Solutions to Some Open Problems About Four Sombor–Index–Like Graph Invariants

Akbar Ali\textsuperscript{a,\,*}, Igor Ž. Milovanović\textsuperscript{b}, Abeer M. Albalahi\textsuperscript{a}
Abdulaziz M. Alanazi\textsuperscript{c}, Amjad E. Hamza\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, Faculty of Science, University of Ha’il, Ha’il, Saudi Arabia
\textsuperscript{b}Faculty of Electronic Engineering, University of Niš, Niš, Serbia
\textsuperscript{c}Department of Mathematics, University of Tabuk, Tabuk, Saudi Arabia
akbarali.maths@gmail.com, igor.Milovanovic@elfak.ni.ac.rs,
a.albalahi@uoh.edu.sa, am.alenezi@ut.edu.sa, boaljod2@hotmail.com

(Received February 23, 2023)

Abstract


1 Introduction and statements of problems

Throughout this paper, only finite graphs are considered. For notation and terminology from (chemical) graph theory, we refer the reader to relevant standard books; for example, [1,2,10,11].

A graph invariant $I_g$ is a function defined on the set of all graphs with the following property: $I_g(G_1) = I_g(G_2)$ whenever $G_1$ and $G_2$ are

\*Corresponding author.
isomorphic. Recently, Gutman [6] proposed six graph invariants in view of geometric considerations and referred them to as Sombor-index-like graph invariants. (Detail about the classical Sombor indices can be found in [4,5,8].) This paper is concerned with four such Sombor-index-like graph invariants, which are defined for a graph $G$ as

$$SO_3(G) = \sqrt{2} \pi \sum_{vw \in E(G)} \frac{d_v^2 + d_w^2}{d_v + d_w},$$

$$SO_4(G) = \frac{\pi}{2} \sum_{uv \in E(G)} \left( \frac{d_v^2 + d_w^2}{d_v + d_w} \right)^2,$$

$$SO_5(G) = 2 \pi \sum_{uv \in E(G)} \frac{|d_v^2 - d_w^2|}{\sqrt{2} + 2 \sqrt{d_v^2 + d_w^2}},$$

$$SO_6(G) = \pi \sum_{uv \in E(G)} \left( \frac{d_v^2 - d_w^2}{\sqrt{2} + 2 \sqrt{d_v^2 + d_w^2}} \right)^2,$$

where $E(G)$ denotes the set of edges of $G$ and $d_u$ represents the degree of a vertex $u$ in $G$.

By a chemical tree, we mean a tree of maximum degree at most 4. Let $T_n$, $CT_n$ and $CG_n$ be the classes of all trees, chemical trees and connected graphs, respectively, of order $n$. Quite recently, Tang et al. [9] investigated some extremal properties of Sombor-index-like graph invariants and posed following open problems.

**Problem 1.** Find the extremal values of the graph invariants $SO_5$ and $SO_6$ in the classes $T_n$ and $CT_n$.

**Problem 2.** Find the maximum value of the graph invariants $SO_4$ in the class $T_n$.

**Problem 3.** Find the extremal values of the graph invariants $SO_5$ and $SO_6$ in the class $CG_n$.

**Problem 4.** Find the maximum values of the graph invariants $SO_3$ and $SO_4$ in the class $CG_n$. 
In this paper, Problems 2 and 4 are solved completely. Also, partial solutions to Problems 1 and 3 are provided; more precisely, Problem 1 is solved for the class \( T_n \) (this solution also resolve the minimal part of Problem 1 regarding \( CT_n \)) and a solution to the minimal part of Problem 3 is indicated.

2 Results

A vertex \( x \) in a graph is said to be a pendent vertex if \( d_x = 1 \). An edge incident to a pendent vertex is known as a pendent edge. For an edge \( vw \) of a graph, define

\[
SO_5(vw) = \frac{2\pi |d_v^2 - d_w^2|}{\sqrt{2} + 2\sqrt{d_v^2 + d_w^2}}.
\]

The following result gives the solution to the minimal part of Problem 1.

**Proposition 1.** If \( T \) is a tree of order at least 3, then

\[
SO_5(T) \geq \frac{12\pi}{\sqrt{2} + 2\sqrt{5}} \quad \text{and} \quad SO_6(T) \geq \frac{18\pi}{(\sqrt{2} + 2\sqrt{5})^2},
\]

where the equality in any of these two inequalities holds if and only if \( T \) is a path graph. Particularly, the path graph \( P_n \) uniquely attains the minimum values of \( SO_5 \) and \( SO_6 \) in the classes \( T_n \) and \( CT_n \) for each \( n \in \{4, 5, 6, \ldots\} \).

**Proof.** Since the proofs of both desired inequalities are similar to each other, we prove one of them; namely, the one involving \( SO_5 \). Consider a pendent vertex \( u \in V(T) \) adjacent to a vertex \( v \). Then

\[
SO_5(uv) = \frac{2\pi (d_v^2 - 1)}{\sqrt{2} + 2\sqrt{d_v^2 + 1}} \geq \frac{6\pi}{\sqrt{2} + 2\sqrt{5}}
\]

where the equation \( SO_5(uv) = 6\pi/(\sqrt{2} + 2\sqrt{5}) \) holds if and only if \( d_v = 2 \). Denote by \( PE(T) \) the set consisting of all pendent edges in \( T \). By utilizing
the definition of $SO_5$, we have

$$SO_5(T) = \sum_{uv \in P_E(T)} SO_5(uv) + \sum_{xy \in E(T) \setminus P_E(T)} SO_5(xy)$$

$$\geq \sum_{uv \in P_E(T)} \frac{6\pi}{\sqrt{2} + 2\sqrt{5}} + \sum_{xy \in E(T) \setminus P_E(T)} (0) \quad (1)$$

$$= \frac{6\pi}{\sqrt{2} + 2\sqrt{5}} |P_E(T)|$$

$$\geq \frac{12\pi}{\sqrt{2} + 2\sqrt{5}}. \quad (2)$$

Certainly, the equality in (1) holds if and only if $\max\{d_v, d_u\} = 2$ for each edge $uv \in P_E(T)$ and $d_y = d_x$ for each edge $xy \in E(T) \setminus P_E(T)$. Note also that the equality in (2) holds if and only if the set $P_E(T)$ has only two elements.

**Remark.** The proof of Proposition 1 suggests slightly general lower bounds on $SO_5$ and $SO_6$ as given below. For a connected graph $G$ with at least two edges and with $n_1$ pendant vertices, the following inequalities hold

$$SO_5(G) \geq \frac{6\pi}{\sqrt{2} + 2\sqrt{5}} n_1 \quad \text{and} \quad SO_6(G) \geq \frac{9\pi}{(\sqrt{2} + 2\sqrt{5})^2} n_1,$$

where the equality in any of these two inequalities holds if and only if $G$ is either a path graph or a regular graph.

Next, we move towards a solution to the maximal part of Problem 1 for the case of trees. For this, we need the following lemma.

**Lemma 1.** The functions $f$ and $g$ defined by

$$f(x, y) = \frac{2\pi|x^2 - y^2|}{\sqrt{2} + 2\sqrt{x^2 + y^2}} \quad \text{and} \quad g(x, y) = \pi \left( \frac{x^2 - y^2}{\sqrt{2} + 2\sqrt{x^2 + y^2}} \right)^2$$

are strictly decreasing in $x$ whenever $1 \leq x \leq y$.

The next lemma not only provides the solution to the maximal part of Problem 1 ($m = n - 1$) concerning the class $T_n$ but also indicates the
trivial solution to the minimal part of Problem 3.

**Lemma 2.** For $n \geq 3$, if $G$ is a connected graph of order $n$ and size $m$, then

$$0 \leq SO_5(G) \leq \frac{2\pi m((n-1)^2 - 1)}{\sqrt{2} + 2\sqrt{(n-1)^2 + 1}}$$

and

$$0 \leq SO_6(G) \leq \pi m \left(\frac{(n-1)^2 - 1}{\sqrt{2} + 2\sqrt{(n-1)^2 + 1}}\right)^2$$

where the left equality in either of the inequalities (3) and (4) holds if and only if $G$ is regular, while the right equality in either of the inequalities (3) and (4) holds if and only if $G$ is the star graph $S_n$.

**Proof.** Since the proofs of both desired inequalities are similar to each other, we prove one of them; namely, the inequality (3). It is obvious that $SO_5(G) \geq 0$ with equality if and only if $G$ is regular. Now, consider an arbitrary edge $uv \in E(G)$ with the condition $d_u \leq d_v$. By utilizing the function $f$ defined in Lemma 1, we have

$$f(d_u, d_v) \leq f(1, d_v) \leq f(1, n-1)$$

where the equation $f(d_u, d_v) = f(1, n-1)$ holds if and only if $(d_u, d_v) = (1, n-1)$. Thus,

$$SO_5(G) = \sum_{uv \in E(G)} f(d_u, d_v) \leq \sum_{uv \in E(G)} f(1, n-1) = m f(1, n-1),$$

where the equation $SO_5(G) = m f(1, n-1)$ holds if and only if $G$ is the star graph $S_n$. 

From Lemma 2, the next result follows.

**Proposition 2.** The star graph $S_n$ uniquely attains the maximum values of $SO_5$ and $SO_6$ in the class $\mathcal{T}_n$ for each $n \in \{4, 5, 6, \ldots\}$.

Now, we pay attention to solving Problem 2. For this, we need the following lemma.
Lemma 3. If $G$ is a connected graph of order at least 3 with the minimum degree 1 and the maximum degree $\Delta$, then

$$SO_4(G) \leq \frac{\pi(\Delta^2 + 1)}{2(\Delta + 1)^2} F(G)$$

with equality if and only if $\{d_v, d_w\} = \{1, \Delta\}$ for every edge $vw \in E(G)$, where $F(G) = \sum_{x \in V(G)} d_x^3$, known as the forgotten topological index [3].

Proof. Note that the function $f$ defined by

$$f(x, y) = \frac{\pi(x^2 + y^2)}{2(x + y)^2},$$

is strictly decreasing in $x$ whenever $1 \leq x \leq y$. Thereby, for any edge $vw \in E(G)$ with $d_v \leq d_w$, it holds that

$$f(d_v, d_w) \leq f(1, d_w) \leq f(1, \Delta) \quad (5)$$

where the equation $f(d_v, d_w) = f(1, \Delta)$ holds if and only if $d_v = 1$ and $d_w = \Delta$. From (5) it follows that

$$\frac{\pi}{2} \left( \frac{d_v^2 + d_w^2}{d_v + d_w} \right)^2 \leq (d_v^2 + d_w^2) f(1, \Delta). \quad (6)$$

Now, applying the summation over the edge set $E(G)$ on (6) yields the desired result. \qed

From a general result of [7], it follows that the inequality $F(T) < F(S_n)$ holds for any tree $T$ of order $n$ different from the star graph $S_n$. Also, for $\Delta \leq n - 1$, it holds that

$$\frac{\pi(\Delta^2 + 1)}{2(\Delta + 1)^2} \leq \frac{\pi((n - 1)^2 + 1)}{2n^2}$$

with equality if and only if $\Delta = n - 1$. These observations together with Lemma 3 give the next result, which provides the solution to Problem 2.
Proposition 3. For \( n \geq 3 \), if \( T \) is a tree of order \( n \) different from the star graph \( S_n \), then

\[
SO_4(T) < \frac{\pi(n-1)((n-1)^2+1)}{2n^2}.
\]

Particularly, the star graph \( S_n \) uniquely attains the maximum value of \( SO_4 \) in the class \( T_n \) for each \( n \in \{4, 5, 6, \ldots\} \).

Finally, in order to providing the solution to Problem 4, we give the following lemma.

Lemma 4. If \( G \) is a connected graph with the maximum degree \( \Delta \) and size \( m \), then

\[
SO_3(G) \leq \sqrt{2} \pi \Delta m \quad \text{and} \quad SO_4(G) \leq \frac{\pi \Delta^2 m}{2},
\]

where the equality in either of the two inequalities holds if and only if \( G \) is a \( \Delta \)-regular graph.

Proof. For any edge \( vw \in E(G) \), it holds that \( d_v^2 + d_w^2 \leq \Delta d_v + \Delta d_w \) and \( (d_v^2 + d_w^2)^2 \leq (\Delta d_v + \Delta d_w)^2 \), where the equality in either of the two inequalities holds if and only if \( d_v = d_w = \Delta \).

The next result gives the solution to Problem 4.

Proposition 4. If \( G \) is a connected graph of order \( n \), then

\[
SO_3(G) \leq \frac{\pi n(n-1)^2}{\sqrt{2}} \quad \text{and} \quad SO_4(G) \leq \frac{\pi n(n-1)^3}{4}
\]

where the equality in either of the two inequalities holds if and only if \( G \) is the complete graph \( K_n \).

Proof. Let \( \Delta \) and \( m \) be the maximum degree and size of \( G \). Then, by utilizing Lemma 4, we get

\[
SO_3(G) \leq \sqrt{2} \pi \Delta m \leq \frac{\pi n(n-1)^2}{\sqrt{2}} \quad \text{and} \quad SO_4(G) \leq \frac{\pi \Delta^2 m}{2} \leq \frac{\pi n(n-1)^3}{4}.
\]
Acknowledgment: This research has been funded by the Scientific Research Deanship, University of Ha'il, Saudi Arabia, through project number RG-22 005.

References


