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Solutions to Some Open Problems About Four Sombor–Index–Like Graph Invariants

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Abstract

This paper gives solutions to most of the open problems posed in the very recent paper [Z. Tang, Q. Li, H. Deng, Trees with extremal values of the Sombor-index-like graph invariants, *MATCH Commun. Math. Comput. Chem.* **90** (2023) 203–222].

1 Introduction and statements of problems

Throughout this paper, only finite graphs are considered. For notation and terminology from (chemical) graph theory, we refer the reader to relevant standard books; for example, [1, 2, 10, 11].

A graph invariant I_g is a function defined on the set of all graphs with the following property: $I_g(G_1) = I_g(G_2)$ whenever G_1 and G_2 are

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isomorphic. Recently, Gutman [6] proposed six graph invariants in view of geometric considerations and referred them to as Sombor-index-like graph invariants. (Detail about the classical Sombor indices can be found in [4, 5, 8].) This paper is concerned with four such Sombor-index-like graph invariants, which are defined for a graph G as

$$SO_3(G) = \sqrt{2}\pi \sum_{vw \in E(G)} \frac{d_v^2 + d_w^2}{d_v + d_w},$$

$$SO_4(G) = \frac{\pi}{2} \sum_{uv \in E(G)} \left(\frac{d_v^2 + d_w^2}{d_v + d_w} \right)^2,$$

$$SO_5(G) = 2\pi \sum_{uv \in E(G)} \frac{\left| d_v^2 - d_w^2 \right|}{\sqrt{2} + 2\sqrt{d_v^2 + d_w^2}},$$

$$SO_6(G) = \pi \sum_{uv \in E(G)} \left(\frac{d_v^2 - d_w^2}{\sqrt{2} + 2\sqrt{d_v^2 + d_w^2}} \right)^2,$$

where $E(G)$ denotes the set of edges of G and d_u represents the degree of a vertex u in G.

By a chemical tree, we mean a tree of maximum degree at most 4. Let $\mathcal{T}_n$, $\mathcal{CT}_n$ and $\mathcal{CG}_n$ be the classes of all trees, chemical trees and connected graphs, respectively, of order n. Quite recently, Tang et al. [9] investigated some extremal properties of Sombor-index-like graph invariants and posed following open problems.

Problem 1. *Find the extremal values of the graph invariants SO_5 and SO_6 in the classes $\mathcal{T}_n$ and $\mathcal{CT}_n$.*

Problem 2. *Find the maximum value of the graph invariants SO_4 in the class $\mathcal{T}_n$.*

Problem 3. *Find the extremal values of the graph invariants SO_5 and SO_6 in the class $\mathcal{CG}_n$.*

Problem 4. *Find the maximum values of the graph invariants SO_3 and SO_4 in the class $\mathcal{CG}_n$.*

In this paper, Problems 2 and 4 are solved completely. Also, partial solutions to Problems 1 and 3 are provided; more precisely, Problem 1 is solved for the class $\mathcal{T}_n$ (this solution also resolve the minimal part of Problem 1 regarding $\mathcal{CT}_n$) and a solution to the minimal part of Problem 3 is indicated.

2 Results

A vertex x in a graph is said to be a pendent vertex if $d_x = 1$. An edge incident to a pendent vertex is known as a pendent edge. For an edge vw of a graph, define

$$SO_5(vw) = \frac{2\pi \left|d_v^2 - d_w^2\right|}{\sqrt{2} + 2\sqrt{d_v^2 + d_w^2}}.$$

The following result gives the solution to the minimal part of Problem 1.

Proposition 1. *If T is a tree of order at least 3, then*

$$SO_5(T) \geq \frac{12\pi}{\sqrt{2} + 2\sqrt{5}} \quad \text{and} \quad SO_6(T) \geq \frac{18\pi}{\left(\sqrt{2} + 2\sqrt{5}\,\right)^2},$$

where the equality in any of these two inequalities holds if and only if T is a path graph. Particularly, the path graph P_n uniquely attains the minimum values of SO_5 and SO_6 in the classes $\mathcal{T}_n$ and $\mathcal{CT}_n$ for each $n \in \{4, 5, 6, \ldots\}$.

Proof. Since the proofs of both desired inequalities are similar to each other, we prove one of them; namely, the one involving SO_5. Consider a pendent vertex $u \in V(T)$ adjacent to a vertex v. Then

$$SO_5(uv) = \frac{2\pi(d_v^2 - 1)}{\sqrt{2} + 2\sqrt{d_v^2 + 1}} \geq \frac{6\pi}{\sqrt{2} + 2\sqrt{5}}$$

where the equation $SO_5(uv) = 6\pi/(\sqrt{2} + 2\sqrt{5})$ holds if and only if $d_v = 2$. Denote by $P_E(T)$ the set consisting of all pendent edges in T. By utilizing the definition of SO_5, we have

$$\begin{aligned}
SO_5(T) &= \sum_{uv\in P_E(T)} SO_5(uv) + \sum_{xy\in E(T)\setminus P_E(T)} SO_5(xy) \\
&\geq \sum_{uv\in P_E(T)} \frac{6\pi}{\sqrt{2}+2\sqrt{5}} + \sum_{xy\in E(T)\setminus P_E(T)} (0) \qquad (1) \\
&= \frac{6\pi}{\sqrt{2}+2\sqrt{5}} |P_E(T)| \\
&\geq \frac{12\pi}{\sqrt{2}+2\sqrt{5}}. \qquad (2)
\end{aligned}$$

Certainly, the equality in (1) holds if and only if $\max\{d_v, d_u\} = 2$ for each edge $uv \in P_E(T)$ and $d_y = d_x$ for each edge $xy \in E(T) \setminus P_E(T)$. Note also that the equality in (2) holds if and only if the set $P_E(T)$ has only two elements. ■

Remark. The proof of Proposition 1 suggests slightly general lower bounds on SO_5 and SO_6 as given below. For a connected graph G with at least two edges and with n_1 pendent vertices, the following inequalities hold

$$SO_5(G) \geq \frac{6\pi}{\sqrt{2}+2\sqrt{5}} n_1 \quad \text{and} \quad SO_6(G) \geq \frac{9\pi}{\left(\sqrt{2}+2\sqrt{5}\right)^2} n_1,$$

where the equality in any of these two inequalities holds if and only if G is either a path graph or a regular graph.

Next, we move towards a solution to the maximal part of Problem 1 for the case of trees. For this, we need the following lemma.

Lemma 1. *The functions f and g defined by*

$$f(x,y) = \frac{2\pi|x^2-y^2|}{\sqrt{2}+2\sqrt{x^2+y^2}} \quad \text{and} \quad g(x,y) = \pi\left(\frac{x^2-y^2}{\sqrt{2}+2\sqrt{x^2+y^2}}\right)^2$$

are strictly decreasing in x whenever $1 \leq x \leq y$.

The next lemma not only provides the solution to the maximal part of Problem 1 (take $m = n-1$) concerning the class $\mathcal{T}_n$ but also indicates the trivial solution to the minimal part of Problem 3.

Lemma 2. *For $n \geq 3$, if G is a connected graph of order n and size m, then*

$$0 \leq SO_5(G) \leq \frac{2\pi m\big((n-1)^2-1\big)}{\sqrt{2}+2\sqrt{(n-1)^2+1}} \tag{3}$$

and

$$0 \leq SO_6(G) \leq \pi m \left(\frac{(n-1)^2-1}{\sqrt{2}+2\sqrt{(n-1)^2+1}}\right)^2 \tag{4}$$

where the left equality in either of the inequalities (3) *and* (4) *holds if and only if G is regular, while the right equality in either of the inequalities* (3) *and* (4) *holds if and only if G is the star graph S_n.*

Proof. Since the proofs of both desired inequalities are similar to each other, we prove one of them; namely, the inequality (3). It is obvious that $SO_5(G) \geq 0$ with equality if and only if G is regular. Now, consider an arbitrary edge $uv \in E(G)$ with the condition $d_u \leq d_v$. By utilizing the function f defined in Lemma 1, we have

$$f(d_u, d_v) \leq f(1, d_v) \leq f(1, n-1)$$

where the equation $f(d_u, d_v) = f(1, n-1)$ holds if and only if $(d_u, d_v) = (1, n-1)$. Thus,

$$SO_5(G) = \sum_{uv \in E(G)} f(d_u, d_v) \leq \sum_{uv \in E(G)} f(1, n-1) = m\, f(1, n-1),$$

where the equation $SO_5(G) = m\, f(1, n-1)$ holds if and only if G is the star graph S_n. ■

From Lemma 2, the next result follows.

Proposition 2. *The star graph S_n uniquely attains the maximum values of SO_5 and SO_6 in the class $\mathcal{T}_n$ for each $n \in \{4, 5, 6, \ldots\}$.*

Now, we pay attention to solving Problem 2. For this, we need the following lemma.

Lemma 3. *If G is a connected graph of order at least 3 with the minimum degree 1 and the maximum degree Δ, then*

$$SO_4(G) \leq \frac{\pi(\Delta^2+1)}{2\,(\Delta+1)^2}F(G)$$

with equality if and only if $\{d_v, d_w\} = \{1, \Delta\}$ for every edge $vw \in E(G)$, where $F(G) = \sum_{x\in V(G)} d_x^3$, known as the forgotten topological index [3].

Proof. Note that the function f defined by

$$f(x,y) = \frac{\pi(x^2+y^2)}{2(x+y)^2},$$

is strictly decreasing in x whenever $1 \leq x \leq y$. Thereby, for any edge $vw \in E(G)$ with $d_v \leq d_w$, it holds that

$$f(d_v, d_w) \leq f(1, d_w) \leq f(1, \Delta) \tag{5}$$

where the equation $f(d_v, d_w) = f(1, \Delta)$ holds if and only if $d_v = 1$ and $d_w = \Delta$. From (5) it follows that

$$\frac{\pi}{2}\left(\frac{d_v^2+d_w^2}{d_v+d_w}\right)^2 \leq (d_v^2+d_w^2)f(1,\Delta). \tag{6}$$

Now, applying the summation over the edge set $E(G)$ on (6) yields the desired result. ∎

From a general result of [7], it follows that the inequality $F(T) < F(S_n)$ holds for any tree T of order n different from the star graph S_n. Also, for $\Delta \leq n-1$, it holds that

$$\frac{\pi(\Delta^2+1)}{2\,(\Delta+1)^2} \leq \frac{\pi((n-1)^2+1)}{2n^2}$$

with equality if and only if $\Delta = n-1$. These observations together with Lemma 3 give the next result, which provides the solution to Problem 2.

Proposition 3. *For $n \geq 3$, if T is a tree of order n different from the star graph S_n, then*

$$SO_4(T) < \frac{\pi(n-1)((n-1)^2+1)^2}{2n^2}.$$

Particularly, the star graph S_n uniquely attains the maximum value of SO_4 in the class $\mathcal{T}_n$ for each $n \in \{4, 5, 6, \ldots\}$.

Finally, in order to providing the solution to Problem 4, we give the following lemma.

Lemma 4. *If G is a connected graph with the maximum degree Δ and size m, then*

$$SO_3(G) \leq \sqrt{2}\pi\Delta m \quad \text{and} \quad SO_4(G) \leq \frac{\pi\Delta^2 m}{2},$$

where the equality in either of the two inequalities holds if and only if G is a Δ-regular graph.

Proof. For any edge $vw \in E(G)$, it holds that $d_v^2 + d_w^2 \leq \Delta d_v + \Delta d_w$ and $(d_v^2 + d_w^2)^2 \leq (\Delta d_v + \Delta d_w)^2$, where the equality in either of the two inequalities holds if and only if $d_v = d_w = \Delta$. ■

The next result gives the solution to Problem 4.

Proposition 4. *If G is a connected graph of order n, then*

$$SO_3(G) \leq \frac{\pi n(n-1)^2}{\sqrt{2}} \quad \text{and} \quad SO_4(G) \leq \frac{\pi n(n-1)^3}{4}$$

where the equality in either of the two inequalities holds if and only if G is the complete graph K_n.

Proof. Let Δ and m be the maximum degree and size of G. Then, by utilizing Lemma 4, we get

$$SO_3(G) \leq \sqrt{2}\pi\Delta m \leq \frac{\pi n(n-1)^2}{\sqrt{2}} \text{ and } SO_4(G) \leq \frac{\pi\Delta^2 m}{2} \leq \frac{\pi n(n-1)^3}{4}.$$

■

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