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Infinite Numbers of Infinite Classes L-Borderenergetic Graphs

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Abstract

The graph G of order n is an L-borderenergetic graph which means it has the same Laplacian energy as the complete graph K_n. In this paper, we find that the combination of complete bipartite graphs and stars can construct infinite numbers of infinite classes L-borderenergetic graphs. We give two infinite numbers of infinite classes L-borderenergetic graphs and two infinite classes L-borderenergetic graphs under the operators union, join and their mixed. This research could provide experience for further study the structural characteristics of L-borderenergetic graphs.

1 Introduction

A graph G is a simple graph that is to say it has at most one edge between two distinct vertices and no edge from one vertice to the same vertice. The undirected graph is a graph that has no direction associated with its edge. In this paper, we are only concerned the simple and undirected graph. Let $G = (V, E)$ be a graph of order $n = |V|$ and $V = \{v_1, v_2, \cdots, v_n\}$,

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the edge $v_xv_y \in E$. The adjacency matrix $A(G)$ of G is the 0-1 matrix, where the entry $a_{xy} = 1$ when $v_xv_y \in E$ and $a_{xy} = 0$ otherwise. The Laplacian matrix $L(G)$ of G is defined as $L(G) = D(G) - A(G)$, where $D(G) = diag(d_1, d_2, \cdots, d_n)$ whose entry d_i is the degree of v_i. The energy of a graph G, introduced by Ivan Gutman [11], is defined by

$$E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where λ_i are the eigenvalues of G. The energy of graph is not only a mathematical problem, but also comes from the chemical concept of quantum chemistry [24], which has certain chemical application significance [19,20].

In this paper, we mainly study the Laplacian energy. The Laplacian energy of G was introduced by Ivan Gutman and Bo Zhou [12]

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i - \overline{d}\right|,$$

where μ_i are the Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. The Laplacian energy has been received a lot of attention, such as [1–4, 13–15, 23, 29, 30].

The Laplacian borderenergetic (L-borderenergetic) graph G of order n is a graph that its Laplacian energy is equal to the complete graph K_n, i.e. $LE(G) = LE(K_n) = 2(n-1)$. The concept of L-borderenergetic graphs was first proposed by Fernando Tura [25]. The research on L-borderenergetic graphs is mainly divided into two aspects, one is search and construction of the L-borderenergetic graphs [10, 17, 26–28], the other is investigate some properties of L-borderenergetic graphs [5–8, 16, 21, 22]. These works are very helpful for our study.

In this paper, we mainly investigate how to construct L-borderenergetic graphs by using the combination of complete bipartite graphs and stars. In the [9], they constructed infinite classes L-borderenergetic graphs, which gives a good inspiration for our works. We find the Laplacian energy of complete bipartite graphs can be greater than the complete graphs and the stars will be less than the complete graphs, so we guess their combination may construct the L-borderenergetic graphs and verified this guess. Firstly, we use the union operator to construct the L-borderenergetic graphs and find the condition for constructing the L-borderenergetic graphs, construct 1 infinite numbers of infinite classes L-borderenergetic graphs and 1 infinite classes L-borderenergetic graphs. Secondly, the join and mixed operators are used to construct the L-borderenergetic graphs and find 1 infinite classes L-borderenergetic graphs and 1 infinite numbers of infinite classes L-borderenergetic graphs.

The paper is organized as follows. In Section 2, some known results about L-borderenergetic graphs are described. In Section 3, we give 2 infinite numbers of infinite classes L-borderenergetic graphs and 2 infinite classes L-borderenergetic graphs by using union, join and mixed operators on complete bipartite graphs and stars. In Section 4, we conclude this paper.

2 Premilinares

Let $K_n = (V_{K_n}, E_{K_n})$ be a complete graph of order n. Then, the complement of $G = (V, E)$ is defined as $\overline{G} = (V, E_{K_n} \backslash E)$. A complete bipartite graph with a bipartition of sizes n_1 and n_2 is denoted by K_{n_1,n_2}. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be undirected graphs without loops or multiple edges. Then the union $G = G_1 \bigcup G_2$ of G_1 and G_2 is defined as $G = (V_1 \bigcup V_2, E_1 \bigcup E_2)$. The join $G = G_1 \nabla G_2$ of G_1 and G_2 is defined as $G = \overline{\overline{G_1} \bigcup \overline{G_2}}$. We use G^n to represent the join of n-copies of G and nG to represent the union of n-copies of G.

The Laplacian spectrum of a join of two graphs can be find in [25] as follows. This is an important result for this paper.

Theorem 1. *Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. Let L_1 and L_2 be the Laplacian matrices for G_1 and G_2, respectively, and let L be the Laplacian matrix for $G_1 \nabla G_2$. If $0 = \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{n_1}$ and $0 = \beta_1 \le \beta_2 \le \cdots \le \beta_{n_2}$ are the eigenvalues of L_1 and L_2, respectively. Then the eigenvalues of L are*

$$\{0, n_2 + \alpha_2, n_2 + \alpha_3, \ldots, n_2 + \alpha_{n_1}, n_1 + \beta_2, n_1 + \beta_3, \ldots, n_1 + \beta_{n_2}, n_1 + n_2\}.$$

3 Construction of L-borderenergetic graphs

In this section, the L-borderenergetic graphs based on complete bipartite graphs and stars are given. We divide these new constructions in two parts. On the one hand, we construct the L-borderenergetic graphs from the union of some complete bipartite graphs and stars. On the other hand, we show L-borderenergetic graphs based on the join or mixed operators of complete bipartite graphs and stars.

3.1 Union of complete bipartite graphs and stars

In this section, we use the union operator to construct the L-borderenergetic graphs and find the condition for constructing the L-borderenergetic graphs, construct 1 infinite numbers of infinite classes L-borderenergetic graphs and 1 infinite classes L-borderenergetic graphs.

Theorem 2. *Let $K_{m,n}$ be the complete bipartite graph and $S_{r(m+n)}$ be a star with order $r(m+n)$. Then $G = K_{m,n} \bigcup S_{r(m+n)}$ is L-borderenergetic and L-noncospectral graph with $K_{(r+1)(m+n)}$ if $r = \frac{m+n-3}{2mn-3m-3n+1}$ is a positive integer and $m, n \geq 2$.*

Proof. It is known to all, the Laplacian spectrum of $K_{m,n}$ and $S_{r(m+n)}$ are $\{0, [m]^{n-1}, [n]^{m-1}, m+n\}$ and $\{0, [1]^{r(m+n)-2}, r(m+n)\}$, respectively. Then the Laplacian spectrum of $G = K_{m,n} \bigcup S_{r(m+n)}$ can be given as follows

$$\left\{[0]^2, [m]^{n-1}, [n]^{m-1}, m+n, [1]^{r(m+n)-2}, r(m+n)\right\}. \tag{1}$$

Let $\overline{d}$ be the average degree of $G = K_{m,n} \bigcup S_{r(m+n)}$. Then we have

$$\overline{d} = \frac{(n-1)m + (m-1)n + m + n + r(m+n) - 2 + r(m+n)}{(r+1)(m+n)}$$

$$= \frac{2mn + 2r(m+n) - 2}{(r+1)(m+n)}.$$

According to (1), it follows that the Laplacian energy of G is

$$LE(G) = 2\overline{d} + (n-1)\left|m-\overline{d}\right| + (m-1)\left|n-\overline{d}\right| + \left|m+n-\overline{d}\right| \\ + (r(m+n)-2)\left|\overline{d}-1\right| + \left|r(m+n)-\overline{d}\right|.$$

Nextly, let's discuss the positive and negative of these absolute values. Firstly, we need to verify the positive and negative of $m-\overline{d}$.

$$m-\overline{d} = m - \frac{2mn+2r(m+n)-2}{(r+1)(m+n)} \\ = \frac{(mr-2r)(m+n)+m^2-mn+2}{(r+1)(m+n)}$$

According to the $r = \frac{m+n-3}{2mn-3m-3n+1}$, it follows that

$$m-\overline{d} = \frac{2mn-m-n+1}{m+n}$$

According to the $r = \frac{m+n-3}{2mn-3m-3n+1}$ is a positive integer and $m,n \geq 2$, it follows that $2mn-3m-3n+1 > 0$ and then $2mn-m-n+1 > 0$. So $m-\overline{d} > 0$. Using the same method, we can carry out $n-\overline{d} > 0$ and then $m+n-\overline{d} > 0$. Furthermore we get $r(m+n)-\overline{d} > 0$. Finally, we will verify the positive and negative of $\overline{d}-1$.

$$\overline{d}-1 = \frac{2mn+2r(m+n)-2}{(r+1)(m+n)} - 1 = \frac{2mn-2+(r-1)(m+n)}{(r+1)(m+n)}$$

According to $r \geq 1$ and $m,n \geq 2$, it follows that $\overline{d}-1 > 0$. Then we have

$$LE(G) - 2(r+1)(m+n) + 2 \\ = 2\overline{d} + (n-1)\left(m-\overline{d}\right) + (m-1)\left(n-\overline{d}\right) + \left(m+n-\overline{d}\right) \\ + (r(m+n)-2)\left(\overline{d}-1\right) + r(m+n) - \overline{d} - 2(r+1)(m+n) + 2 \\ = (rm+rn-m-n)\overline{d} + 2mn + 2 - 2(r+1)(m+n) + 2 \\ = (rm+rn-m-n)\left(\frac{2mn+2r(m+n)-2}{(r+1)(m+n)}\right) + 2mn + 4 \\ - 2(r+1)(m+n)$$

$$= \frac{(2m+2n)r^2+(4mn-2m-2n)r+4}{r+1} - 2(r+1)(m+n)+2$$
$$= \frac{(4mn-6m-6n+2)r+(6-2m-2n)}{r+1}$$

From $r = \frac{m+n-3}{2mn-3m-3n+1}$, we have $LE(G) - 2(r+1)(m+n)+2 = 0$. It is now obvious that the theorem holds. ■

Next, we give some examples to verify the Theorem 2.

Example 1. The following examples are all $r = \frac{m+n-3}{2mn-3m-3n+1} \in \mathbb{Z}^+$ and $m, n \geq 2$.

1. $G_1 = K_{2,6} \bigcup S_{5(2+6)}$ of order $n_1 = 48$.
2. $G_2 = K_{3,4} \bigcup S_{(3+4)}$ of order $n_2 = 14$.
3. $G_3 = K_{3,3} \bigcup S_{3(3+3)}$ of order $n_3 = 24$.

Theorem 3. *Let $K_{2,2t+4}$ be the complete bipartite graph and S_d be a star with order d. Then $G = pK_{2,2t+4} \bigcup (tp)S_d$ is L-borderenergetic and L-noncospectral graph with $K_{(td+2t+6)p}$ if $d = \frac{2pt^2+8pt-2t+8p-6}{t}$ is a positive integer and $t, p \geq 1$.*

Proof. It is known to all, the Laplacian spectrum of $K_{2,2t+4}$ and S_d are $\left\{0, [2]^{2t+3}, [2t+4]^1, 2t+6\right\}$ and $\left\{0, [1]^{d-2}, d\right\}$, respectively. Then the Laplacian spectrum of $G = pK_{2,2t+4} \bigcup (tp)S_d$ can be given as follows

$$\left\{0^{tp+p}, [2]^{p(2t+3)}, [2t+4]^p, [2t+6]^p, [1]^{tp(d-2)}, [d]^{tp}\right\}.$$

Let $\overline{d}$ be the average degree of $G = pK_{2,2t+4} \bigcup (tp)S_d$. Then we have

$$\overline{d} = \frac{(4t+6)p}{(td+2t+6)p} + \frac{(2t+4)p}{(td+2t+6)p} + \frac{(2t+6)p}{(td+2t+6)p} + \frac{t(d-2)p}{(td+2t+6)p} + \frac{tdp}{(td+2t+6)p} = \frac{2td+6t+16}{td+2t+6}.$$

Then we have

$$LE(G) - LE(K_{(td+2t+6)p})$$
$$= \overline{d} + (2t+3)p\left|2-\overline{d}\right| + p\left|2t+4-\overline{d}\right| + p\left|2t+6-\overline{d}\right|$$
$$+ tp(d-2)\left|1-\overline{d}\right| + tp\left|d-\overline{d}\right| - 2(td+2t+6)p+2$$

$$= \frac{2td+6t+16}{td+2t+6} + (2t+3)p\frac{2t+4}{td+2t+6} + p\frac{2dt^2+2td+4t^2+14t+18}{td+2t+6}$$
$$+ p\frac{2dt^2+4td+4t^2+18t+20}{td+2t+6} + tp(d-2)\frac{td+4t+10}{td+2t+6}$$
$$+ tp\frac{td^2+6d-6t-16}{td+2t+6} - \frac{2(td+2t+6)^2p-2td-4t-12}{td+2t+6}$$
$$= \frac{2pd^2t^2+8pdt^2+24pdt+4pt^2+32pt+56p}{td+2t+6}$$
$$- \frac{2(td+2t+6)^2p-2t\frac{2pt^2+8pt-2t+8p-6}{t}-4t-12}{td+2t+6}$$
$$= \frac{2pd^2t^2+8pdt^2+24pdt+4pt^2+32pt+56p}{td+2t+6}$$
$$- \frac{2pd^2t^2+8pdt^2+24pdt+4pt^2+32pt+56p}{td+2t+6} = 0.$$

Hence, $G = pK_{2,2t+4}\bigcup(tp)S_d$ is L-borderenergetic and L-noncospectral graph with $K_{(td+2t+6)p}$. ■

From the Theorem 3, we can obtain infinite numbers of infinite classes of L-borderenergetic graphs. For example, the infinite classes of L-borderenergetic graphs can be listed as follows by using Theorem 3.

Example 2. The following examples are all $d = \frac{2pt^2+8pt-2t+8p-6}{t} \in \mathbb{Z}^+$ and $t, p \geq 1$.

1. $t = 1, d = 18p-8, G_1 = pK_{2,6}\bigcup pS_{18p-8}$ is L-borderenergetic graphs with K_{18p^2}.
2. $t = 2, d = 16p-10, G_2 = pK_{2,8}\bigcup 2pS_{16p-10}$ is L-borderenergetic graphs with K_{32p^2-10p}.
3. $t = 3, d = \frac{50p-12}{3}, G_3 = pK_{2,10}\bigcup 3pS_{\frac{50p-12}{3}}$ is L-borderenergetic graphs with K_{50p^2}.

That is to say, for any $t \in \mathbb{Z}^+$, an infinite classes can be obtain according to the relation between d and p.

Proposition 4. *Let $K_{2,5}$ be the complete bipartite graph and S_d be a star with order d. Then $G = pK_{2,5}\bigcup S_d$ is L-borderenergetic and L-noncospectral graph with K_{7p+d} if $d = \frac{5p^2+9p-2}{p-1}$ is a positive integer and $p \geq 2$.*

Proof. It is known to all, the Laplacian spectrum of $K_{2,5}$ and S_d are $\{0, [2]^4, 5, 7\}$ and $\{0, 1^{d-2}, d\}$, respectively. Then the Laplacian spectrum of $G = pK_{2,5} \bigcup S_d$ can be given as follows

$$\left\{0^{p+1}, [2]^{4p}, [5]^{p}, [7]^{p}, 1^{d-2}, d\right\}.$$

Let $\overline{d}$ be the average degree of $G = pK_{2,5} \bigcup S_d$. Then we have

$$\overline{d} = \frac{8p + 5p + 7p + d - 2 + d}{7p + d} = \frac{20p + 2d - 2}{7p + d}.$$

According to $d = \frac{5p^2+9p-2}{p-1}$, it follows that

$$\begin{aligned}
&LE(G) - LE(K_{7p+d}) \\
&= (p+1)\overline{d} + 4p\left|2 - \overline{d}\right| + p\left|5 - \overline{d}\right| + p\left|7 - \overline{d}\right| + (d-2)\left|1 - \overline{d}\right| + \left|d - \overline{d}\right| \\
&- 2(7p + d) + 2 \\
&= (p+1)\frac{20p + 2d - 2}{7p + d} + 4p\frac{6p - 2}{7p + d} + p\frac{15p + 3d + 2}{7p + d} + p\frac{29p + 5d + 2}{7p + d} \\
&+ (d-2)\frac{13p + d - 2}{7p + d} + \frac{10p^3 + 18p^2 + 9p - 1}{2p^2 - p - 1} - 2(7p + \frac{5p^2 + 9p - 2}{p - 1}) + 2 \\
&= \frac{24p^2 + 2p - 2}{p - 1} - \frac{24p^2 + 2p - 2}{p - 1} = 0
\end{aligned}$$

Therefore, $G = pK_{2,5} \bigcup S_d$ is L-borderenergetic and L-noncospectral graph with K_{7p+d}. ■

3.2 Join and mixed of complete bipartite graphs and stars

In this section, we use the join and mixed operators to construct the L-borderenergetic graphs and find 1 infinite classes L-borderenergetic graphs and 1 infinite numbers of infinite classes L-borderenergetic graphs.

Proposition 5. *Let $K_{m,n}$ be the complete bipartite graph and S_1 be a star with order 1. Then $G = K_{m,n} \nabla S_1$ is L-borderenergetic and L-noncospectral graph with K_{m+n+1} if $m, n \in \mathbb{Z}^+$ and $m = n + 1$ or $n = m + 1$.*

Proof. It is known to all, the Laplacian spectrum of $K_{m,n}$ and S_1 are $\left\{0, [m]^{n-1}, [n]^{m-1}, m + n\right\}$ and $\{0\}$, respectively. Then the Laplacian spectrum of $G = K_{m,n} \nabla S_1$ can be given as follows

$$\left\{0, [m+1]^{n-1}, [n+1]^{m-1}, [m+n+1]^2\right\}.$$

Let $\overline{d}$ be the average degree of $G = K_{m,n} \nabla S_1$. Then we have

$$\begin{aligned}\overline{d} &= \frac{(m+1)(n-1)+(m-1)(n+1)+2(m+n+1)}{m+n+1} \\ &= \frac{2mn+2m+2n}{m+n+1}.\end{aligned}$$

According to $m = n+1$, it follows that

$$\begin{aligned}&LE(G) - LE(K_{m+n+1}) \\ &= \overline{d} + (n-1)\left|m+1-\overline{d}\right| + (m-1)\left|n+1-\overline{d}\right| + 2\left|m+n+1-\overline{d}\right| \\ &- 2(m+n+1)+2 \\ &= \frac{n^2+3n+1}{n+1} + (n-1)\frac{1}{n+1} + n\frac{n}{n+1} + 2\frac{n^2+n+1}{n+1} - 4n - 2 \\ &= \frac{4n^2+6n+2}{n+1} - 4n - 2 = 4n+2-4n-2 = 0\end{aligned}$$

Therefore, $G = K_{n+1,n} \nabla S_1$ is L-borderenergetic and L-noncospectral graph with K_{2n+2}. The condition of $n = m+1$ can be proved in the same way as shown before. ■

Firstly, we give a lemma which is needed in the following Theorem 6.

Lemma 1. *Let $K_{m,n}$ be the complete bipartite graph of order $m+n$ and $p, i \in \mathbb{Z}^+$. Then $(pK_{m,n})^i$ has Laplacian spectrum*

$$\left\{0, [p(i-1)(m+n)]^{i(p-1)}, [m+p(i-1)(m+n)]^{ip(n-1)}, \right.$$
$$\left. [n+p(i-1)(m+n)]^{ip(m-1)}, [(1+p(i-1))(m+n)]^{ip}, [ip(m+n)]^{i-1}\right\}$$

Proof. The proof is by induction on i. When $i=1$, the Laplacian spectrum of $pK_{m,n}$ is $\left\{0^p, [m]^{p(n-1)}, [n]^{p(m-1)}, [m+n]^p\right\}$. Hence, it holds for $i=1$. We assume that the lemma is true for i. Then when $i+1$, the graph

$(pK_{m,n})^{i+1} = (pK_{m,n})^i \nabla (pK_{m,n})$ has Laplacian spectrum

$$\Big\{0, [pi(m+n)]^{p-1}, [m+pi(m+n)]^{p(n-1)}, [n+pi(m+n)]^{p(m-1)},$$
$$[m+n+pi(m+n)]^{p}, [ip(m+n)]^{i(p-1)}, [m+pi(m+n)]^{ip(n-1)},$$
$$[n+pi(m+n)]^{ip(m-1)}, [(1+ip)(m+n)]^{(i+1)p}, [(i+1)p(m+n)]^{i}\Big\}$$
$$= \Big\{0, [p(i-1)(m+n)]^{(i+1)(p-1)}, [m+p(i-1)(m+n)]^{(i+1)p(n-1)},$$
$$[n+ip(m+n)]^{(i+1)p(m-1)}, [(1+ip)(m+n)]^{(i+1)p}, [(i+1)p(m+n)]^{i}\Big\}$$

by Theorem 1. Then the lemma holds for $i+1$. Therefore, we are done. ∎

Theorem 6. *Let $K_{1,1}$ be the complete bipartite graph and S_{2r} be a star with order $2r$. Then $G = K_{1,1}^i \bigcup qS_{2r}$ is L-borderenergetic and L-noncospectral graph with $K_{2(qr+i)}$ if $q = \frac{4ri^2-2i^2+3i-8ri+3r-1}{2r-1}$ is a positive integer and $r, i \geq 1$.*

Proof. From Lemma 1, the Laplacian spectrum of $K_{1,1}^i$ and qS_{2r} are $\{0, [2i]^{2i-1}\}$ and $\{0^q, [1]^{2rq-2q}, [2r]^q\}$, respectively. Then the Laplacian spectrum of $G = K_{1,1}^i \bigcup qS_{2r}$ can be obtain as follows

$$\left\{[0]^{q+1}, [1]^{2rq-2q}, [2i]^{2i-1}, [2r]^q\right\}. \tag{2}$$

Let $\overline{d}$ be the average degree of $G = K_{1,1}^i \bigcup qS_{2r}$. Then we have

$$\overline{d} = \frac{2i(2i-1)+2qr-2q+2qr}{2i+2qr} = \frac{2qr+2i^2-q-i}{i+qr}.$$

According to (2), it follows that the Laplacian energy of G is

$$\begin{aligned} LE(G) &= (q+1)\overline{d} + (2qr-2q)\left|1-\overline{d}\right| + (2i-1)\left|2i-\overline{d}\right| + q\left|2r-\overline{d}\right| \\ &= (q+1)\frac{2qr+2i^2-q-i}{i+qr} + (2qr-2q)\left|1-\frac{2qr+2i^2-q-i}{i+qr}\right| \\ &+ (2i-1)\left|2i-\frac{2qr+2i^2-q-i}{i+qr}\right| + q\left|2r-\frac{2qr+2i^2-q-i}{i+qr}\right| \\ &= (q+1)\frac{2qr+2i^2-q-i}{i+qr} + (2qr-2q)\frac{q(r-1)+2i(i-1)}{i+qr} \end{aligned}$$

$$+ (2i-1)\frac{2qr(i-1)+q+i}{i+qr} + q\left|\frac{2qr(r-1)+2ri-2i^2+q+i}{i+qr}\right|$$

Let's substitute the $q = \frac{4ri^2-2i^2+3i-8ri+3r-1}{2r-1}$ into the $2ri-2i^2+q+i$, then we have

$$\begin{aligned}&2ri-2i^2+q+i\\ =&\frac{2i(r-i)(2r-1)+i(2r-1)+4ri^2-2i^2+3i-8ri+3r-1}{2r-1}\\ =&\frac{8ri(i-1)+4ir^2+2i+3r-1}{2r-1}>0.\end{aligned}$$

So we come to the conclusion $2r-\overline{d}>0$. Therefore, the Laplacian energy of G minus Laplacian energy of $K_{2(qr+i)}$ is

$$\begin{aligned}&LE(G)-4i-4qr+2\\ =&\frac{((2-4r)q+(6r+6i-16ri+8ri^2-4i^2-2))q}{rq+i}\\ =&\frac{(-2(4ri^2-2i^2+3i-8ri+3r-1)}{rq+i}\\ +&\frac{(6r+6i-16ri+8ri^2-4i^2-2))q}{rq+i}=0\end{aligned}$$

Hence, it is proven that $G = K_{1,1}^i \bigcup qS_{2r}$ is L-borderenergetic and L-noncospectral graph with $K_{2(qr+i)}$.

■

From the Theorem 6, we can obtain infinite numbers of infinite classes of L-borderenergetic graphs. For example, the infinite classes of L-borderenergetic graphs can be listed as follows by using Theorem 6.

Example 3. The following examples are all $q = \frac{4ri^2-2i^2+3i-8ri+3r-1}{2r-1} \in \mathbb{Z}^+$ and $r, i \geq 1$.
1. $r=1, q=2i^2-5i+2, G_1 = K_{1,1}^i \bigcup (2i^2-5i+2)S_2$ is L-borderenergetic graphs with K_{4i^2-8i+4}.
2. $r=2, q=2i^2-\frac{13}{3}i+\frac{5}{3}, G_2 = K_{1,1}^i \bigcup (2i^2-\frac{13}{3}i+\frac{5}{3})S_4$ is L-borderenergetic graphs with $K_{8i^2-\frac{46}{3}i+\frac{20}{3}}$.
3. $r=3, q=2i^2-\frac{21}{5}i+\frac{8}{5}, G_3 = K_{1,1}^i \bigcup (2i^2-\frac{21}{5}i+\frac{8}{5})S_6$ is L-borderenergetic

graphs with $K_{12i^2-\frac{116}{5}i+\frac{48}{5}}$.

That is to say, for any $r \in \mathbb{Z}^+$, an infinite classes can be obtain according to the relation between q and i.

4 Conclusion

In this paper, we use the complete bipartite graphs and stars to construct the L-borderenergetic graphs under the operators union, join and mixed. We find two infinite numbers of infinite classes L-borderenergetic graphs and two infinite classes L-borderenergetic graphs. On the one hand, our structure provided the possibility for other graphs to construct infinite numbers of infinite classes L-borderenergetic graphs. On the other hand, these results give some new structures of L-borderenergetic graphs.

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