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Resonance Graphs and a Binary Coding of Perfect Matchings of Outerplane Bipartite Graphs

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Abstract

The aim of this paper is to investigate resonance graphs of 2-connected outerplane bipartite graphs, which include various families of molecular graphs. Firstly, we present an algorithm for a binary coding of perfect matchings of these graphs. Further, 2-connected outerplane bipartite graphs with isomorphic resonance graphs are considered. In particular, it is shown that if two 2-connected outerplane bipartite graphs are evenly homeomorphic, then its resonance graphs are isomorphic. Moreover, we prove that for any 2-connected outerplane bipartite graph G there exists a cata-condensed even ring systems H such that the resonance graphs of G and H are isomorphic. We conclude with the characterization of 2-connected outerplane bipartite graphs whose resonance graphs are daisy cubes.

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1 Introduction

Kekulé structures of aromatic hydrocarbons reflect the positions of double bonds in a molecule. In graph theory, Kekulé structures are modelled by perfect matchings of the corresponding molecular graph. On the other hand, the interaction between Kekulé structures is reflected in the resonance graph of a given molecule. Resonance graphs were independently introduced by chemists (El-Basil [3, 4], Gründler [9]) and also by mathematicians (Zhang, Guo, and Chen [15]) under the name Z-transformation graph.

Initially, various properties of resonance graphs of hexagonal systems were established in [15]. Later, the concept of resonance graphs was generalized to all plane (elementary) bipartite graphs (for example, see [17,18]).

In [13, 16], a binary coding procedure of vertices of resonance graphs of catacondensed hexagonal systems was developed. Later [1], this binary coding was generalized to catacondensed even ring systems (CERS), which form a subfamily of 2-connected outerplane bipartite graphs (see also [14]). In recent years, various structural properties of resonance graphs of 2-connected (outer)plane bipartite graphs were deduced [5–8]. For example, in [8] all plane bipartite graphs whose resonance graphs can be constructed from an edge by a sequence of peripheral convex expansions are characterized.

The paper is organized as follows. Firstly, in Section 3 we generalize the binary coding procedure of perfect matchings from CERS [1] to all 2-connected outerplane bipartite graphs. Next, in Section 4 we study 2-connected outerplane bipartite graphs with isomorphic resonance graphs. In particular, we prove that if G and H are evenly homeomorphic, then its resonance graphs are isomorphic. Furthermore, in Section 5 we prove that for any 2-connected outerplane bipartite graph G there exists a CERS H such that the resonance graphs of G and H are isomorphic. Finally, we characterize 2-connected outerplane bipartite graphs whose resonance graphs are daisy cubes, which extends results from [19] and [2].

2 Preliminaries

The *distance* $d_G(u,v)$ between vertices u and v of a graph G is defined as the usual shortest path distance. The distance between two edges e and f of G, denoted by $d_G(e,f)$, is defined as the distance between corresponding vertices in the line graph $L(G)$ of G.

The *hypercube* Q_n of dimension n is defined in the following way: all vertices of Q_n are presented as n-tuples $x_1x_2\ldots x_n$ where $x_i \in \{0,1\}$ for each $i \in \{1,\ldots,n\}$, and two vertices of Q_n are adjacent if the corresponding n-tuples differ in precisely one position. A subgraph H of a graph G is an *isometric subgraph* if for all $u,v \in V(H)$ it holds $d_H(u,v) = d_G(u,v)$. If a graph is isomorphic to an isometric subgraph of G, we say that it can be *isometrically embedded* in G. Any isometric subgraph of a hypercube is called a *partial cube* [11].

If G is a plane graph, then an edge e of G that belongs to two inner faces of G will be called an *inner edge*. We say that two faces of G are *adjacent* if they have an edge in common. An inner face adjacent to the outer face is called a *peripheral face*. In addition, we denote the edges lying on some face s of G by $E(s)$. The subgraph induced by the edges in $E(s)$ is the *periphery of* s and the periphery of the outer face is also called the *periphery of* G. Moreover, for a peripheral face s and the outer face s_0, the subgraph induced by the edges in $E(s) \cap E(s_0)$ is called the *common periphery* of s and G. The vertices of G that belong to the outer face are called *peripheral vertices* and the remaining vertices are *interior vertices*. Furthermore, an *outerplane graph* is a plane graph in which all vertices are peripheral vertices.

The following definitions can be found, for example, in [5]. A bipartite graph G is *elementary* if and only if it is connected and each edge is contained in some perfect matching of G. A peripheral face s of a plane elementary bipartite graph G is called *reducible* if the subgraph H of G obtained by removing all internal vertices (if exist) and edges on the common periphery of s and G is elementary.

An *even ring system* is a 2-connected plane bipartite graph with all interior vertices of degree 3 and all peripheral vertices of degree 2 or 3.

Moreover, an outerplane even ring system is called *catacondensed even ring system* or shortly CERS [14].

A *1-factor* of a graph G is a spanning subgraph of G such that every vertex has degree one. The edge set of a 1-factor is called a *perfect matching* of G, which is a set of independent edges covering all vertices of G. In chemical literature, perfect matchings are known as Kekulé structures (see [10] for more details).

Let G be a 2-connected plane bipartite graph. The *resonance graph* $R(G)$ of G is the graph whose vertices are the perfect matchings of G, and two perfect matchings M_1, M_2 are adjacent whenever their symmetric difference forms the edge set of exactly one inner face s of G, i.e. $M_1 \oplus M_2 = E(s)$.

Next, we state the definition of a reducible face decomposition, see [17] and [5, 6]. Firstly, we introduce the *bipartite ear decomposition* of a plane elementary bipartite graph G with n inner faces. Starting from an edge e of G, join its two end-vertices by a path P_1 of odd length and proceed inductively to build a sequence of bipartite graphs as follows. If $G_{i-1} = e + P_1 + \cdots + P_{i-1}$ has already been constructed, add the ith ear P_i of odd length by joining any two vertices belonging to different bipartition sets of G_{i-1} such that P_i has no internal vertices in common with the vertices of G_{i-1}. A bipartite ear decomposition of a plane elementary bipartite graph G is called a *reducible face decomposition* (shortly RFD) if G_1 is a periphery of a finite face s_1 of G, and the ith ear P_i lies in the exterior of G_{i-1} such that P_i and a part of the periphery of G_{i-1} surround a finite face s_i of G for all $i \in \{2, \ldots, n\}$. For such a decomposition, we use notation $RFD(G_1, G_2, \ldots, G_n)$, where $G_n = G$.

Furthermore, if G is a graph and $X \subseteq V(G)$, then the notation $G[X]$ is used to denote the subgraph of G induced by the set X.

3 Binary coding of perfect matchings

In this section, we develop an algorithm for constructing binary codes of perfect matchings of 2-connected outerplane bipartite graphs. This represents a generalization of the result from [1]. For this purpose, we firstly need several auxiliary results.

Let G be a 2-connected outerplane bipartite graph and H_1, H_2 two induced subgraphs of G such that $V(H_1) \cup V(H_2) = V(G)$ and $|E(H_1) \cap E(H_2)| = 1$. If $e \in E(H_1) \cap E(H_2)$, we say that H_1 and H_2 are *e-subgraphs* of graph G. Moreover, let M be a perfect matching of G. We say that a vertex $x \in V(H_i)$ is *M-covered* in H_i, $i \in \{1, 2\}$, if there exists a vertex $y \in V(H_i)$ such that $xy \in M$. Furthermore, an edge f is M-covered in H_i if its end-vertices are both M-covered in H_i.

Proposition 1. *Let G be a 2-connected outerplane bipartite graph and H_1, H_2 two induced subgraphs of G such that $V(H_1) \cup V(H_2) = V(G)$ and $e = uv$ is the only edge in the set $E(H_1) \cap E(H_2)$. Moreover, let M be a perfect matching of G. Then e is M-covered in H_1 or H_2.*

Proof. Suppose that u is M-covered in H_1 but not in H_2 and v is M-covered in H_2 but not in H_1. It is easy to see that H_1 is again a 2-connected outerplane bipartite graph and therefore, it has an even number of vertices. Let $H_1' = H_1 - v$. Then, $M \setminus E(H_2)$ is a perfect matching of the graph H_1'. However, graph H_1' has an odd number of vertices, which is a contradiction with the existence of a perfect matching. ■

Proposition 2. *Let G be a 2-connected outerplane bipartite graph and $e = uv \in E(G)$ an edge belonging to two inner faces s and s' of G. Also, let H be the e-subgraph of G containing s', $f \neq e$ an edge of face s, and H_f the f-subgraph of G not containing e. Suppose that M is a perfect matching of G such that e is M-covered in H. Then $d_G(e, f)$ is even if and only if f is M-covered in H_f.*

Proof. Let $P = (e, f_1, f_2, \ldots, f_k = f)$ be a shortest path in G between e and f. Moreover, let H_i be the f_i-subgraph of G that does not contain s. Since e is M-covered in H, by Proposition 1 it follows that f_1 is not M-covered in H_1. Using the same argument, the edge f_2 must be M-covered in H_2. Inductively, we obtain that f_i is M-covered in H_i if and only if i is even. As a consequence, f is M-covered in H_f if and only if $k = d_G(e, f)$ is even. ■

Remark. *Let s be an inner face of a 2-connected outerplane bipartite graph G. Then s is reducible if and only if it is adjacent to exactly one inner face of G [6].*

Let G be a 2-connected outerplane bipartite graph with n inner faces, s a reducible face of G, and $e = uv$ the edge of s that belongs to exactly two inner faces of G. Moreover, let G' be the graph obtained from G by removing face s. In addition, we denote by H the subgraph of G induced on the vertices of s. We partition the perfect matchings of G into the sets $\mathcal{M}_e(G)$, $\mathcal{M}_{\overline{e}}^{G'}(G)$, and $\mathcal{M}_{\overline{e}}^{H}(G)$. More precisely, $\mathcal{M}_e(G)$ is the set of all perfect matchings of G that contain edge e and $\mathcal{M}_{\overline{e}}^{G'}(G)$ is the set of all perfect matchings M of G such that $e \notin M$ and e is M-covered in G'. Similarly, $\mathcal{M}_{\overline{e}}^{H}(G)$ is the set of all perfect matchings M of G such that $e \notin M$ and e is M-covered in H.

It is straightforward to see that the subgraph of $R(G)$ induced by the vertices from $\mathcal{M}_e(G) \cup \mathcal{M}_{\overline{e}}^{G'}(G)$, denoted as $R(G)[\mathcal{M}_e(G) \cup \mathcal{M}_{\overline{e}}^{G'}(G)]$, is isomorphic to $R(G')$. Moreover, $R(G)[\mathcal{M}_e(G)]$ and $R(G)[\mathcal{M}_{\overline{e}}^{H}(G)]$ are also isomorphic. By using these facts, we can obtain binary codes of length n of perfect matchings of G, where n is the number of inner faces of G. Then, G' contains $n-1$ inner faces and suppose that we have already obtained binary codes of perfect matchings of G'.

Every perfect matching M' of G' with binary code $b(M')$ can be in the unique way extended to a perfect matching M of G, see Figure 1 (a). Binary code $b(M)$ is obtained by concatenation of 0 to $b(M')$. In this way, we obtain the binary codes for perfect matchings in the set $\mathcal{M}_e(G) \cup \mathcal{M}_{\overline{e}}^{G'}(G)$.

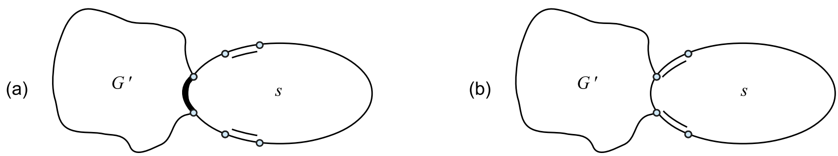

Figure 1. Two possibilities for the perfect matchings of G. The common edge of G' and s is bold iff it is M-covered in G'.

On the other hand, let M' be a perfect matching of G' such that $e \in M'$. We define M as the unique perfect matching of G such that $M' \setminus \{e\} \subseteq M$

and $e \notin M$, see Figure 1 (b). Binary code $b(M)$ is then obtained by concatenation of 1 to $b(M')$. Here, we obtain the binary codes for perfect matchings in the set $\mathcal{M}_{\overline{e}}^{H}(G)$.

The obtained binary coding procedure of perfect matchings of a 2-connected outerplane bipartite graph G follows a peripheral convex expansion described in [6]. Therefore, by Theorem 3.2 [6] our procedure gives an isometric embedding of the resonance graph $R(G)$ into the hypercube of dimension n, where n is the number of inner faces of G. Consequently, two perfect matchings M_1 and M_2 of G are adjacent in $R(G)$ if and only if their binary codes differ in exactly one position.

In [1, 13] the algorithms for binary coding of perfect matchings of benzenoid graphs and CERS were presented. The mentioned algorithms are here generalized to 2-connected outerplane bipartite graphs. We first extend the following definition from [1] to a larger family of graphs.

Definition 1. *Let s, s', s'' be three inner faces of a 2-connected outerplane bipartite graph such that s and s' have common edge e and s', s'' have common edge f. The triple (s, s', s'') is called an **adjacent triple of inner faces**. Moreover, (s, s', s'') is **regular** if the distance $d_G(e, f)$ is an even number and **irregular** otherwise.*

To show an example, consider the triple (s_i, s_j, s_{r+1}) from Figure 2. The mentioned triple is regular in case (a) and irregular in case (b).

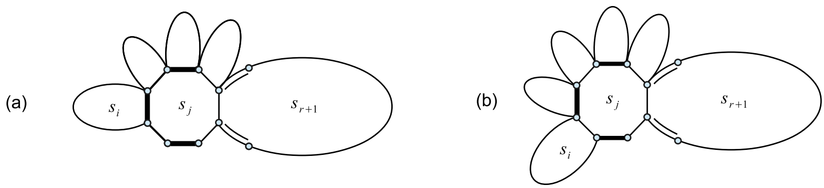

Figure 2. Perfect matchings of G_{r+1} with respect to the regularity of triple (s_i, s_j, s_{r+1}). An edge f is bold iff it is M-covered in the f-subgraph that does not contain face s_j.

Suppose G is a 2-connected outerplane bipartite graph. Moreover, let $RFD(G_1, G_2, \ldots, G_n)$, where $G_n = G$, be a reducible face decomposition associated with a sequence of inner faces $s_1, s_2, \ldots, s_n$. The set of all binary codes for the perfect matchings of G_r will be denoted as B_r for every $r \in \{1, \dots, n\}$.

If G has only two faces, s_1 and s_2, we define the binary codes $B_2 = \{00, 01, 10\}$ in the following way: code 00 represents the perfect matching that contains the common edge of s_1 and s_2. Further, let 01 be the perfect matching obtained from 00 by rotating the edges in s_2, and 10 the remaining perfect matching, see Figure 3.

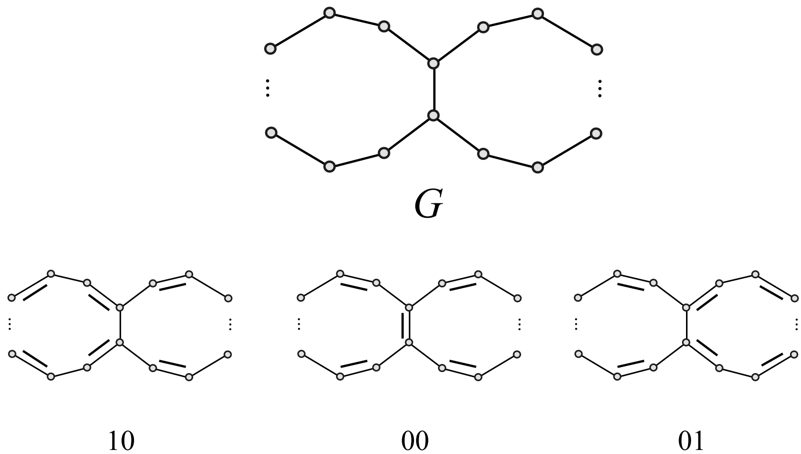

Figure 3. Binary coding of perfect matchings of a graph G with two inner faces.

Assume that B_r is the set of all the binary codes for perfect matchings of the graph G_r, which is composed of faces $s_1, \dots, s_r$. Graph G_{r+1} is then obtained from G_r by adding a new face s_{r+1}. Let s_j, $j \in \{1, \dots, r\}$, be the unique face adjacent to s_{r+1}. Moreover, let s_i be the inner face adjacent to s_j with the smallest index $i \in \{1, \dots, r\}$.

The set B_{r+1} of all binary codes for perfect matchings of the graph G_{r+1} then contains all the strings that are obtained by concatenating 0 to every $x = x_1 x_2 \dots x_r \in B_r$. Moreover, the set B_{r+1} also contains additional codes, which are due to Proposition 2 obtained in one of the following ways:

(a) If (s_i, s_j, s_{r+1}) is regular, then B_{r+1} also contains all the strings that are obtained by concatenating 1 to every $x = x_1 x_2 \dots x_r \in B_r$ with

$x_j = 0$, see Figure 2 (a).

(b) If (s_i, s_j, s_{r+1}) is irregular, then B_{r+1} also contains all the strings that are obtained by concatenating 1 to every $x = x_1x_2 \dots x_r \in B_r$ with $x_j = 1$, see Figure 2 (b).

Finally, we present the procedure for binary coding of perfect matchings for a 2-connected outerplane bipartite graph, see Algorithm 1. In the algorithm, we denote $B := B_r$ and $B' := B_{r+1}$.

Algorithm 1: Binary coding of perfect matchings of a 2-connected outerplane bipartite graph.

Input: $RFD(G_1, G_2, \dots, G_n)$ of a graph G associated with a sequence $s_1, \dots, s_n$.

Output: Binary codes for all perfect matchings of G.

```
B := {00, 01, 10}
for r = 2, ..., n − 1 do
    B' := ∅
    set j ∈ {1, ..., r} such that s_j is adjacent to s_{r+1}
    i = min{l | s_l is adjacent to s_j}
    if (s_i, s_j, s_{r+1}) is regular then
        for each x ∈ B do
            B' := B' ∪ {x0}
            if x_j = 0 then
                B' := B' ∪ {x1}
            end
        end
    else
        for each x ∈ B do
            B' := B' ∪ {x0}
            if x_j = 1 then
                B' := B' ∪ {x1}
            end
        end
    end
    B := B'
end
```

We now apply Algorithm 1 on graph G from Figure 4. Its faces are denoted as $s_1, \dots, s_4$. As usual, by G_k we denote the subgraph of G that contains faces $s_1, \ldots, s_k$, where $k \in \{2, 3, 4\}$, and therefore $G_4 = G$. The resonance graphs obtained by Algorithm 1 are shown in Figure 4.

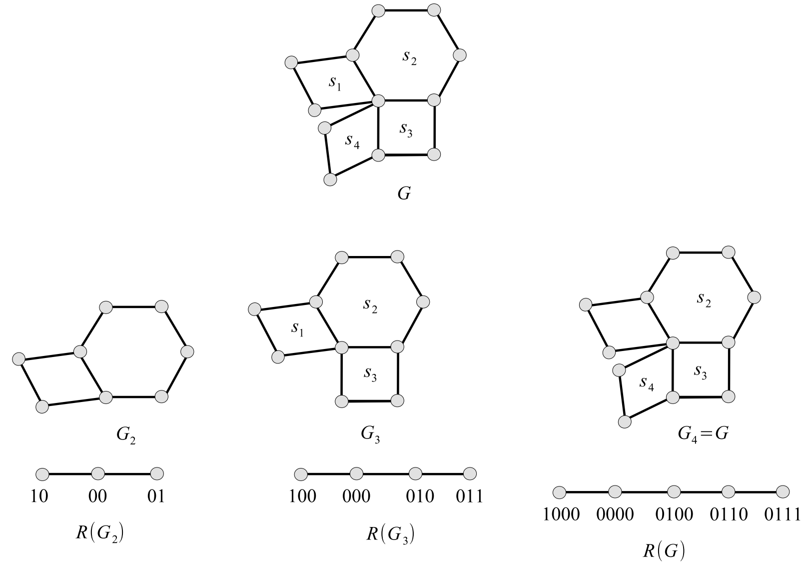

Figure 4. Binary coding procedure of perfect matchings of a graph G together with resonance graphs.

4 Evenly homeomorphic 2-connected outerplane bipartite graphs

In this section, we consider 2-connected outerplane bipartite graphs with isomorphic resonance graphs. The main result of the section represents a generalization of a result from [1]. Firstly, we need to define two transformations. As usual, for a graph G we denote by $\deg u$ the degree of a vertex $u \in V(G)$.

Transformation 1. *Let G be a 2-connected outerplane bipartite graph and $P = (x, y, z)$ a path on three vertices in G such that $\deg y = 2$ and the face containing P is not a 4-cycle. The graph G' is obtained from G by deleting y and identifying vertices x and z, see Figure 5.*

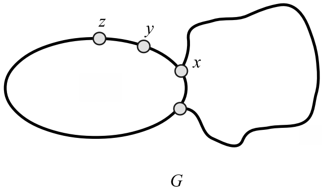

Figure 5. Transformation 1.

Transformation 2. *Let G be a 2-connected outerplane bipartite graph and $v \in V(G)$ such that* $\deg v = k$*. Then v belongs to exactly $k-1$ inner faces of G. Moreover, let $u_1, \ldots, u_k$ be the neighbours of v ordered in the clockwise direction such that vu_1 and vu_k belong to the outer face.*

(*i*) *If $k \geq 3$, then the graph G' is obtained from G be deleting vertex v, adding the path (v_1, v_2, v_3) and inserting edges v_1u_1, v_1u_2, and v_3u_i for any $i \in \{3, \ldots, k\}$, see Figure 6 (i).*

(*ii*) *If $k = 2$, then the graph G' is obtained from G be deleting vertex v, adding the path (v_1, v_2, v_3) and inserting edges v_1u_1 and v_3u_2, see Figure 6 (ii).*

(i)

(ii)

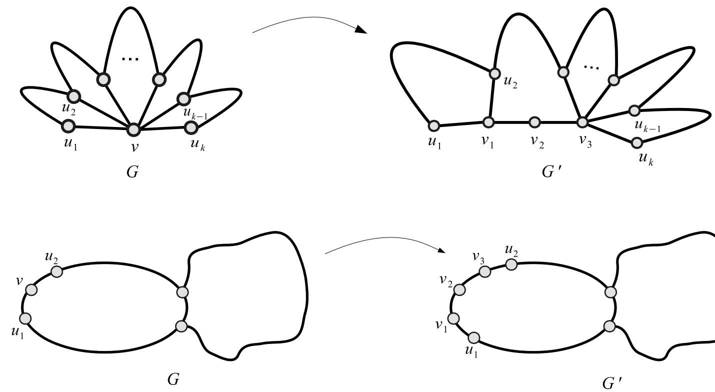

Figure 6. Transformation 2.

Note that if $\deg(v) = k \geq 4$, then after applying Transformation 2 the maximum degree of v_1 and v_3 in graph G' is $k-1$. It is also obvious that the graph G' obtained by Transformation 1 or Transformation 2 is again a 2-connected outerplane bipartite graph.

In the following definition, we generalize the concept of evenly homeomorphic CERS [2] to all 2-connected outerplane bipartite graphs.

Definition 2. *Let G and H be two 2-connected outerplane bipartite graphs. Then G is* ***evenly homeomorphic*** *to H if it is possible to successively apply Transformation 1 or 2 on G and H to obtain graphs G' and H', respectively, such that G' and H' are isomorphic. In such a case we write $G \overset{R}{\sim} H$.*

It is obvious that the relation $\overset{R}{\sim}$ is an equivalence relation on the set of all 2-connected outerplane bipartite graphs. Moreover, if G and H are evenly homeomorphic, then both graphs have the same number of inner faces.

The following two lemmas are also needed.

Lemma 1. *Let G and G' be 2-connected outerplane bipartite graphs such that G' is obtained from G by applying Transformation 1 or Transformation 2. Then any two inner edges $e, f \in E(G)$ are also in $E(G')$ and it holds $d_{G'}(e, f) - d_G(e, f) \in \{-2, 0, 2\}$.*

Proof. Obviously, if we apply Transformation 1, then the distance between two inner edges e and f remains the same or decreases by 2. On the other hand, after using Transformation 2 the distance between e and f remains the same or increases by 2. ■

Lemma 2. *Let G and H be evenly homeomorphic 2-connected outerplane bipartite graphs and let (s_1, s_2, s_3) be an adjacent triple of inner faces in G. If (s'_1, s'_2, s'_3) denotes the corresponding adjacent triple of inner faces in H, then the triple (s_1, s_2, s_3) is regular if and only if the triple (s'_1, s'_2, s'_3) is regular.*

Proof. Let $e \in E(s_1) \cap E(s_2)$, $f \in E(s_2) \cap E(s_3)$, $e' \in E(s'_1) \cap E(s'_2)$, and $f' \in E(s'_2) \cap E(s'_3)$. By the definitions of Transformations 1, 2 and Lemma 1, it holds that $d_G(e, f)$ is even if and only if $d_H(e', f')$ is even. Therefore, the triple (s_1, s_2, s_3) is regular if and only if (s'_1, s'_2, s'_3) is regular. ■

Finally, we can state the main result of this section.

Theorem 1. *Let G and H be 2-connected outerplane bipartite graphs. If G and H are evenly homeomorphic, then the resonance graph $R(G)$ is isomorphic to the resonance graph $R(H)$.*

Proof. Suppose G is a 2-connected outerplane bipartite graph. Moreover, let $RFD(G_1, G_2, \ldots, G_n)$, where $G_n = G$, be a reducible face decomposition associated with the sequence of inner faces $s_1, s_2, \ldots, s_n$. Also, denote by s'_i, $i \in \{1, \ldots, n\}$, the corresponding inner faces of graph H, which give the reducible face decomposition $RFD(H_1, H_2, \ldots, H_n)$ such that $H_n = H$.

We show that for any $r \in \{2, \ldots, n\}$, the set of binary codes B_r of the graph G_r obtained by Algorithm 1 coincides with the set of binary codes B'_r of the graph H_r. Consequently, the resonance graphs $R(G_r)$ and $R(H_r)$ are isomorphic for all $r \in \{2, \ldots, n\}$, which implies that $R(G)$ and $R(H)$ are isomorphic. We proceed by induction on the number of inner faces.

Obviously, the sets of binary codes B_2 and B'_2 are equal. Next, assume that for some $r \geq 2$ the sets of codes B_r and B'_r coincide. Let s_j be the face of G_{r+1} from the set $\{s_1, \ldots, s_r\}$ that is adjacent to s_{r+1}. In addition, define s_i as the face with the smallest index among all the adjacent inner faces of s_j. Analogously, we also define s'_j and s'_i in the graph H_{r+1}. By Lemma 2 we obtain that the adjacent triple of inner faces (s_i, s_j, s_{r+1}) is regular if and only if (s'_i, s'_j, s'_{r+1}) is regular. Hence, by Algorithm 1 we obtain $B_{r+1} = B'_{r+1}$. ∎

We conclude the section with the following open problem.

Problem. *Characterize 2-connected outerplane bipartite graphs with isomorphic resonance graphs.*

5 Resonance graphs of 2-connected outerplane bipartite graphs and CERS

In this final section, we firstly show that the set of all resonance graphs of 2-connected outerplane bipartite graphs coincides with the set of all resonance graphs of CERS. Next, we consider 2-connected outerplane bipartite graphs whose resonance graphs are daisy cubes.

Theorem 2. *For any* 2*-connected outerplane bipartite graph* G *there exists a CERS* H *such that* G *and* H *are evenly homeomorphic. Consequently, the resonance graphs* $R(G)$ *and* $R(H)$ *are isomorphic.*

Proof. Let G be a 2-connected outerplane bipartite graph such that G is not a CERS. Then there exists a vertex $v \in V(G)$ for which $\deg v = k \geq 4$. After applying Transformation 2 on v, we obtain a 2-connected outerplane bipartite graph G_1 with three new vertices v_1, v_2, v_3, see Figure 6 (i). It is easy to see that $\deg v_1 = 3$, $\deg v_2 = 2$, and $\deg v_3 = k-1$. Note that G and G_1 are evenly homeomorphic and by Theorem 1 the resonance graphs $R(G)$ and $R(G_1)$ are isomorphic. Then, we repeat the same procedure until every vertex of the transformed graph has degree at most 3. Consequently, we obtain a sequence of graphs $G_1, G_2, \ldots, G_m$, where G and G_m are evenly homeomorphic and the resonance graphs $R(G)$ and $R(G_m)$ are isomorphic. Let $H = G_m$. Since H is a 2-connected outerplane bipartite graph with the degree of every vertex at most 3, it is a CERS. ■

Next, we characterize 2-connected outerplane bipartite graphs whose resonance graphs are daisy cubes [12]. Therefore, some additional definitions are needed.

Let $B = \{0, 1\}$ and B^n the set of all binary strings of length n. Moreover, let $\leq$ be a partial order on B^n defined with $(u_1, \ldots, u_n) \leq (v_1, \ldots, v_n)$ if and only if $u_i \leq v_i$ holds for all $i \in \{1, \ldots, n\}$. For $X \subseteq B^n$, we define the graph $Q_n(X)$ as the subgraph of Q_n as $Q_n(X) = Q_n\left[\{u \in B^n \mid u \leq x \,\text{for some}\, x \in X\}\right]$ and say that $Q_n(X)$ is a *daisy cube* (generated by X).

Furthermore, we generalize the concept of regular CERS from [2] to all 2-connected outerplane bipartite graphs.

Definition 3. *If a* 2*-connected outerplane bipartite graph* G *has at most two inner faces or if every adjacent triple of inner faces of* G *is regular, then* G *is called* ***regular****.*

The following result was proved in [2].

Theorem 3. *[2] If G is a CERS, then G is regular if and only if the resonance graph $R(G)$ is a daisy cube.*

Finally, we generalize the above result to all 2-connected outerplane bipartite graphs.

Theorem 4. *If G is a 2-connected outerplane bipartite graph, then G is regular if and only if the resonance graph $R(G)$ is a daisy cube.*

Proof. Let G be a 2-connected outerplane bipartite graph. By Theorem 2, there exists a CERS H such that G and H are evenly homeomorphic and the resonance graphs $R(G)$ and $R(H)$ are isomorphic. By Lemma 2, G is regular if and only if H is regular. Also, by Theorem 3, H is regular if and only if $R(H)$ is a daisy cube. Therefore, G is regular if and only if the resonance graph $R(H)$ is a daisy cube and this is further equivalent to $R(G)$ being a daisy cube. ■

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