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A Note on an Inequality Between Energy and Sombor Index of a Graph

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Abstract

The Sombor index of graph G is defined as $\sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$, where d_u and d_v are the degree of vertices u and v in G, respectively. The energy of G is defined as the sum of absolute values of all eigenvalues of its adjacency matrix and denoted by $\mathcal{E}(G)$. It was proved that if G is a graph of order at least 3, then $\mathcal{E}(G) < SO(G)$. In this paper, we strengthen this result by showing that if G is a connected graph of order n which is not $P_n (n \leq 8)$, then $\mathcal{E}(G) \leq \frac{SO(G)}{2}$.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ denote the set of its vertices and edges, respectively. By the *size* of G, we mean the number of its edges. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph G is called *r-regular*

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whenever $\Delta(G) = \delta(G) = r$. A path of order n is denoted by P_n. The *star graph* of order n, denoted by S_n. Also, we denote the degree of vertex $u \in G$ by d_u.

In this paper, the *energy* of a graph G, is shown by $\mathcal{E}(G)$ and is defined as the sum of the absolute values of all eigenvalues of its adjacency matrix. Hence,

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

In [7] the energy of the path P_n was calculated as below,

$$\mathcal{E}(P_n) = \begin{cases} \dfrac{2}{sin\dfrac{\pi}{2(n+1)}} - 2 & if\ n \equiv 0 \pmod 2 \\ \\ \dfrac{2cos\dfrac{\pi}{2(n+1)}}{sin\dfrac{\pi}{2(n+1)}} - 2 & if\ n \equiv 1 \pmod 2 \end{cases}$$

As a pioneer in 1978 the energy of a graph was defined by Ivan Gutman in [4]. Next, the concept of Sombor index was introduced by him in [3] in the chemical graph theory. The Sombor index of G, $SO(G)$, is defined as $\sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$. Recently, the relation between Sombor index and some parameters of a graph have been extensively studied by many authors, for instance see [5], [6], [8], [9] and [10].

In [11] Ülker et al obtained the bound $\mathcal{E}(G)\Delta(G)^3 \geq SO(G)$ for a graph G, and for Δ-regular graphs they showed that, $\mathcal{E}(G)\Delta(G)^2 \geq SO(G)$. Also they proved that if G is a regular graph, then $\mathcal{E}(G) \leq SO(G)^2$ and the equality holds if $G \cong K_2$.

In this paper we attempt to improve the bounds of energy in terms of $SO(G)$ and show that if G is a connected graph of order at least 9, then $\mathcal{E}(G) \leq \dfrac{SO(G)}{2}$.

2 Preliminaries

In the following, we state a lemma which is used in our proofs.

Lemma 1. *[1] Let G be a graph and $H_1, \ldots, H_k$ be a partition of $E(G)$. Then $\mathcal{E}(G) \leq \sum_{i=1}^{k} \mathcal{E}(H_i)$.*

The following result shows that the energy of a graph with minimum degree at least 2, does not exceed its Sombor index.

Lemma 2. *[11] Let G be a connected graph with minimum vertex degree $\delta \geq 2$. Then $\mathcal{E}(G) \leq SO(G)$.*

The following result gives an inequality between energy and Sombor index of a graph.

Theorem 1. *[12] Let G be a connected graph with n vertices. If $n \geq 3$, then $\mathcal{E}(G) < SO(G)$.*

Theorem 2. *[2] For a graph G with vertices $v_1, \ldots, v_n$ of degrees $d_1, \ldots, d_n$ we have*

$$\mathcal{E}(G) \leq \sum_{i=1}^{n} \sqrt{d_i} \leq \sqrt{2mn}.$$

The second inequality holds if and only if G is a regular graph.

3 Results

Here, we improve the bound given in Theorem 1. We start by the following remark.

Remark 1. *By a computer computation one can see that for every connected graph G of order at most 10, $\mathcal{E}(G) \leq \dfrac{SO(G)}{2}$, except for, $P_r, r = 2, \ldots, 8$. Also, we have,*

Now, we are in a position to prove our main theorem.

Theorem 3. *If G is a connected graph of order n which is not $P_n (n \leq 8)$, then $\mathcal{E}(G) \leq \dfrac{SO(G)}{2}$.*

Table 1

n	$\mathcal{E}(P_n) - \frac{SO(P_n)}{2}$	n	$\mathcal{E}(P_n) - \frac{SO(P_n)}{2}$
2	≈ 1.3	3	≈ 0.6
4	≈ 0.83	5	≈ 0.4
6	≈ 0.51	7	≈ 0.17
8	≈ 0.22		

Proof. By Remark 1, we may assume that $n \geq 11$. We prove the assertion by induction on the size of G.

Case 1. G is not a tree.
Let C be a cycle in G. Suppose that $v \in V(C)$ has maximum degree among all vertices of C. If $d_v = 2$, then $G = C_n$ and by Theorem 2, $\mathcal{E}(G) \leq n\sqrt{2} = \frac{SO(G)}{2}$. So assume that $d_v \geq 3$. If $d_v \geq 4$, then let $e \in E(C)$ and e is incident with v. Let $G' = G \setminus e$. By Lemma 1 and induction hypothesis, we have

$$\mathcal{E}(G) \leq \mathcal{E}(G') + 2 \leq \frac{SO(G') + 4}{2} \leq \frac{SO(G') + \sqrt{20}}{2} \leq \frac{SO(G)}{2}.$$

Therefore, assume that $d_v = 3$ and e is an edge as before. Clearly, we have

$$\begin{aligned}\mathcal{E}(G) &\leq \mathcal{E}(G') + 2 \leq \frac{SO(G') + 4}{2} \\ &\leq \frac{SO(G') + \sqrt{13} + (\sqrt{13} - \sqrt{10})}{2} \leq \frac{SO(G)}{2}.\end{aligned}$$

Case 2. G is a tree.
If $G = P_n$, then define $G' = G \setminus E(P_3)$. Now, by Lemma 1 and induction hypothesis, we find that

$$\begin{aligned}&\mathcal{E}(G) \leq \mathcal{E}(G') + 2\sqrt{2} \\ &\quad \leq \frac{SO(G') + 4\sqrt{2}}{2} = \frac{SO(G') + \sqrt{5} + 2\sqrt{2} + (2\sqrt{2} - \sqrt{5})}{2} = \frac{SO(G)}{2},\end{aligned}$$

as desired. Thus assume that $\Delta(G) \geq 3$.

First suppose that $\Delta(G) \geq 5$ and $d_v \geq 5$. If $G = S_n$, then

$$\mathcal{E}(G) = 2\sqrt{n-1} \leq \frac{1}{2}(n-1)\sqrt{(n-1)^2+1} = \frac{SO(G)}{2}.$$

Thus we may assume that G contains an edge e incident with v such that e is not a pendant edge. Let $G' = G \setminus e$ and G_1 and G_2 be two components of G'. Now, by Lemma 1, Remark 1 and induction hypothesis, we have

$$\begin{aligned}\mathcal{E}(G) &\leq \mathcal{E}(G') + 2 = \mathcal{E}(G_1) + \mathcal{E}(G_2) + 2 \\ &\leq \frac{SO(G_1) + SO(G_2) + 4 + 1.3}{2} \qquad \text{(See Table 1)} \\ &\leq \frac{SO(G_1) + SO(G_2) + \sqrt{29}}{2} \leq \frac{SO(G)}{2},\end{aligned}$$

as desired.

Now, let $\Delta(G) = 4$ and $d_v = 4$. Since $|V(G)| \geq 11$, there is an edge e incident with v such that no component of $G' = G \setminus e$ is P_2. Let G_1 and G_2 be two components of G'.

So by Lemma 1, Remark 1 and induction hypothesis, we have

$$\begin{aligned}\mathcal{E}(G) &\leq \mathcal{E}(G') + 2 = \mathcal{E}(G_1) + \mathcal{E}(G_2) + 2 \\ &\leq \frac{SO(G_1) + SO(G_2) + 4 + (2 \times 0.83)}{2} \qquad \text{(See Table 1)} \\ &\leq \frac{SO(G_1) + SO(G_2) + \sqrt{20} + 3(\sqrt{32} - \sqrt{25})}{2} \leq \frac{SO(G)}{2}.\end{aligned}$$

Thus assume that $\Delta(G) = 3$ and $d_v = 3$. If G contains two pendant incident edges e and f, then define $G' = G \setminus \{e, f\}$. Now, by Lemma 1, Remark 1 and induction hypothesis, we have

$$\begin{aligned}\mathcal{E}(G) \leq \mathcal{E}(G') + 2\sqrt{2} &\leq \frac{SO(G') + 4\sqrt{2}}{2} \\ &\leq \frac{SO(G') + 2\sqrt{10}}{2} \leq \frac{SO(G)}{2},\end{aligned}$$

as desired. Hence suppose that G has no two incident pendant edges. Now, define

$$s := min\{d(x,y)\,|\,d_x = 1\,,\, d_y = 3\,,\, x,y \in V(G)\}.$$

Let $s = d(u,v)$, where $d_u = 1$ and $d_v = 3$. Three following cases can be considered:

(i) $s = 1$. Let $G' = G \setminus u$. By Lemma 1 and induction hypothesis, we find that

$$\begin{aligned}\mathcal{E}(G) \leq \mathcal{E}(G') + 2 &\leq \frac{SO(G') + 4}{2} \\ &\leq \frac{SO(G') + \sqrt{10} + 2(\sqrt{18} - \sqrt{13})}{2} \leq \frac{SO(G)}{2},\end{aligned}$$

as desired.

(ii) $s = 2$. Let $G' = G \setminus \{e, f\}$, where e and f are two incident edges such that f is a pendant edge and e is incident with v. By Lemma 1 and induction hypothesis, we have

$$\begin{aligned}\mathcal{E}(G) \leq \mathcal{E}(G') + 2\sqrt{2} &\leq \frac{SO(G') + 4\sqrt{2}}{2} \\ &\leq \frac{SO(G') + \sqrt{13} + \sqrt{5}}{2} \leq \frac{SO(G)}{2},\end{aligned}$$

as desired.

(iii) $s \geq 3$. Let e, f, h are three edges incident with v. Let $G' = G \setminus \{e, f, h\}$ and G_1, G_2, G_3 be connected components of G'. If $G_i \simeq P_r$ for some i, then clearly $r \geq 3$.
By Lemma 1, one may see that $\mathcal{E}(G) \leq \mathcal{E}(G') + 2\sqrt{3} = \sum_{i=1}^{3} \mathcal{E}(G_i) + 2\sqrt{3}$. So, by induction hypothesis and Remark 1, we conclude that

$$\begin{aligned}\mathcal{E}(G) &\leq \frac{\sum_{i=1}^{3} SO(G_i) + (6 \times 0.83) + 4\sqrt{3}}{2} \qquad \text{(See Table 1)} \\ &\leq \frac{\sum_{i=1}^{3} SO(G_i) + 3\sqrt{13} + 3(\sqrt{18} - \sqrt{13})}{2} \leq \frac{SO(G)}{2},\end{aligned}$$

as desired and the proof is complete. ■

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