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Estrada and *L*-Estrada Indices of a Graph and Their Relationship with the Number of Spanning Trees

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Abstract

Let G be a $n-$vertex simple graph. Suppose $A(G)$ and $L(G) = \Delta(G) - A(G)$ are adjacency and Laplacian matrix of G, respectively, where $\Delta(G)$ is degree matrix of G. $EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$ and $LEE(G) = \sum_{i=1}^{n} e^{\mu_i}$ are called Estrada and Laplacian Estrada index of G, where λ_i and μ_i, $1 \leq i \leq n$, denote the eigenvalues of $A(G)$ and $L(G)$. In this paper, some new upper and lower bounds for $EE(G)$ and $LEE(G)$ are given. Moreover, some relations between $EE(G)$ and $LEE(G)$, and the number of spanning trees are established.

1 Introduction

Let $G = (V, E)$ be a simple undirected $n-$vertex graph, where $V(G)$ and $E(G)$ denote the vertex and edge set of G, respectively. Let $A(G)$, $L(G) =$

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$\Delta(G) - A(G)$, and $Q(G) = \Delta(G) + A(G)$ are adjacency, Laplacian and signless Laplacian matrix of G, respectively, where $\Delta(G)$ is degree matrix of G. The Estrada, Laplacian Estrada and signless Laplacian Estrada indices of G are defined as $EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$ and $LEE(G) = \sum_{i=1}^{n} e^{\mu_i}$ where λ_i and μ_i, $1 \leq i \leq n$, denote the eigenvalues of $A(G)$ and $L(G)$, see [8, 13]. To see some previous results, you can refer to [1, 2, 15]. The Estrada index can be used to measure the folding degree of long-chain protein [6].

Let $\{v_1, v_2, \cdots v_n\}$ be vertex set of graph G. The degree of vertex v_i denoted by $d(v_i) = d_i$. The first Zagreb index of graph G, is defined as $\sum_{i=1}^{n} d_i^2$ and denoted by $M_1 = M_1(G)$, see [7, 11, 14]. In this paper, we denote the $n-$vertex complete graph, path and cycle by K_n, P_n and C_n and complete bipartite graph by $K_{m,n}$.

This paper is organized as follows: In the second Section, we express the required results of the past. In third Section, Part 1, some new bounds for $EE(G)$ are obtained in terms of the number of fixed edges of $n-$vertex graph G. Furthermore, we give some new bounds for $EE(G)$ in terms of the number of spanning trees. In third Section, Part 2, we give some new bounds for $LEE(G)$ in terms of the number of edges and spanning trees.

Throughout this paper our notations are standard and can be taken from [3, 4, 9, 10].

2 Preliminaries

An alternating sequence of the form $v_0 e_1 v_1 e_2 \ldots v_k e_k v_k$, of vertices and edges, such that for any $i = 0, 1, \ldots, k-1, v_{i-1}, v_i = e_i($ the edge $e_i)$ is called a walk of length k in a graph G from v_0 to v_k. We invite the interested readers to see papers [12, 18] standard results about walk on graphs.

In this paper we denote by $w_k(ij)$ the number of all walks of length k starting from v_i and terminating at v_j.

Theorem 1. *[3] Let A be the adjacency matrix of a graph G and v_i and v_j be two arbitrary vertices of G. Then $[A^k]_{ij} = w_k(ij)$.*

Theorem 2. *[4] Let A be the adjacency matrix of a graph G. Then the $k-th$ spectral moment of G, $Sm(G,k) = \sum_{i=1}^{n} \lambda_i^k$, is equal to $tr(A^k) = \sum_{i=1}^{n} w_k(ii)$.*

In the next theorem, $\Phi(H)$ represents the number of subgraphs of type H in graph G.

Theorem 3. *[18] Let G be a simple graph. Then $Sm(G,1) = 0$, $Sm(G,2) = 2m$, $Sm(G,3) = 6T(G)$, $Sm(G,4) = 2M_1 - 2m + 8q$, $Sm(G,5) = 10P(G) + 30T(G) + 10T'(G)$ and $Sm(G,6) = 2\Phi(P_2) + 12\Phi(P_3) + 6\Phi(P_4) + 12\Phi(K_{1,3}) + 12\Phi(H_2) + 36\Phi(H_3) + 24\Phi(H_4) + 24T(G) + 48q(G) + 12\Phi(C_6)$, where $T(G)$, M_1, $q(G)$, $P(G)$ and $T'(G)$ are the number of triangles, the first Zagreb index, the number of pentagons and the number of triangles with a pendant edge.*

Let L be Laplacian matrix of a graph G. Then the $k-$th Laplacian spectral moment of G, $Lsm(G,k)$ is defined as $\sum_{i=1}^{n} \mu_i^k$.

Theorem 4. *[17] Let G be a simple graph. Then $Lsm(G,1) = 2m$, $Lsm(G,2) = 2m + M_1$, $Lsm(G,3) = -6T(G) + 3M_1 + \sum_{i=1}^{n} d_i^3$, $Lsm(G,4) = 2M_1 - 2m + 4\sum_{i=1}^{n} d_i^3 + \sum_{i=1}^{n} d_i^4 + 4\sum_{v_iv_j \in E(G)}^{m} d_i d_j + 8q$ and $Lsm(G,5) = -5M_1 + 5\sum_{i=1}^{n} d_i^3 + 5\sum_{i=1}^{n} d_i^4 + \sum_{i=1}^{n} d_i^5 + 30T(G) - 10P(G) - 10\sum_{i=1}^{n} d_i^2 t_i + 5\sum_{v_i \sim v_j}^{m} (d_i^2 d_j + d_i d_j^2) + 10\sum_{v_i \sim v_j}^{m} d_i d_j - 10\sum_{i=1}^{n} d_i t_i + 10\sum_{i=1}^{n} d_i q_i - 10\sum_{v_i \sim v_j}^{m} CN(v_i, v_j) d_i d_j$ where $CN(v_i, v_j)$ is the number of common nighbors of v_i and v_j in G.*

3 Main results

This section has two parts. The first part includes relationships between the number of spanning trees and Estrada index. In addition, we fined new bounds for Estrada index of graphs. The second part includes relationships between the number of spanning trees and Laplacian Estrada index. All graphs in this section are connected.

3.1 Estrada index

In this part, we fined some bounds based on the number of vertices, edges and spanning trees of graphs. We improve the results of paper [5], authored by Jose and others, in the next theorem.

Theorem 5. *Let G be a graph of order n, size m and q quadrangles. Then*

$$EE(G) \geq n-1+\frac{M_1}{12}-\frac{2m^2}{n^2}-\frac{2m^4}{3n^4}-\frac{4m^6}{45n^6}+\frac{11m}{12}+\frac{q}{3}+\frac{Sm(G,6)}{6!}+\cosh(2m/n),$$

with equlity if and only if G is an edgeless graph.

Proof. Since $EE(G)=\sum_{k=0}^{\infty}\frac{Sm(G,k)}{k!}$, $EE(G)\geq\sum_{k=0}^{\infty}\frac{Sm(G,2k)}{(2k)!}=n+m+\frac{2M_1-2m+8q}{4!}+\frac{Sm(G,6)}{6!}+\sum_{k=4}^{\infty}\frac{Sm(G,2k)}{(2k)!}$, with equality if and only if G is bipartite. Its clear that for $k\geq 4$, $Sm(G,2k)\geq\lambda_{Max}^{2k}$ with equality if and only if G is an edgeless graph. $\lambda_{Max}^{2k}\geq(\frac{2m}{n})^{2k}$, with equality if and only if $G=K_2$ or an edgeless graph. Thus $Sm(G,2k)\geq(\frac{2m}{n})^{2k}$ with equality if and only if G is an edgeless graph. Then

$$EE(G)\geq n+m+\frac{2M_1-2m+8q}{4!}+\frac{Sm(G,6)}{6!}+\sum_{k=4}^{\infty}\frac{(\frac{2m}{n})^{2k}}{(2k)!}$$

$$=n-1+\frac{M_1}{12}-\frac{2m^2}{n^2}-\frac{2m^4}{3n^4}-\frac{4m^6}{45n^6}+\frac{11m}{12}+\frac{q}{3}+\frac{Sm(G,6)}{6!}+\cosh(2m/n),$$

with equlity if and only if G is an edgeless graph. ■

Theorem 6. *Let G be a graph of order n and size m. Then*

$$e^{-\sqrt{\frac{2m(n-1)}{n}}}+(n-1)e^{\sqrt{\frac{2m}{n(n-1)}}}\leq EE(G)\leq(n-1)e^{-\sqrt{\frac{2m}{n(n-1)}}}+e^{\sqrt{\frac{2m(n-1)}{n}}},$$

with equlity in left and right if and only if G is K_2 and K_n, respectively.

Proof. First, we find the extremum of the function $f(x_1,x_2,\cdots,x_n)=\sum_{i=1}^{n}e^{x_i}$ with respect to the conditions $\sum_{i=1}^{n}x_i=0$ and $\sum_{i=1}^{n}x_i^2=2m$. By the method of Lagrange multipliers with two constraints, we have $e^{x_i}=\lambda+2\mu x_i$, for some $\lambda\neq 0$ and $\mu\neq 0$. It's clear that this equation has at most two solutions x_1 and x_2. Thus $n_0x_1+(n-n_0)x_2=0$ and $n_0x_1^2+(n-n_0)x_2^2=2m$, for a n_0. Then $x_1=\sqrt{\frac{2m(n-n_0)}{nn_0}}$ and $x_2=-\sqrt{\frac{2mn_0}{n(n-n_0)}}$. This means that $f(x_1,x_2,\cdots,x_n)=n_0e^{x_1}+(n-n_0)e^{x_2}$. Now by substituting 1 to $n-1$ instead of n_0, we have $f_{Max}=(n-1)e^{-\sqrt{\frac{2m}{n(n-1)}}}+e^{\sqrt{\frac{2m(n-1)}{n}}}$ and

$f_{Min} = e^{-\sqrt{\frac{2m(n-1)}{n}}} + (n-1)e^{\sqrt{\frac{2m}{n(n-1)}}}$. Thus by choosing $x_i = \lambda_i$, we have

$$e^{-\sqrt{\frac{2m(n-1)}{n}}} + (n-1)e^{\sqrt{\frac{2m}{n(n-1)}}} \leq EE(G) \leq (n-1)e^{-\sqrt{\frac{2m}{n(n-1)}}} + e^{\sqrt{\frac{2m(n-1)}{n}}}.$$

■

In the next theorem, we find a relationship between $EE(G)$ and the number of spanning trees of $k-$regular graph G. In [16], we can see Geometric−Arithmetic inequality(more briefly the AM−GM inequality) and its reverse as follows: let $x_1, x_2, ..., x_n$ are positive real numbers, $h = \frac{Max(x_i)}{Min(x_i)}$ and $S(1) = 1$ and $S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{eLn(h)}, h \neq 1$,

$$\sqrt[n]{\Pi_{i=1}^{n} x_i} \leq \frac{\sum_{i=1}^{n} x_i}{n} \leq S(h)\sqrt[n]{\Pi_{i=1}^{n} x_i},$$

with equality if and only if $x_1 = x_2 = ... = x_n$.

Theorem 7. *Let G be a $k-$regular bipartite $n-$vertex graph and $t(G)$ be the number of spanning trees in G. Then*

$$2\cosh(k) + e^{-k}(n-2)e^{\sqrt[n-2]{\frac{nt(G)}{2k}}} \leq EE(G) \leq 2\cosh(k) + S(h)e^{-k}(n-2)e^{\sqrt[n-2]{\frac{nt(G)}{2k}}},$$

$h = \frac{Max(k-\lambda_i)}{Min(k-\lambda_i})$, $-k < \lambda_i < k$, *with equlity if and only if G is compelete bipartite graph.*

Proof. Since G is $k-$regular and bipartite, k and $-k$ are eigenvalues of G. Then $EE(G) = e^k + e^{-k} + \sum_{-k<\lambda_i<k} e^{\lambda_i} = 2\cosh(k) + \sum_{-k<\lambda_i<k} e^{\lambda_i}$. Because of that G is bipartite, $\sum_{-k<\lambda_i<k} e^{\lambda_i} = \sum_{-k<\lambda_i<k} e^{-\lambda_i}$. So

$$EE(G) = 2\cosh(k) + e^{-k} \sum_{-k<\lambda_i<k} e^{k-\lambda_i}$$

$$= 2\cosh(k) + e^{-k} \sum_{r=0}^{\infty} \frac{\sum_{-k<\lambda_i<k}(k-\lambda_i)^r}{r!}.$$

By AM−GM inequality,

$$\sum_{-k<\lambda_i<k} (k-\lambda_i)^r \geq (n-2)\sqrt[n-2]{\Pi_{-k<\lambda_i<k}(k-\lambda_i)^r},$$

with equality if and only if $(k-\lambda_i)^r = (k-\lambda_j)^r$ if and only if $\lambda_i = \lambda_j$. By matrix−tree theorem $\Pi_{-k<\lambda_i<k}(k-\lambda_i)^r = \frac{nt(G)}{2k}$. Therefore $EE(G) \geq 2\cosh(k) + e^{-k}(n-2)\sum_{r=0}^{\infty}\frac{(\sqrt[n-2]{\frac{nt(G)}{2k}})^r}{r!} = 2\cosh(k) + e^{-k}(n-2)e^{\sqrt[n-2]{\frac{nt(G)}{2k}}}$. The inequality is sharp if and only if $\lambda_i = \lambda_j$ if and only if $\lambda_i = 0$ for $\lambda_i \neq k, -k$ if and only if G has three distinct eigenvalue if and only if G is complete bipartite graph.

To prove the inequality on the right, you can apply the reverse of AM−GM inequality. ■

3.2 Laplacian Estrada index

In this part, at first we compute $Lsm(G,6)$ and use it in some inequalities. Next, we obtain inequalities according to the number of vertices, edges and spanning trees. We know that $L = \Delta - A(G)$ and $Lsm(G,6) = tr(L^6)$.

$$\begin{aligned} tr(L^6) &= tr(A^6) + tr(\Delta^6) + 6tr(\Delta^2A^4) + 6tr(\Delta^4A^2) - 6tr(\Delta^5A) - 6tr(\Delta A^5) \\ &- 6tr(\Delta^3A^3) + 6tr(\Delta^3A\Delta A) + 6tr(A^3\Delta A\Delta) - 6tr(\Delta^2A^2\Delta A) \\ &- 6tr(A^2\Delta^2A\Delta) + 3tr(A^2\Delta A^2\Delta) + 3tr(A\Delta^2A\Delta^2) - 2tr(\Delta A\Delta A\Delta A) \end{aligned}$$

$$\begin{aligned} tr(\Delta^2A\Delta A^2) &= \sum_{i=1}^{n}\sum_{k=1}^{n}(\Delta^2A)_{ik}(\Delta A^2)_{ki} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n}d_i^2(A)_{ik}d_k(A^2)_{ki} = \sum_{v_i\sim v_k}[d_i^2d_k + d_k^2d_i]t_{ik}, \end{aligned}$$

where t_{ik} is the number of triangles containing edge v_iv_k.

$$tr(\Delta^4A^2) = \sum_{i=1}^{n}\sum_{k=1}^{n}(\Delta^4)_{ik}(A^2)_{ki} = \sum_{i=1}^{n}d_i^4.(A^2)_{ii} = \sum_{i=1}^{n}d_i^5$$

$$tr(\Delta^2A^4) = \sum_{i=1}^{n}\sum_{k=1}^{n}(\Delta^2)_{ik}(A^4)_{ki} = \sum_{i=1}^{n}d_i^2(A^4)_{ii}$$

$$= \sum_{i=1}^{n} d_i^2 \left(d_i + S(v_i) - d_i + 2q_i + 2\binom{d(v_i)}{2} \right)$$
$$= \sum d_i^2 \big(S(v_i) + 2q_i + d_i(d_i - 1) \big),$$

where q_i and $S(v_i)$ are the number of quadrangle containing v_i and $S(v_i)$ is the sum of degrees of the neighbors of vertex v_i.

$$tr(\Delta^6) = \sum_{i=1}^{n} (\Delta^6)_{ii} = \sum_{i=1}^{n} d_i^6$$
$$tr(\Delta^3 A^3) = \sum_{i=1}^{n} d_i^3 (A^3)_{ii} = 2\sum_{i=1}^{n} d_i^3 t_i,$$

where t_i is the number of triangles containing vertex v_i.

$$tr(\Delta^2 A \Delta^2 A) = \sum_{i=1}^{n}\sum_{k=1}^{n} (\Delta^2 A)_{ik} (\Delta^2 A)_{ki} = \sum_{i=1}^{n} d_i^2 (A)_{ik}.d_k^2.(A)_{ki}$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{n} d_i^2.d_k^2 (A)_{ik}^2 = 2 \sum_{v_i \sim v_k} d_i^2.d_k^2$$

$$tr(\Delta A^3 \Delta A) = \sum_{i=1}^{n}\sum_{k=1}^{n} (\Delta A^3)_{ik} (\Delta A)_{ki} = 2 \sum_{v_i \sim v_k} d_i d_k (A^3)_{ik} a_{ki}$$
$$= 2 \sum_{v_i \sim v_k} d_i d_k (q_{ik} + d_k + d_i - 1),$$

where q_{ik} is the number of quadragles containing edge $v_i v_k$.

$$tr(\Delta^5 A) = \sum_{i=1}^{n}\sum_{k=1}^{n} (\Delta^5)_{ik} (A)_{ki} = 0$$

$$tr(\Delta A^2 \Delta A^2) = \sum_{i=1}^{n}\sum_{k=1}^{n} d_i (A^2)_{ik}.d_k (A^2)_{ik}$$
$$= 2\sum_{i=1}^{n} d_i^4 + 2 \sum_{1 \le i < j \le n} d_i d_j CN^2(i,j),$$

where $CN(i,j)$ is the number of common neighbors of vertices v_i and v_j.

$$tr(\Delta A\Delta^3 A) = \sum_{i=1}^{n}\sum_{k=1}^{n}(\Delta A)_{ik}(\Delta^3 A)_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} d_i a_{ik}.d_k^3 a_{ki}$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{n} d_i d_k^3 a_{ik} = \sum_{v_i\sim v_k}(d_i d_k^3 + d_i^3 d_k)$$

$$tr(\Delta^2 A^2\Delta A) = \sum_{i=1}^{n}\sum_{k=1}^{n}(\Delta^2 A^2)_{ik}(\Delta A)_{ki}$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{n} d_i(A^2)_{ik}.d_k a_{ki} = \sum_{v_i\sim v_k}(d_i^2 d_k + d_k^2 d_i)t_{ik}$$

$$tr(\Delta A\Delta A\Delta A) = \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n} d_r d_i d_k a_{kr} a_{ri} a_{ik}$$
$$= 3! \sum_{T(v_r,v_i,v_k)} d_r d_i d_k,$$

where $T(v_r, v_i, v_k)$ is a triangle containing vertices v_r, v_j and v_k.

$$tr(A^5\Delta) = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}^5 \Delta_{ki} = \sum_{i-1}^{n} d_i(A^5)_{ii}$$
$$= \sum_{i=1}^{n} d_i(2p_i + 10t_i + 2N(\tau') + 4N(\tau'') + 2N(\tau''')),$$

where p_i is the number of pentagones containing vertex v_i and τ', τ'' and τ''' depicted in Figure 1.

$$tr(L^6) = Sm(G,6) + \sum_{i=1}^{n} d_i^6 + 6\sum_{i=1}^{n} d_i^2(S(v_i) + 2q_i + d_i(d_i - 1))$$
$$+ 6\sum_{i=1}^{n} d_i^5 - 12\sum_{i=1}^{n} d_i^3 t_i + 6\sum_{v_i\sim v_k}(d_i d_k^3) + d_k d_i^3$$

$$
\begin{aligned}
&+ 12 \sum_{v_i \sim v_k} d_i d_k (q_{ik} + d_k + d_i - 1) - 6 \sum_{v_i \sim v_k} (d_i^2 d_k + d_k^2 d_i) t_{ik} \\
&+ 3 \left(\sum d_i^4 + 2 \sum_{1 \le i < j \le n} d_i d_j CN^2(i,j) \right) + 6 \sum_{v_i \sim v_k} d_i^2 d_k^2 \\
&- 18 \sum_{T(v_r, v_i, v_k)} d_r d_i d_k - 6 \sum_{i=1}^{n} d_i (2p_i + 10t_i + 2\tau' + 4\tau'' + 2\tau''') \\
&- 6 \sum (d_i^2 d_k + d_k^2 d_i) t_{ik}
\end{aligned}
$$

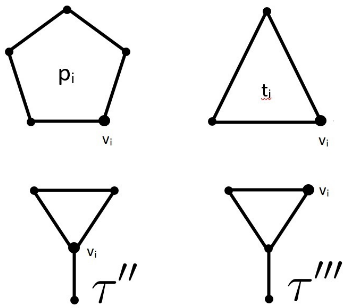

Figure 1. Graphs generated by closed walk of length 5 from vertex v_i.

Theorem 8. *Let G be an $n-$vertex and $m-$edge graph. Then*

$$
LEE(G) \ge 1 + \sum_{i=2}^{6} \left(\frac{Lsm(G,i)}{i!} - \frac{(2m)^i}{i!(n-1)^{i-1}} \right) + (n-1)e^{\frac{2m}{n-1}},
$$

with equality if and only if G is K_2.

Proof. Since zero is one of the eigenvalues of G, $LEE(G) = 1 + \sum_{i=1}^{n-1} e^{\mu_i} = 1 + \sum_{k=0}^{\infty} \frac{\sum_{i=1}^{n-1} \mu_i^k}{k!} = 1 + \sum_{k=0}^{\infty} \frac{Lsm(G,k)}{k!}$. The minimum value of $Lsm(G,k)$ is equal to $(n-1)(\frac{2m}{n-1})^k$. Then $LEE(G) = 1 + \sum_{k=0}^{6} \frac{Lsm(G,k)}{k!} + \sum_{k=7}^{\infty} \frac{Lsm(G,k)}{k!} \ge 1 + \sum_{k=0}^{6} \frac{Lsm(G,k)}{k!} + (n-1) \sum_{k=7}^{\infty} \frac{(\frac{2m}{n-1})^k}{k!} = 1 + \sum_{i=2}^{6} (\frac{Lsm(G,i)}{i!} - \frac{(2m)^i}{i!(n-1)^{i-1}}) + (n-1)e^{\frac{2m}{n-1}}$. With equality if and only if G has two eigenvalues 0 and $\frac{2m}{n-1}$ if and only if G is K_2. ■

In the next theorem, the upper and lower bounds of Laplacian Estrada index of graphs with a fixed number of edges and a fixed the first Zagreb index are provided. The next theorem can be compared with the results of paper [19].

Theorem 9. *Let G be a $n-$vertex and $m-$edge graph. If $\Delta = (n-1)(2m+\sum_{i=1}^{n} d_i^2) - 4m^2$, then*

$$1 + e^{\frac{2m}{n-1} - \frac{1}{n-1}\sqrt{(n-2)\Delta}} + (n-2)e^{\frac{2m}{n-1} + \frac{1}{n-1}\sqrt{\frac{\Delta}{n-2}}} \le LEE(G) \le$$

$$1 + (n-2)e^{\frac{2m}{n-1} - \frac{1}{n-1}\sqrt{\frac{\Delta}{n-2}}} + e^{\frac{2m}{n-1} + \frac{1}{n-1}\sqrt{(n-2)\Delta}}\ .$$

Proof. Consider the function $f(x_1, x_2, \cdots, x_{n-1}) = \sum_{i=1}^{n-1} x_i^k$. We find the extremum of f with respect to conditions $x_1 + x_2 + \cdots + x_{n-1} = 2m$ and $x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = 2m + \sum_{i=1}^{n} d_i^2 (:= c)$. By the method of Lagrange multipliers with two constraints, we have $e^{x_i} = \lambda + 2\mu x_i$, for some $\lambda \neq 0$ and $\mu \neq 0$. This equation has at most two solutions x_1 and x_2. Thus $ax_1 + (n-1-a)x_2 = 2m$ and $ax_1^2 + (n-1-a)x_2^2 = c$, for a a. Consider $n-1-a = b$ and $\Delta = (n-1)(2m + \sum_{i=1}^{n} d_i^2) - 4m^2$. Then $x_1 = \frac{2m}{n-1} + \frac{1}{n-1}\sqrt{\frac{b\Delta}{a}}$ and $x_2 = \frac{2m}{n-1} - \frac{1}{n-1}\sqrt{\frac{a\Delta}{b}}$. This means that $f(x_1, x_2, \cdots, x_{n-1}) = ae^{x_1} + be^{x_2}$. Now by substituting 1 to $n-2$ instead of a, we have $f_{Max} = (n-2)e^{\frac{2m}{n-1} - \frac{1}{n-1}\sqrt{\frac{\Delta}{n-2}}} + e^{\frac{2m}{n-1} + \frac{1}{n-1}\sqrt{(n-2)\Delta}}$ and $f_{Min} = e^{\frac{2m}{n-1} - \frac{1}{n-1}\sqrt{(n-2)\Delta}} + (n-2)e^{\frac{2m}{n-1} + \frac{1}{n-1}\sqrt{\frac{\Delta}{n-2}}}$. Thus by choosing $x_i = \mu_i$, the result is obtained. ■

In the next theorem, We find a relationship between spanning trees and Laplacian Estrada index of graphs.

Theorem 10. *Let G be a connected graph. Then*

$$(n-1)e^{\sqrt[n-1]{nt(G)}} \le LEE(G) - 1 - \sum_{k=1}^{6} \left(\frac{Lsm(G,k) - (n-1)\sqrt[n-1]{nt(G)}^{\,k}}{k!} \right)$$

$$\le (n-1)S(h)e^{\sqrt[n-1]{nt(G)}}.$$

Proof. $LEE(G) = 1 + \sum_{k=0}^{6} \frac{Lsm(G,k)}{k!} + \sum_{k=7}^{\infty} \frac{Lsm(G,k)}{k!}$. By geometric-arithmetic inequality,

$$(n-1)\sqrt[n-1]{nt(G)}^{k} \leq Lsm(G,k) \leq (n-1)S(h)\sqrt[n-1]{nt(G)}^{k}.$$

Then

$$(n-1)\sum_{k=7}^{\infty}\frac{\sqrt[n-1]{nt(G)}^{k}}{k!} \leq \sum_{k=7}^{\infty}\frac{Lsm(G,k)}{k!} \leq (n-1)S(h)\sum_{k=7}^{\infty}\frac{\sqrt[n-1]{nt(G)}^{k}}{k!}.$$

Since $\sum_{k=7}^{\infty}\frac{\sqrt[n-1]{nt(G)}^{k}}{k!} = e^{\sqrt[n-1]{nt(G)}} - \sum_{k=0}^{6}\frac{\sqrt[n-1]{nt(G)}^{k}}{k!}$, the proof is complete. ■

Conclusion. To improve the bounds obtained for Estrada index with the above method, three conditions $\sum_{i=1}^{n}\lambda_i = 0$, $\sum_{i=1}^{n}\lambda_i^2 = 2m$ and $\sum_{i=1}^{n}\lambda_i^3 = 6T(G)$ can be used. Also, this can be improved by adding condition $\sum_{i=1}^{n}\lambda_i^4 = 2M_1 - 2m$. The same method can be used to improve the bounds of Laplacian Estrada index.

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