}}} \\ &= \frac{(1+t^q)^{\frac{1}{q}-1}(1+t^p)^{\frac{1}{p}-1} t^{q-1}(t^{p-q} - 1)}{(1+t^q)^{\frac{2}{q}}}. \end{aligned} \tag{19}$$

Since $p > q$, from (19) we obtain that f is monotonically decreasing in $\left[\frac{\delta}{\Delta}, 1\right]$ and monotonically increasing on $\left[1, \frac{\Delta}{\delta}\right]$. Thus the minimum of f will attained by $t = 1$ and the maximum will attained at $t \in \left\{\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right\}$. Therefore,

$$\frac{2^{\frac{1}{p}}}{2^{\frac{1}{q}}} \leq f(t) \leq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{(\delta^q + \Delta^q)^{\frac{1}{q}}}; \tag{20}$$

$$\frac{1}{2^{\frac{1}{q}-\frac{1}{p}}} \leq \frac{(d_u^p + d_v^p)^{\frac{1}{p}}}{(d_u^q + d_v^q)^{\frac{1}{q}}} \leq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{(\delta^q + \Delta^q)^{\frac{1}{q}}} \quad \text{(by putting } t = \tfrac{d_u}{d_v}\text{)};$$

$$\frac{1}{2^{\frac{p-q}{pq}}} (d_u^q + d_v^q)^{\frac{1}{q}} \leq (d_u^p + d_v^p)^{\frac{1}{p}} \leq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{(\delta^q + \Delta^q)^{\frac{1}{q}}} (d_u^q + d_v^q)^{\frac{1}{q}}. \tag{21}$$

Taking summation over $uv \in E(G)$, and using $SO_p(G) = \sum_{uv \in E(G)} (d_u^p + d_v^p)^{\frac{1}{p}}$ and $SO_q(G) = \sum_{uv \in E(G)} (d_u^q + d_v^q)^{\frac{1}{q}}$, we get

$$\frac{1}{2^{\frac{p-q}{pq}}} SO_q(G) \leq SO_p(G) \leq \frac{(\delta^p + \Delta^p)^{\frac{1}{p}}}{(\delta^q + \Delta^q)^{\frac{1}{q}}} SO_q(G). \tag{22}$$

The left side equality in (20) holds if and only if $t = 1$ and so left side equality of (21) holds if and only if $\frac{d_u}{d_v} = 1$, i.e., $d_u = d_v$. Thus the left side equality of (22) holds if and only if $d_u = d_v$ for all $uv \in E(G)$, i.e., if and only if G is regular. Again right side equality in (20) hold if and only if $t \in \left\{\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right\}$ and so right side equality of (21) is hold if and only if $\frac{d_u}{d_v} \in \left\{\frac{\delta}{\Delta}, \frac{\Delta}{\delta}\right\}$, i.e., if and only if $d_u, d_v \in \{\delta, \Delta\}$. Thus the right-side equality in (22) hold if and only if $d_u, d_v \in \{\delta, \Delta\}$ for all $uv \in E(G)$, i.e., if and only if G is bi-regular. ■

Corollary. *For a simple graph G with minimum degree $\delta > 0$ and maximum degree Δ,*

$$\frac{1}{\sqrt{2}} SO_1(G) \quad \leq \quad SO_2(G) \leq \frac{(\delta^2 + \Delta^2)^{\frac{1}{2}}}{(\delta + \Delta)} SO_1(G).$$

Also, the left-side equality holds if and only if G is regular, and the right-side equality holds if and only if G is a bi-regular graph.

Corollary. *[18] For a simple graph G with minimum degree $\delta > 0$ and maximum degree Δ,*

$$2\, ISI(G) \leq M_1(G) \leq \frac{(\delta + \Delta)^2}{2\,\delta\Delta} ISI(G).$$

Also, the left-side equality holds if and only if G is regular, and the right-side equality holds if and only if G is a bi-regular graph.

Proof. Putting $p = 1$ and $q = -1$ in Theorem 7, we have

$$\frac{1}{2^{-2}} SO_{-1}(G) \quad \leq \quad SO_1(G) \leq \frac{(\delta + \Delta)}{\left(\frac{1}{\delta} + \frac{1}{\Delta}\right)^{-1}} SO_{-1}(G).$$

Since $ISI(G) = 2SO_{-1}(G)$, so we get

$$2\, ISI(G) \leq M_1(G) \leq \frac{(\delta + \Delta)^2}{2\,\delta\Delta} ISI(G),$$

which completes the proof. ■

Corollary. *For a simple graph G with minimum degree $\delta > 0$ and maximum degree Δ,*

$$\sqrt{2}\, ISI(G) \leq SO_2(G) \leq \frac{\sqrt{\delta^2+\Delta^2}(\delta+\Delta)}{2\,\delta\Delta}\, ISI(G).$$

Also, the left-side equality holds if and only if G is regular, and the right-side equality holds if and only if G is a bi-regular graph.

Proof. Putting $p = 2$ and $q = -1$ in Theorem 7, we have

$$\frac{1}{2^{-3/2}} SO_{-1}(G) \leq SO_2(G) \leq \frac{(\delta^2+\Delta^2)^{1/2}}{\left(\frac{1}{\delta}+\frac{1}{\Delta}\right)^{-1}} SO_{-1}(G).$$

The results follows immediately as $ISI(G) = 2SO_{-1}(G)$. ∎

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