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Construction of Equienergetic Trees

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Abstract

The energy $\mathcal{E}(G)$ of a graph G is the sum of the absolute values of all eigenvalues of G. Two graphs of the same order are said to be equienergetic if their energies are equal. As pointed out by Gutman, it is not known how to systematically construct any pair of equienergetic, non-cospectral trees until now. Inspired by the research of integral trees, we proposed a construction of infinite pairs of equienergetic trees of diameter 4.

1 Introduction

In this paper all graphs are simple and undirected. Let $G = (V, E)$ be such a graph on $n(G)$ vertices, and let λ_i be the i-th largest eigenvalues of the adjacency matrix of G. Denote the characteristic polynomial of G as $P(G, x)$, and it is known that the zeroes of $P(G, x)$ are the eigenvalues of G. The *energy* of graph is defined [4, 5] as $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. Two graphs of the same order are said to be *equienergetic* if their energies are equal. For obvious reasons one is interested only in noncospectral equienergetic graphs. Equienergetic trees were considered in [1, 7]. However, there is

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no general method to constructing pairs of equienergetic trees until now. This problem was confirmed by Gutman in [6]: *Although it is known that there are numerous pairs of equienergetic, non-cospectral trees, it is not known how to systematically construct any such pair.*

A graph G is called integral if all eigenvalues of G are integers. Many different classes of integral trees have been constructed in the past decades [2, 3, 8, 9]. Luckily, we found two pairs of equienergetic trees in [2] and its related webpage Small integral trees, which are listed in Table 1.

name	spectrum	n	$\mathcal{E}(G)$
$SK_{1,35}$	$6, 1^{34}, 0$	71	80
$T(1,4) * T(2,1,3) * T(4,3,1) * T(1,13,1)$	$4, 3, 2^6, 1^{21}, 0^{13}$	71	80
$K_{1,31} \sim SK_{1,28}$	$6, 5, 1^{27}, 0^{31}$	89	76
$ST(6) * T(15,4) * T(1,3,1)$	$5, 2^{15}, 1^3, 0^{51}$	89	76

Table 1. Two pairs of equienergetic integral trees: the spectrum contains only the nonnegative half, and multiplicities are written as exponents.

Inspired by this observation, we try to construct equienergetic tree pairs with integral energies, where the task seems easier than ones in general cases.

2 Construction of a family of equienergetic tree pairs

Let $K_{a,b}$ be a complete bipartite graph on $a+b$ vertices. The star graph S_n is a tree with a vertex of degree $n-1$, that is, $S_n = K_{1,n-1}$. A tree T is called balanced if the vertices at the same distance from the center of T have the same degree. $T(n_k, n_{k-1}, \cdots, n_1)$ is a balanced tree of diameter $2k$, where $n_k, n_{k-1}, \cdots, n_1$ denote the number of successors of a vertex at distance $k-j$ from the center. Let the tree $K_{1,s} \bullet T(m,t)$ of diameter 4 be obtained by identifying the center w of $K_{1,s}$ and the center v of $T(m,t)$. The following Lemma can be found in [8].

Lemma 1.

$$P[K_{1,s} \bullet T(m,t), x] = x^{m(t-1)+(s-1)}(x^2-t)^{m-1}[x^4-(m+t+s)x^2+st].$$

Extending a method for calculation energy of trees in [1], we get the following lemma:

Lemma 2. *Let p, q be positive real numbers with $p^2 \geq 4q$. Then the sum of absolute value of roots of $x^4 - px^2 + q = 0$ is $2\sqrt{p + 2\sqrt{q}}$.*

Proof. The roots of $x^4 - px^2 + q = 0$ are $x_{1,2} = \pm\sqrt{\frac{p+\sqrt{p^2-4q}}{2}}$ and $x_{3,4} = \pm\sqrt{\frac{p-\sqrt{p^2-4q}}{2}}$, all of which are real numbers since $p^2 \geq 4q$. Therefore the sum of absolute value of the roots is

$$\sum_{i=1}^{4} |x_i| = 2\left(\sqrt{\frac{p+\sqrt{p^2-4q}}{2}} + \sqrt{\frac{p-\sqrt{p^2-4q}}{2}}\right)$$

$$= 2\sqrt{\left(\sqrt{\frac{p+\sqrt{p^2-4q}}{2}} + \sqrt{\frac{p-\sqrt{p^2-4q}}{2}}\right)^2} = 2\sqrt{p+2\sqrt{q}}$$

as desired. ■

Let $mz(G)$ denote the multiplicity of eigenvalue zero of G.

Lemma 3. *Let $G = K_{1,s} \bullet T(m,t)$. Then $n(G) = m(t-1) + (s-1) + 2(m-1) + 4$, $\mathcal{E}(G) = 2\sqrt{t}(m-1) + 2\sqrt{(m+t+s) + 2\sqrt{st}}$, And $mz(G) = m(t-1) + (s-1)$.*

Proof. By Lemma 1 and 2, the assertion holds easily. ■

With the aid of computer, we find a way to construct a family of equienergetic tree pairs of diameter 4 within tree class $K_{1,s} \bullet T(m,t)$.

Theorem 1. *Let k be a positive integer. Let tree $T_1 = K_{1,s_1} \bullet T(m_1, t_1)$ with $s_1 = [(2k-1)(6k-1)]^2$, $t_1 = 1$, $m_1 = 48k^3 - 28k^2 + 8k$. Let tree $T_2 = K_{1,s_2} \bullet T(m_2, t_2)$ with $s_2 = 1$, $t_2 = (3k-1)^2$, $m_2 = 16k^2$. Then T_1 and T_2 are non-cospectral equienergetic graphs, that is, $n(T_1) = n(T_2)$, $mz(T_1) \neq mz(T_2)$, and $\mathcal{E}(T_1) = \mathcal{E}(T_2)$.*

Proof. By Lemma 3 and simple computation, we get $n(T_1) = n(T_2) = 144k^4 - 96k^3 + 32k^2 + 2$, $mz(T_1) - mz(T_2) = -96k^3 + 88k^2 - 16k < 0$ for all k, $\mathcal{E}(T_1) = 2[(2k-1)(6k-1)(48k^3 - 28k^2 + 8k - 1) +$

$\sqrt{144k^4 - 144k^3 + 60k^2 - 8k + 2 + 2(2k-1)(6k-1)}] = 2[(2k-1)(6k-1)(48k^3 - 28k^2 + 8k - 1) + 2(6k^2 - 3k + 1)] = 96k^3 - 32k^2 + 4k + 2$, and $\mathcal{E}(T_2) = 2[(3k-1)(16k^2-1) + \sqrt{1 + (3k-1)^2 + 16k^2 + 2(3k-1)}] = 2[(3k-1)(16k^2-1) + 5k] = 96k^3 - 32k^2 + 4k + 2 = \mathcal{E}(T_1)$. Therefore the proof completes clearly. ■

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