Extreme Sombor Spectral Radius of Unicyclic Graphs

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Abstract

Let G be a graph, the Sombor matrix \( S(G) \) of G was recently introduced by Wang et al. It is a new matrix based on Sombor index, where the \((i, j)\) entry \( S_{ij} = \sqrt{d_i^2 + d_j^2} \) if vertices \( i \) and vertices \( j \) are adjacent in G, and \( S_{ij} = 0 \) for other cases. Xueliang Li and Junming Wang solved the conjecture for the upper and lower bounds of the ABC spectral radius for unicyclic graphs by Ghorbani et all. Inspired by this, we investigate the spectral radius on Sombor matrix of unicyclic graphs. In the paper, we use the method of classified discussion and Cauchy-Schwartz inequality to determine the external Sombor spectral radius of unicyclic graphs and provide the conditions for the equality.

1 Introduction

Let \( G = (V, E) \) be a simple connected graph of order \( n \), where vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = \{e_1, e_2, \ldots, e_m\} \). A graph G is called unicyclic graph if \( m = n \). We denote by \( d_i \) the degree of vertex \( v_i \), by \( \delta \) the minimum degree of G, and by \( A = (A_{ij})_{n \times n} \) the adjacency matrix

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of a graph $G$. The eigenvalues of the adjacency matrix $A$ are denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, and $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The spectral radius of $G$ is the largest eigenvalue of the adjacency matrix, which is denoted by $\lambda_1(G)$. For more definitions, the reader can refer to [1,2].

The Sombor matrix $S = S(G) = (s_{ij})_{n \times n}$ of $G$ is defined as

$$s_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2}, & \text{if } v_i v_j \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of the Sombor matrix $S(G)$ are denoted by $\mu_1(G), \mu_2(G), \ldots, \mu_n(G)$, and $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$. The Sombor spectral radius of $G$ is the largest eigenvalue of the Sombor matrix, which is denoted by $\mu_1(G)$. This matrix was recently introduced by Wang et al [3] based on the concept of Sombor index by Gutman [4] in the chemical graph theory. This new matrix is related to the forgotten index [5] that is a well-known degree-based topological index. Up to date, the study of the Sombor matrix of graphs mainly focuses on the Sombor index of chemical graphs [6–8] and graph operators [9]. For more related results, the reader can refer to [10,13,14,17,18].

In 2020, Ghorbani et al posed a conjecture for the upper and lower bounds of the ABC spectral for unicyclic graphs [11], and Xueliang Li et al solved the conjecture [15]. In this paper, for unicyclic graphs we investigate the Sombor matrix and calculate the largest and smallest Sombor spectral radius and characterize the corresponding extremal graphs.

## 2 Preliminary

**Lemma 1.** [12] [16] Let $T \geq 0$ be an irreducible matrix. Then $\bar{R} \leq \mu_1(T) \leq R_{\text{max}}$, where $\bar{R}$ is the average value of row sums of $T$ and $R_{\text{max}}$ is the value of the largest row sum. Either equality holds if and only if the row sums are equal.

**Lemma 2.** (Perron-Frobenius Theory) Let $T \geq 0$ be an irreducible matrix with an eigenvalue $\theta_0$. Suppose $t \in R, x \in R^n, x \geq 0, x \neq 0$. If $Tx \leq tx,$
then $t \geq \theta_0$.

**Lemma 3.** (*Cauchy-Schwarz Inequality*) Let $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ be positive real numbers. Then

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2},$$

where the equality holds if and only if $a_i = k b_i, 1 \leq i \leq n$.

## 3 Extrend Sombor spectral radius

For convenience’s sake, we denote the star graph, cycle graph of order $n$ by $S_n$ and $C_n$ respectively. Let $S^+_3$ denote a unicyclic graph obtained by attaching $n - 3$ vertices to some fixed vertex of $C_3$. Let $U_i(1 \leq i \leq 5)$ and $G_i(1 \leq i \leq 4)$ denote some special graphs in Figure 1 and Figure 2.

![Diagram](a) $U_1$  (b) $U_2$  (c) $U_3$  (d) $U_4$  (e) $U_5$

**Figure 1.** $U_i(1 \leq i \leq 5)$

![Diagram](a) $G_1$  (b) $G_2$  (c) $G_3$  (d) $G_4$

**Figure 2.** $G_i(1 \leq i \leq 4)$

It is well known that there is only one type $C_3$ of unicyclic graph for $n = 3$, and its Sombor spectral radius is $\mu_1(C_3) = 4\sqrt{2}$. Then we calculate Sombor spectral radius of unicyclic graphs for $4 \leq n \leq 6$(see Figure 3, Figure 4 and Figure 5), and display the computed result in Tab.1. In the following section, we consider the case of unicyclic graphs for $n \geq 7$. 
Figure 3. Unicyclic graphs with $n = 4$

Figure 4. Unicyclic graphs with $n = 5$

Figure 5. Unicyclic graphs with $n = 6$

Table 1. Sombor spectral radius of the unicyclic graphs for $4 \leq n \leq 6$.

<table>
<thead>
<tr>
<th>$n = 4$</th>
<th>$a$</th>
<th>$b$</th>
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</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>$4\sqrt{2}$</td>
<td>7.2538</td>
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<tr>
<th>$n = 5$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
</tr>
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<tbody>
<tr>
<td>$\mu_1$</td>
<td>$4\sqrt{2}$</td>
<td>6.9797</td>
<td>7.4872</td>
<td>9.5328</td>
<td>8.4853</td>
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<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
<th>$D_6$</th>
<th>$D_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>$4\sqrt{2}$</td>
<td>6.8356</td>
<td>7.2079</td>
<td>9.1369</td>
<td>8.0039</td>
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<th>$n = 6$</th>
<th>$D_8$</th>
<th>$D_9$</th>
<th>$D_{10}$</th>
<th>$D_{11}$</th>
<th>$D_{12}$</th>
<th>$D_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>9.7225</td>
<td>8.1108</td>
<td>7.5467</td>
<td>8.6409</td>
<td>9.5341</td>
<td>10.4129</td>
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</table>
3.1 The lower bound for the Sombor spectral radius

Theorem 1. If $G$ is the unicyclic graph with $n$ vertices, then
\[ \mu_1(G) \geq \mu_1(C_n) = 4\sqrt{2}, \]
where the equality holds if and only if $G \cong C_n$.

Proof. Since $G$ is the unicyclic graph with $n$ vertices, we have
\[ d_1 + d_2 + \cdots + d_n = 2n. \]

According to Lemma 1, we have
\[ \mu_1(G) \geq \frac{2}{n} \sum_{v_i v_j \in E(G)} \sqrt{d_i^2 + d_j^2} = \frac{\sqrt{2}}{n} \sum_{v_i v_j \in E(G)} \sqrt{2} \sqrt{d_i^2 + d_j^2}. \]

By Cauchy-Schwarz Inequality,
\[ \sqrt{2} \sqrt{d_i^2 + d_j^2} \geq d_i + d_j, \]
\[ n \sum_{i=1}^{n} d_i^2 \geq \left( \sum_{i=1}^{n} d_i \right)^2 = 4n^2. \] (1)

Then
\[ \mu_1(G) \geq \frac{\sqrt{2}}{n} \sum_{v_i v_j \in E(G)} (d_i + d_j) = \frac{\sqrt{2}}{n} \sum_{i=1}^{n} d_i^2 \geq 4\sqrt{2}. \]

It is easy to check that $\mu_1(C_n) = 4\sqrt{2}$. So
\[ \mu_1(G) \geq \mu_1(C_n) = 4\sqrt{2}. \] (2)

If the equation in (2) holds, then all the above inequalities must be equalities. By Lemma 3, the equality in (1) holds if and only if $d_i = d_j$ for any $1 \leq i, j \leq n$. Since $G$ is a unicyclic graph, it implies that $G \cong C_n$.

Conversely, if $G \cong C_n$, then $d_i = 2$ for $1 \leq i \leq n$, and so $\mu_1(C_n) = 4\sqrt{2}$. □
3.2 The upper bound for the Sombor spectral radius

Let \( \Delta_{\text{adj}}(G) = \max\{d_v + d_u|uv \in E(G)\} \), \( X = (\sqrt{d_1}, \sqrt{d_2}, \cdots, \sqrt{d_n})^T \).

**Lemma 4.** Let \( G \) be an unicyclic graph with \( n \) vertices. If \( \Delta_{\text{adj}}(G) \leq n-1 \), then \( \mu_1(G) \leq \sqrt{n+1}\sqrt{n^2 - 6n + 13} \).

**Proof.** Since \( \Delta_{\text{adj}}(G) \leq n-1 \), we have \( 1 \leq d_i \leq n-3, i = 1, 2, \cdots, n \), and \( d_{j_1} + d_{j_2} + \cdots + d_{j_{d_i}} \leq n + 1 \), where \( v_{j_1}, v_{j_2}, \cdots, v_{j_{d_i}} \) denote the adjacent vertices to the vertex \( v_i \).

\[
(SX)_i = \sum_{v_i, v_j \in E(G)} \sqrt{d_i^2 + d_j^2} \sqrt{d_j} \leq \sqrt{d_i^2 + (n-1-d_i)^2} \sum_{v_i, v_j \in E(G)} \sqrt{d_j}.
\]

By Cauchy-Schwarz Inequality,

\[
\sum_{v_i, v_j \in E(G)} \sqrt{d_j} \leq \sqrt{\sum_{v_i, v_j \in E(G)} d_j} \sqrt{d_i} \leq \sqrt{n+1} \sqrt{d_i},
\]

So

\[
(SX)_i \leq \sqrt{2d_i^2 - 2(n-1)d_i + (n-1)^2} \sqrt{n+1} \sqrt{d_i}
\]

\[
\leq \sqrt{n^2 - 6n + 13} \sqrt{n+1} \sqrt{d_i}.
\]

Namely, \( SX \leq \sqrt{n+1} \sqrt{n^2 - 6n + 13} X \).

Therefore, by Lemma 2, we can get \( \mu_1(G) \leq \sqrt{n+1} \sqrt{n^2 - 6n + 13} \). \( \blacksquare \)

**Lemma 5.** Let \( f(x) = x^2 + x + \frac{n^2 + 2n + 3}{x} - 2n - 1 \), \( x \in [3, n-3] \) and \( n \geq 7 \). Then \( f(x) \leq n^2 - 6n + 15 \).

**Proof.** First, for \( x \in [3, n-3] \),

\[
f(x) \leq x^2 + n - 3 + \frac{n^2 + 2n + 3}{x} - 2n - 1\]

\[
= x^2 + \frac{n^2 + 2n + 3}{x} - n - 4 \triangleq g(x).
\]

Second,

\[
g'(x) = 2x - \frac{n^2 + 2n + 3}{x^2}.
\]
Let \( g'(x) = 0 \). Then, we have

\[
x = \sqrt[3]{\frac{n^2 + 2n + 3}{2}},
\]

\[
g \left( \sqrt[3]{\frac{n^2 + 2n + 3}{2}} \right) = 3 \left( \frac{n^2 + 2n + 3}{2} \right)^\frac{2}{3} - n - 4,
\]

\[
g(3) = \frac{1}{3} n^2 - \frac{n}{3} + 6,
\]

\[
g(n - 3) = n^2 - 6n + 10 + \frac{18}{n - 3} \leq n^2 - 6n + 15.
\]

So, for \( x \in [3, n - 3] \),

\[
\max g(x) = g(n - 3).
\]

Thus, for \( x \in [3, n - 3] \), \( f(x) \leq n^2 - 6n + 15, n \geq 7. \)

\[\Box\]

**Lemma 6.** [15] Let \( G \) be an unicyclic graphs with \( n \) vertices. If there exist two adjacent vertices \( u, v \in V(G) \) such that \( d_u + d_v \geq n \), then \( G \in U = \{U_1, U_2, U_3, U_4, U_5\} \), see Figure 1.

**Lemma 7.** If \( G \in U_1 \setminus \{G_1, G_2\} \), then \( \mu_1(G) \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15}. \)

**Proof.** Since \( G \in U_1 \setminus \{G_1, G_2\} \), we have that \( d_1 + d_2 = n + 1, d_3 = 2, d_4 = \cdots = d_n = 1 \), and \( 4 \leq d_1, d_2 \leq n - 3 \). Then

\[
(SX)_1 = \sum_{v_1 v_j \in E(G)} \sqrt{d_1^2 + d_j^3 \sqrt{d_j}}
\]

\[
= \sqrt{d_1^2 + d_2^2} \sqrt{d_2} + \sqrt{d_1^2 + d_3^3 \sqrt{d_3}} + (d_1 - 2) \sqrt{d_1^3 + 1}.
\]

By Lemmas 3 and 5, we have

\[
(SX)_1 \leq \sqrt{d_1^2 + d_2^2 + d_3^2 + (d_1 - 2)(d_1^2 + 1) \sqrt{d_2 + d_3 + d_1 - 2}}
\]

\[
= \sqrt{d_1^3 + d_1^2 - (2n + 1)d_1 + n^2 + 2n + 3 \sqrt{n + 1}}
\]

\[
= \sqrt{n + 1} \sqrt{d_1^2 + d_1 - 2n - 1 + n^2 + 2n + 3 \sqrt{d_1}}
\]

\[
\leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15 \sqrt{d_1}}.
\]
Similarly,

\[ (SX)_2 \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15} \sqrt{d_2}. \]

For \( i = 3 \), we have

\[
(SX)_3 = \sum_{v_3 v_j \in E(G)} \sqrt{d_3^2 + d_j^2} 
= \sqrt{4 + d_1^2 \sqrt{d_1} + 4 + d_2^2 \sqrt{d_2}} 
\leq \sqrt{8 + d_1^2 + d_2^2 \sqrt{d_1} + d_2} 
= \sqrt{n + 1} \sqrt{d_1^2 - (n + 1)d_1 + \frac{n^2 + 2n + 9}{2} \sqrt{d_3}} 
\leq \sqrt{n + 1} \sqrt{\frac{1}{2} (n^2 - 6n + 33) \sqrt{d_3}} 
\leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15} \sqrt{d_3}.
\]

For \( 4 \leq i \leq n \), we have

\[
(SX)_i = \sum_{v_i v_j \in E(G)} \sqrt{d_i^2 + d_j^2} \sqrt{d_j} 
= \sqrt{d_i^2 + d_k^2} \sqrt{d_k} 
= \sqrt{1 + d_k^2} \sqrt{d_k} 
\leq \sqrt{(n - 3)(n^2 - 6n + 10)} 
\leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15}, \quad k = 1, 2.
\]

In summary, we have \( SX \leq \sqrt{(n + 1)(n^2 - 6n + 15)} X \), and so \( \mu_1(G) \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15} \) \( \square \)

**Lemma 8.** If \( G \in \mathcal{U}_2 \setminus \{G_2\} \), then \( \mu_1(G) \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15} \).

**Proof.** Since \( G \in \mathcal{U}_2 \setminus \{G_2\} \), we have \( d_1 + d_2 = n, d_3 = 3, d_4 = \cdots = d_n = 1 \), and \( 3 \leq d_1, d_2 \leq n - 3 \). Then

\[
(SX)_1 = \sum_{v_1 v_j \in E(G)} \sqrt{d_1^2 + d_j^2} \sqrt{d_j}
\]
\[ \begin{align*}
&= \sqrt{d_1^2 + d_2^2} \sqrt{d_2} + \sqrt{d_2^2 + d_3^2} \sqrt{d_3} + (d_1 - 2) \sqrt{d_1^2 + 1} \\
&\leq \sqrt{d_1^2 + d_2^2 + d_2^2 + d_3^2} + (d_1 - 2)(d_2^2 + 1) \sqrt{d_2 + d_3 + d_1 - 2} \\
&= \sqrt{n + 1} \sqrt{d_1^2 + d_2^2 + 1 + 2(n - 1) + n^2} \\
&= \sqrt{n + 1} \sqrt{d_1^2 + d_2^2 + 1 - 2n + \frac{n^2 + 7}{d_1}} \sqrt{d_1} \\
&\leq \sqrt{d_2^2 + d_1 - 2n - 1 + \frac{n^2 + 2n + 3}{d_1}} \sqrt{d_1} \\
&\leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15} \sqrt{d_1}.
\end{align*} \]

Similarly,

\[ (SX)_2 \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15} \sqrt{d_2}. \]

For \( i = 3 \), we have

\[ (SX)_3 = \sum_{v_3 v_j \in E(G)} \sqrt{d_3^2 + d_j^2} \sqrt{d_j} \]

\[ = \sqrt{d_3^2 + d_1^2} \sqrt{d_1} + \sqrt{d_3^2 + d_2^2} \sqrt{d_2} + \sqrt{d_3^2 + 1} \sqrt{1} \]

\[ \leq \sqrt{3d_3^2 + d_1^2 + d_2^2 + 1} \sqrt{n + 1} \]

\[ = \sqrt{2d_1^2 - 2nd_1 + n^2 + 28} \sqrt{n + 1} \]

\[ = \sqrt{\frac{1}{3} (2d_1^2 - 2nd_1 + n^2 + 28)} \sqrt{n + 1} \sqrt{d_3} \]

\[ \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15} \sqrt{d_3}. \]

For \( 4 \leq i \leq n \), we have

\[ (SX)_i = \sqrt{d_i^2 + d_k^2} \sqrt{d_k} \]

\[ = \sqrt{1 + d_k^2} \sqrt{d_k} \]

\[ \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15}, k = 1, 2, 3. \]

In summary, we have \( SX \leq \sqrt{(n + 1)(n^2 - 6n + 15)} X \), and so \( \mu_1(G) \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 15} \).
Lemma 9. If $G \in \mathcal{U}_3 \setminus \{G_3\}$, then $\mu_1(G) \leq \sqrt{n + 1}\sqrt{n^2 - 6n + 14}$.

Proof. Since $G \in \mathcal{U}_3 \setminus \{G_3\}$, we have $d_1 + d_2 = n, d_3 = d_4 = 2, d_5 = \cdots = d_n = 1$, and $3 \leq d_1, d_2 \leq n - 3$. Then

$$(SX)_1 = \sum_{v_1v_j \in E(G)} \sqrt{d_i^2 + d_j^2} \sqrt{d_j}$$

$$= \sqrt{d_1^2 + d_2^2} \sqrt{d_2} + \sqrt{d_1^2 + d_3^2} \sqrt{d_3} + \sqrt{d_1^2 + d_4^2} \sqrt{d_4}$$

$$+(d_1 - 3) \sqrt{d_1^2 + 1}$$

$$\leq \sqrt{3d_1^2 + d_2^2 + d_3^2 + d_4^2 + (d_1 - 3)(d_1^2 + 1)} \sqrt{d_2 + d_3 + d_4 + d_1 - 3}$$

$$= \sqrt{d_1^3 + d_1^2 + (1 - 2n)d_1 + n^2 + 5\sqrt{n + 1}}$$

$$= \sqrt{d_1^2 + d_1 + 1 - 2n + \frac{n^2 + 5}{d_1} \sqrt{n + 1} \sqrt{d_1}}$$

$$\leq \sqrt{n + 1}\sqrt{n^2 - 6n + 14}\sqrt{d_1}.$$  

Similarly,

$$(SX)_2 \leq \sqrt{n + 1}\sqrt{n^2 - 6n + 14}\sqrt{d_2}.$$  

For $i = 3, 4, 5$, we have

$$(SX)_3 = \sum_{v_3v_j \in E(G)} \sqrt{d_3^2 + d_j^2} \sqrt{d_j}$$

$$= \sqrt{d_3^2 + d_1^2} \sqrt{d_1} + \sqrt{d_3^2 + d_2^2} \sqrt{d_2}$$

$$\leq \sqrt{2d_3^2 + d_1^2 + d_2^2} \sqrt{d_1 + d_2}$$

$$= \sqrt{8 + d_1^2 + (n - d_1)^2} \sqrt{n}$$

$$= \sqrt{d_1^2 - nd_1 + \frac{n^2}{2} + 4\sqrt{n}\sqrt{2}}$$

$$\leq \sqrt{\frac{n^2}{2} - 3n + 13\sqrt{n}\sqrt{2}}$$

$$\leq \sqrt{n^2 - 6n + 14\sqrt{n} + 1\sqrt{d_3}},$$
$$(SX)_4 = \sum_{v_i v_j \in E(G)} \sqrt{d_i^2 + d_j^2} \sqrt{d_j}$$

$$= \sqrt{d_4^2 + d_1^2} \sqrt{d_1} + \sqrt{d_4^2 + d_5^2} \sqrt{d_5}$$

$$= \sqrt{4 + d_1^2} \sqrt{d_1} \sqrt{5}$$

$$\leq \sqrt{n^2 - 6n + 14} \sqrt{n + 1} \sqrt{2}$$

$$\leq \sqrt{n^2 - 6n + 14} \sqrt{n + 1} \sqrt{d_4},$$

and

$$(SX)_5 = \sqrt{d_5^2 + d_4^2} \sqrt{d_4} = 2 \sqrt{1 + 4} \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 14}.$$

For $6 \leq i \leq n$, we have

$$(SX)_i = \sqrt{d_i^2 + d_k^2} \sqrt{d_k}$$

$$= \sqrt{1 + d_k^2} \sqrt{d_k}$$

$$\leq \sqrt{(n_3)^2 + 1} \sqrt{n - 3}$$

$$\leq \sqrt{n + 1} \sqrt{n^2 - 6n + 14}.$$

In summary, we have $SX \leq \sqrt{(n + 1)(n^2 - 6n + 14)} X$, and so $\mu_1(G) \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 14}$. 

\begin{flushright}
\textbf{Lemma 10.} \textit{If } G \in U_4 \setminus \{G_1, G_3\}, \textit{ then } \mu_1(G) \leq \sqrt{n + 1} \sqrt{n^2 - 6n + 14}.
\end{flushright}

\textit{Proof.} Since $G \in U_4 \setminus \{G_1, G_3\}$, we have $d_1 + d_2 = n, d_3 = d_4 = 2, d_5 = \cdots = d_n = 1$, and $3 \leq d_1, d_2 \leq n - 3$. Then

$$(SX)_1 = \sum_{v_1 v_j \in E(G)} \sqrt{d_1^2 + d_j^2} \sqrt{d_j}$$

$$= \sqrt{d_1^2 + d_2^2} \sqrt{d_2} + \sqrt{d_1^2 + d_3^2} \sqrt{d_3} + \sqrt{d_1^2 + d_4^2} \sqrt{d_4}$$

$$+ (d_1 - 3) \sqrt{d_1^2 + 1}$$

$$\leq \sqrt{3d_1^2 + d_2^2 + d_3^2 + d_4^2 + (d_1 - 3)(d_1^2 + 1)} \sqrt{d_2 + d_3 + d_4 + d_1 - 3}$$

$$= \sqrt{d_1^3 + d_1^2 + (1 - 2n)d_1 + n^2 + 5 \sqrt{n + 1}}$$
\[ \sqrt{d_i^2 + d_1 + 1 - 2n + \frac{n^2 + 5}{d_1}\sqrt{n + 1\sqrt{d_1}}} \leq \sqrt{n + 1\sqrt{n^2 - 6n + 14\sqrt{d_1}}} \]

Similarly,
\[(SX)_2 \leq \sqrt{n + 1\sqrt{n^2 - 6n + 14\sqrt{d_2}}}.\]

For \(i = 3, 4\), we have
\[(SX)_3 = \sqrt{d_3^2 + d_1^2 + d_4^2 + d_4^2} \leq \sqrt{2d_3^2 + d_1^2 + d_4^2}\sqrt{d_1 + d_4} = \sqrt{d_1^2 + 2d_1^2 + d_1^2 + 2}\]
\[\leq \sqrt{n^2 - 6n + 14\sqrt{n + 1\sqrt{2}},}\]

and
\[(SX)_4 \leq \sqrt{n^2 - 6n + 14\sqrt{n + 1\sqrt{2}}}.\]

For \(5 \leq i \leq n\), we have
\[(SX)_i = \sqrt{d_i^2 + d_k^2}\sqrt{d_k} = \sqrt{1 + d_k^2}\sqrt{d_k} \leq \sqrt{n + 1\sqrt{n^2 - 6n + 14}}, \quad k = 1, 2.\]

In summary, we have \(SX \leq \sqrt{(n + 1)(n^2 - 6n + 14)X}\), and so \(\mu_1(G) \leq \sqrt{n + 1\sqrt{n^2 - 6n + 14}}.\)

**Lemma 11.** If \(G \in U_5 \setminus \{G_5\}\), then \(\mu_1(G) \leq \sqrt{n + 1\sqrt{n^2 - 6n + 13}}.\)

**Proof.** Since \(G \in U_5 \setminus \{G_4\}\), we have \(d_1 + d_2 = n, d_3 = d_4 = 2, d_5 = \cdots = d_n = 1, \text{ and } 3 \leq d_1, d_2 \leq n - 3.\) Then
\[(SX)_1 = \sum_{v_1, v_j \in E(G)} \sqrt{d_i^2 + d_j^2}\sqrt{d_j} = \sqrt{d_1^2 + d_2^2}\sqrt{d_2} + \sqrt{d_1^2 + d_4^2}\sqrt{d_4} + (d_1 - 2)\sqrt{d_1^2 + 1} \quad \Box \]
\[
\begin{align*}
\leq & \sqrt{2d_1^2 + d_2^2 + d_4^2 + (d_1 - 2)(d_1^2 + 1)}d_2 + d_4 + d_1 - 2 \\
= & \sqrt{d_1^3 + d_1^2 + (1 - 2n)d_1 + n^2 + 2\sqrt{n}} \\
= & \sqrt{d_1^2 + d_1 + 1 - 2n + \frac{n^2 + 2}{d_1}\sqrt{n\sqrt{d_1}}} \\
\leq & \sqrt{n + 1}\sqrt{n^2 - 6n + 13}\sqrt{d_1}.
\end{align*}
\]

Similarly,
\[
(SX)_2 \leq \sqrt{n + 1}\sqrt{n^2 - 6n + 13}\sqrt{d_2}.
\]

For \(i = 3, 4\), we have
\[
(SX)_3 = \sqrt{d_3^2 + d_2^2}\sqrt{d_2} + \sqrt{d_3^2 + d_4^2}\sqrt{d_4} \\
\leq \sqrt{2d_3^2 + d_2^2 + d_4^2}\sqrt{d_2 + d_4} \\
= \sqrt{d_2^2 + 12\sqrt{d_2} + 2} \\
\leq \sqrt{n^2 - 6n + 13}\sqrt{n + 1}\sqrt{2},
\]

and
\[
(SX)_4 \leq \sqrt{n^2 - 6n + 13}\sqrt{n + 1}\sqrt{2}.
\]

For \(5 \leq i \leq n\), we have
\[
(SX)_i = \sqrt{d_i^2 + d_k^2}\sqrt{d_k} = \sqrt{1 + d_k^2}\sqrt{d_k} \\
\leq \sqrt{1 + (n - 3)^2}\sqrt{n - 3} \\
\leq \sqrt{n + 1}\sqrt{n^2 - 6n + 13}, \quad k = 1, 2.
\]

In summary, we have \(SX \leq \sqrt{(n + 1)(n^2 - 6n + 13)}X\), and so \(\mu_1(G) \leq \sqrt{n + 1}\sqrt{n^2 - 6n + 13}\).

**Lemma 12.** Let \(F = \{G_1, G_2, G_3, G_4\}\). Then \(G_1\) has the largest Sombor spectral radius among \(F\).

**Proof.** For \(G_1\), we have
\[ M(G_1) = \begin{bmatrix}
0 & \beta & \beta & \alpha & \cdots & a \\
\beta & 0 & 2\sqrt{2} & 0 & \cdots & 0 \\
\beta & 2\sqrt{2} & 0 & 0 & \cdots & 0 \\
\alpha & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha & 0 & 0 & 0 & \cdots & 0 
\end{bmatrix}, \]

where
\[
\alpha = \sqrt{(n-1)^2 + 1}, \beta = \sqrt{(n-1)^2 + 4}.
\]

Then
\[
f_{G_1}(\lambda) = \det(\lambda I - M(G_1)) = (\lambda + 2\sqrt{2})\lambda^{n-4}f_1(\lambda),
\]

where
\[
f_1(\lambda) = \lambda^3 - 2\sqrt{2}\lambda^2 - (n^3 - 3n^2 + 4n + 4)\lambda + 2\sqrt{2}(n^3 - 5n^2 + 8n - 6).
\]

It is easy that
\[
f_1(-n\sqrt{n+1}) < 0, \quad f_1(0) > 0,
\]
\[
f_1(\sqrt{n+1}\sqrt{n^2 - 5n + 8}) < 0, \quad f_1(n\sqrt{n+1}) > 0.
\]

By zero point theorem, we have
\[
\mu_1(G_1) \in (\sqrt{n+1}\sqrt{n^2 - 5n + 8}, n\sqrt{n+1}).
\]

For \(G_2\), we have
\[ M(G_2) = \begin{bmatrix}
0 & c & b & 0 & a & \cdots & a \\
c & 0 & \sqrt{13} & \sqrt{10} & 0 & \cdots & 0 \\
b & \sqrt{13} & 0 & 0 & 0 & \cdots & 0 \\
0 & \sqrt{10} & 0 & 0 & 0 & \cdots & 0 \\
a & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a & 0 & 0 & 0 & 0 & \cdots & 0 
\end{bmatrix}, \]
where
\[ a = \sqrt{(n-2)^2 + 1}, b = \sqrt{(n-2)^2 + 4}, c = \sqrt{(n-2)^2 + 9}. \]

Then
\[ f_{G_2}(\lambda) = \det(\lambda I - M(G_2)) = \lambda^{n-4}f_2(\lambda), \]
where
\[ f_2(\lambda) = \lambda^4 - (n^3 - 6n^2 + 13n + 24)\lambda^2 + 23n^3 - 174n^2 + 443n - 380. \]

It is easy that
\[ f_2(-\sqrt{n+1}\sqrt{n^2 - 5n + 8}) > 0, \]
\[ f_2\left(-\sqrt{\frac{1}{2}(n+1)(n^2 - 7n + 20)}\right) < 0, \]
\[ f_2(0) > 0, \]
\[ f_2\left(\sqrt{\frac{1}{2}(n+1)(n^2 - 7n + 20)}\right) < 0, \]
\[ f_2(\sqrt{n+1}\sqrt{n^2 - 5n + 8}) > 0. \]

By zero point theorem, we have
\[ \mu_1(G_2) \leq \sqrt{n+1}\sqrt{n^2 - 5n + 8}. \]

For \( G_3 \), we have
\[
M(G_3) = \begin{bmatrix}
0 & b & b & b & 0 & a & \cdots & a \\
b & 0 & 2\sqrt{2} & 0 & 0 & 0 & \cdots & 0 \\
b & 2\sqrt{2} & 0 & 0 & 0 & 0 & \cdots & 0 \\
b & 0 & 0 & 0 & \sqrt{5} & 0 & \cdots & 0 \\
0 & 0 & 0 & \sqrt{5} & 0 & 0 & \cdots & 0 \\
a & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix},
\]
where 
\[ a = \sqrt{(n - 2)^2 + 1}, \quad b = \sqrt{(n - 2)^2 + 4}, \]

Then 
\[ f_{G_3}(\lambda) = \text{det}(\lambda I - M(G_3)) = (\lambda + 2\sqrt{2})\lambda^{n-6}f_3(\lambda), \]

where
\[
f_3(\lambda) = \lambda^5 - 2\sqrt{2}\lambda^4 - (n^3 - 6n^2 + 13n + 4)\lambda^3 + 2\sqrt{2}(n^3 - 8n^2 + 21n - 12)\lambda^2 + 5(n^3 - 7n^2 + 17n - 9)\lambda - 10\sqrt{2}(n^3 - 9n^2 + 25n - 25).
\]

It is easy that
\[
\begin{align*}
    f_3(-\sqrt{n + 1})\sqrt{n^2 - 5n + 8} &< 0, \\
    f_3(-\sqrt{(n + 1)(n^2 - 9n)}) &> 0, \\
    f_3(0) &< 0, \\
    f_3(4\sqrt{2}) &> 0, \\
    f_3(\sqrt{(n + 1)(n^2 - 8n)}) &< 0, \\
    f_3(\sqrt{n + 1}\sqrt{n^2 - 5n + 8}) &> 0.
\end{align*}
\]

By zero point theorem, we have
\[ \mu_1(G_3) \leq \sqrt{n + 1}\sqrt{n^2 - 5n + 8}. \]

For \( G_4 \), we have
\[ M(G_4) = \begin{bmatrix}
0 & b & 0 & b & a & a & \cdots & a \\
b & 0 & 2\sqrt{2} & 0 & 0 & 0 & \cdots & 0 \\
0 & 2\sqrt{2} & 0 & 2\sqrt{2} & 0 & 0 & \cdots & 0 \\
b & 0 & 2\sqrt{2} & 0 & 0 & 0 & \cdots & 0 \\
a & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \]
where 
\[ a = \sqrt{(n - 2)^2 + 1}, \quad b = \sqrt{(n - 2)^2 + 4}. \]

Then
\[ f_{G_4}(\lambda) = \det(\lambda I - M(G_4)) = \lambda^{n-4} f_4(\lambda), \]

where
\[ f_4(\lambda) = \lambda^4 - (n^3 - 6n^2 + 13n + 12)\lambda^2 + 16(n^3 - 8n^2 + 21n - 20). \]

It is easy that
\[ f_4(\pm\sqrt{n^2 - 5n + 8}) > 0, \]
\[ f_4(\pm 4) < 0, \]
\[ f_4(0) > 0. \]

By zero point theorem, we have
\[ \mu_1(G_4) \leq \sqrt{n + 1}\sqrt{n^2 - 5n + 8}. \]

In summary, the Sombor spectral radius of $G_1$ among $F$ is largest. 

**Theorem 2.** If $G$ is the unicyclic graph of order $n \geq 7$, then
\[ \mu_1(G) \leq \mu_1(S_3^+), \]

with equality if and only if $G \cong S_3^+$. 

**Proof.** By Lemma 4 and Lemmas 6-11, for all unicyclic graphs of order $n \geq 7$ except $G \in \{G_1, G_2, G_3, G_4\}$, we have
\[ \mu_1(G) \leq \sqrt{n + 1}\sqrt{n^2 - 5n + 8}. \]

By Lemma 12, we have
\[ \mu_1(G_1) \geq \sqrt{n + 1}\sqrt{n^2 - 5n + 8} \geq \mu_1(G_i), \quad i = 2, 3, 4. \]
Therefore, $G_1$ is the unicyclic graph with the largest Sombor spectral radius, which is isomorphic to $S^+_3$.

\section{Conclusion}

Since, evidently, $C_3$ be a special form of $S^+_3$, we obtain our main result as follows.

\textbf{Theorem 3.} \textit{If $G$ is the unicyclic graph of order $n \geq 3$, then}

\[ 4\sqrt{2} = \mu_1(C_n) \leq \mu_1(G) \leq \mu_1(S^+_3), \]

\textit{where the lower bound is achieved by $C_n$ and the upper bound is achieved by $S^+_3$.}

Recently, Li et al. proved the conjecture for the upper and lower bounds of the ABC spectral radius for unicyclic graphs. In this paper, we have researched the spectral radius of the Sombor matrix from an algebraic viewpoint, and characterize the corresponding extremal graphs as Theorem 3. Further, we conjecture that for an unicyclic graph of order $n \geq 3$, the spectral radius $\rho$ of the corresponding matrix based on degree of vertices must satisfy that $\rho(C_n) \leq \rho(G) \leq \rho(S^+_3)$, with equality if and only if $G \cong C_n$ for the lower bound, and $G \cong S^+_3$ for the upper bound.

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\textbf{References}


