MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **90** (2023) 247–258
ISSN: 0340–6253
doi: 10.46793/match.90-1.247V

Laplacian Energy of a Graph with Self-Loops

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(Received December 5, 2022)

Abstract

The purpose of this paper is to extend the concept of Laplacian energy from simple graph to a graph with self-loops. Let G be a simple graph of order n, size m and G_S is the graph obtained from G by adding σ self-loops. We define Laplacian energy of G_S as $LE(G_S) = \sum_{i=1}^{n} \left|\mu_i(G_S) - \frac{2m+\sigma}{n}\right|$ where $\mu_1(G_S), \mu_2(G_S), \ldots, \mu_n(G_S)$ are eigenvalues of the Laplacian matrix of G_S. In this paper some basic proprties of Laplacian eigenvalues and bounds for Laplacian energy of G_S are investigated. This paper is limited to bounds in analogy with bounds of $E(G)$ and $LE(G)$ but with some significant differences, more sharper bounds can be found.

1 Introduction

Let $G = (V, K)$ be a simple, undirected graph of order n and size m. In 1978, I. Gutman defined the energy [7] of adjacency matrix $A(G)$ as,

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \tag{1}$$

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where $\lambda_1, \lambda_2, \ldots, \lambda_n$, are the eigenvalues of adjacency matrix $A(G)$.

The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix and the Laplacian energy LE(G) of G is defined [10] as,

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \tag{2}$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $L(G)$ and $D(G) = diag(d_1, d_2 \ldots, d_n)$ is the diagonal matrix of vertex degrees of a graph G. For more on terminologies we refer [3, 4]

Let $S \subseteq V$ with $|S| = \sigma$. A graph with self-loops G_S with vertex set V and edge set $K(G_S)$ is obtained from simple graph G, by attaching a self-loop to each of its vertices belonging to S.

Since graph with self-loops find its application in Chemistry, [8,9,13,14] Gutman et al. defined adjacency matrix $A(G_S)$ [6] of graph G_S in 2021. The matrix $A(G_S)$ is a symmetric matrix of order n whose $(i,j)^{th}$ element is defined as,

$$A(G_S)_{ij} = \begin{cases} 1 & \text{if the vertices } v_i \sim v_j, \\ 0 & \text{if the vertices } v_i \nsim v_j, \\ 1 & \text{if } i = j \text{ and } v_i \in S, \\ 0 & \text{if } i = j \text{ and } v_i \notin S. \end{cases}$$

Then the energy $E(G_S)$ of graph G_S defined as [6],

$$E(G_S) = \sum_{i=1}^{n} \left| \lambda_i(G_S) - \frac{\sigma}{n} \right|. \tag{3}$$

where $\lambda_1(G_S), \lambda_2(G_S), \ldots, \lambda_n(G_S)$ are the eigenvalues of $A(G_S)$. For more information on energy, Laplacian energy and energy of graph with self-loops, refer [2, 5, 11, 12, 15].

The Laplacian matrix $L(G_S)$ of a graph G_S was defined by B. Acikmese [1] in 2015 as,

$$L(G_S) = L(G) + \sum_{(i,i) \in K(G_S)} e_i {e_i}^T \tag{4}$$

Where $L(G) = \sum\limits_{(i,j)\in K(G_S), i\neq j} (e_i - e_j)(e_i - e_j)^T$ and e_i is a vector with its i^{th} entry $+1$ and the others zeros.

We now rephrase the Laplacian matrix $L(G_S)$ of the graph G_S as a symmetric matrix $L(G_S)$ of order n, whose $(i,j)^{th}$ element is defined as

$$L(G_S)_{ij} = \begin{cases} -1 & \text{if the vertices } v_i \sim v_j, \\ 0 & \text{if the vertices } v_i \nsim v_j, \\ d_i + 1 & \text{if } i = j \text{ and } v_i \in S, \\ d_i & \text{if } i = j \text{ and } v_i \notin S. \end{cases}$$

i.e.,

$$L(G_S) = D(G_S) - A(G_S) \tag{5}$$

where $D(G_S) = diag(d_1(G_S), d_2(G_S), \ldots, d_n(G_S))$ is the diagonal matrix of vertex degrees of G_S.

Since $L(G_S)$ is a matrix with non-zero diagonal, similarly as in Eq. (2) and Eq. (3) we now define the Laplacian energy of a graph G_S as

$$LE(G_S) = \sum_{i=1}^{n} \left| \mu_i(G_S) - \frac{2m + \sigma}{n} \right|. \tag{6}$$

where $\mu_1(G_S), \mu_2(G_S), \ldots, \mu_n(G_S)$ are the eigenvalues of $L(G_S)$.

Let $\gamma_i(G_S) = \mu_i(G_S) - \frac{2m+\sigma}{n}$, $i = 1, 2, \ldots, n$ denote the auxiliary eigenvalues of $L(G_S)$. Then,

$$LE(G_S) = \sum_{i=1}^{n} |\gamma_i(G_S)|. \tag{7}$$

In the present paper, we discuss the properties of Laplacian eigenvalues, Laplacian energy and its bounds of graph G_S.

2 Laplacian eigenvalues of G_S

The eigenvalues of $L(G)$ and $A(G_S)$ satisfy the following relation [6, 10] :

$$\sum_{i=1}^{n} \mu_i = 2m; \qquad \sum_{i=1}^{n} \mu_i^2 = 2m + \sum_{i=1}^{n} d_i^2. \tag{8}$$

$$\sum_{i=1}^{n} \lambda_i(G_S) = \sigma; \qquad \sum_{i=1}^{n} \lambda_i^2(G_S) = 2m + \sigma. \tag{9}$$

Lemma 1. *Let $G(V, K)$ be a graph with n vertices and m edges. If $S \subseteq V$ with $|S| = \sigma$, then the eigenvalues $\mu_1(G_S), \mu_2(G_S), \ldots, \mu_n(G_S)$ of $L(G_S)$ satisfy,*

1. $\sum_{i=1}^{n} \mu_i(G_S) = 2m + \sigma$
2. $\sum_{i=1}^{n} \mu_i^2(G_S) = 2m + \sigma + \sum_{i=1}^{n} d_i^2(G_S) - 2 \sum_{v_i \in S} d_i(G_S)$

where $d_i(G_S), i = 1, 2, \ldots, n$ denote vertex degree of G_S.

Proof. 1. We have,

$$\begin{aligned}\sum_{i=1}^{n} \mu_i(G_S) &= \sum_{i=1}^{n} [L(G_S)]_{ii} \\ &= \sum_{i=1}^{n} [D(G_S)]_{ii} - \sum_{i=1}^{n} [A(G_S)]_{ii} \\ &= 2m + \sigma.\end{aligned}$$

2. Also,

$$\begin{aligned}\sum_{i=1}^{n} \mu_i^2(G_S) &= \sum_{i=1}^{n} [L(G_S)^2]_{ii} \\ &= \sum_{i=1}^{n} \left[D(G_S) - A(G_S)\right]^2{}_{ii} \\ &= \sum_{i=1}^{n} [D(G_S)^2]_{ii} + \sum_{i=1}^{n} [A(G_S)^2]_{ii} - 2 \sum_{i=1}^{n} [D(G_S)A(G_S)]_{ii}\end{aligned}$$

$$= \sum_{i=1}^{n} d_i^2(G_S) + 2m + \sigma - 2\sum_{v_i \in S} d_i(G_S).$$

■

Lemma 2. *Let $G(V,K)$ be a graph with n vertices and m edges. If $S \subseteq V$ with $|S| = \sigma$, then the auxiliary eigenvalues $\gamma_1(G_S), \gamma_2(G_S), \ldots, \gamma_n(G_S)$ of $L(G_S)$ satisfy,*

1. $\sum_{i=1}^{n} \gamma_i(G_S) = 0$
2. $\sum_{i=1}^{n} \gamma_i^2(G_S) = 2M$

where $M = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i(G_S) - \frac{2m+\sigma}{n}\right)^2 + \sigma\left(\frac{1}{2} + \frac{2m+\sigma}{n}\right) - \sum_{v_i \in S} d_i(G_S).$

Proof. 1. We have,

$$\begin{aligned}\sum_{i=1}^{n}\gamma_i(G_S) &= \sum_{i=1}^{n}\left(\mu_i(G_S) - \frac{2m+\sigma}{n}\right)\\ &= \sum_{i=1}^{n}\mu_i(G_S) - \sum_{i=1}^{n}\frac{2m+\sigma}{n}\\ &= 2m+\sigma-(2m+\sigma)\\ &= 0.\end{aligned}$$

2. Also, $\sum_{i=1}^{n}\gamma_i^2(G_S) = \sum_{i=1}^{n}\left(\mu_i(G_S) - \frac{2m+\sigma}{n}\right)^2$

$$\begin{aligned}&= \sum_{i=1}^{n}\mu_i^2(G_S) + \sum_{i=1}^{n}\left(\frac{2m+\sigma}{n}\right)^2 - 2\left(\frac{2m+\sigma}{n}\right)\sum_{i=1}^{n}\mu_i(G_S)\\ &= \sum_{i=1}^{n}\left(d_i(G_S) - \frac{2m+\sigma}{n}\right)^2 + 2m + \sigma + 2\sigma\left(\frac{2m+\sigma}{n}\right) -\\ &\qquad 2\sum_{v_i \in S} d_i(G_S)\\ &= 2M.\end{aligned}$$

■

Lemma 3. *[4]*

1. *A symmetric diagonally dominant real matrix with non-negative diagonal entries is positive semidefinite.*

2. *Let B be a symmetric real matrix with non-negative row sums and non-positive off-diagonal entries. Define a graph Γ on the index set of the rows of B, where two distinct indices i, j are adjacent when $b_{ij} \neq 0$. The multiplicity of the eigenvalue 0 of B equals the number of connected components C of Γ such that all rows $i \in C$ have zero row sum.*

Theorem 1. 1. *$L(G_S)$ is positive semidefinite.*

2. *If graph G_S has $p \geq 1$ components such that each of $q \leq p$ number of components with at least one self-loop and n Lapalcian eigenvalues are arranged in non-increasing order $\mu_1(G_S) \geq \mu_2(G_S) \geq \cdots \geq \mu_n(G_S)$ then $\mu_{n-i}(G_S) = 0$ for $i = 0, 1, \ldots, p - q - 1$ and $\mu_{n-p+q}(G_S) > 0$.*

Proof. 1. By Lemma 3, it follows that $L(G_S)$ being a symmetric diagonally dominant real matrix with non-negative diagonal entries is positive semidefinite.

2. Suppose graph G_S has $p \geq 1$ components. By Lemma 3, obtain the graph Γ which is defined on the index set of the rows of $L(G_S)$. By the definition of Γ it is clear that each connected component of G_S will contribute exactly one connected component C of Γ. Hence Γ has p connected components as G_S has p components. Suppose no self-loop is attached to any of the p components, then $L(G_S) = L(G)$. We know that each row of $L(G)$ will have zero row sum so is each connected components C of Γ. If at least one self-loop is attached to at least one of the component, row corresponding to the added self-loop will have nonzero row sum in $L(G_S)$ which in turn results in corresponding row to have non-zero row sum in connected components C of Γ. Hence that component will not contribute in the multiplicity of eigenvalue 0. Similaryly if G_S has at least one

self-loop in each of q components, then these q components will not contribute in the algebraic multiplicity of eigenvalue 0. Therefore q components will have positive eigenvalues. So multiplicity of 0 is $p-q$. Also it follows that if the Lapalcian eigenvalues are labeled so that $\mu_1(G_S) \geq \mu_2(G_S) \geq \cdots \geq \mu_n(G_S)$, then $\mu_{n-i}(G_S) = 0$ for $i = 0, 1, \ldots, p-q-1$ and $\mu_{n-p+q}(G_S) > 0$. ■

3 Laplacian energy of G_S

From literature [10], it is observed that Laplacian energy $LE(G)$ of graph G satisfies $LE(G) \leq \sqrt{2nM}$ and if $2m \geq n$, $2\sqrt{M} \leq LE(G) \leq 2M$, where $M = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2$. For Laplacian matrix $L(G_S)$ by Lemma 2, we have $\sum_{i=1}^{n}\gamma_i^2(G_S) = 2M$
where $M = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i(G_S) - \frac{2m+\sigma}{n}\right)^2 + \sigma\left(\frac{1}{2} + \frac{2m+\sigma}{n}\right) - \sum_{v_i \in S} d_i(G_S)$.
Using Cauchy-Schwarz inequality we get,

$$LE(G_S) \leq \sqrt{2nM} \tag{10}$$

For $2m \geq n$, Squaring $LE(G_S)$ and using Eq. 10 we get,

$$2\sqrt{M} \leq LE(G_S) \leq 2M \tag{11}$$

Theorem 2. *If graph G_S has p components $(p \geq 1)$ out of which q components $(q \leq p)$ have at least one self-loop in each of its component. Then,*
$$LE(G_S) \leq (p-q)\frac{2m+\sigma}{n} + \sqrt{(n-p+q)\left(2M - (p-q)\left(\frac{2m+\sigma}{n}\right)^2\right)}.$$

Proof. Consider a graph G_S with p components $(p \geq 1)$. Let each of $q \leq p$ components has at least one self-loop. Then by Theorem 1, $\gamma_{n-i}(G_S) = -\frac{2m+\sigma}{n}$ for $i = 0, 1, \ldots, p-q-1$.

Consider,

$$\sum_{i=1}^{n-p+q}\sum_{j=1}^{n-p+q}\left(|\gamma_i(G_S)|-|\gamma_j(G_S)|\right)^2 \geq 0$$

$$\sum_{i=1}^{n-p+q}\left(\sum_{j=1}^{n-p+q}|\gamma_i(G_S)|^2+\sum_{j=1}^{n-p+q}|\gamma_j(G_S)|^2-2\sum_{j=1}^{n-p+q}|\gamma_i(G_S)||\gamma_j(G_S)|\right)\geq 0$$

$$2(n-p+q)\left(2M-(p-q)\left(\frac{2m+\sigma}{n}\right)^2\right)\geq 2\left(LE(G_S)-(p-q)\left(\frac{2m+\sigma}{n}\right)\right)^2$$

Thus,

$$(p-q)\left(\frac{2m+\sigma}{n}\right)+\sqrt{(n-p+q)\left(2M-(p-q)\left(\frac{2m+\sigma}{n}\right)^2\right)}\geq LE(G_S).$$

■

Theorem 3. *For complete graph K_n with σ self-loops,*

$$LE(G_S)=\frac{(n^2-3n+2\sigma)}{n}+\sqrt{(n+1)^2-4\sigma}.$$

Proof. Let $(K_n)_S$ be a complete graph with self-loops so that $|S|=\sigma$. Since all vertices are adjacent, position of self-loop will not change the Laplacian enegry. Then,

$$L(K_n)_S=\begin{bmatrix}((n+1)I-J)_{\sigma\times\sigma} & -J_{\sigma\times(n-\sigma)}\\ -J_{(n-\sigma)\times\sigma} & (nI-J)_{(n-\sigma)\times(n-\sigma)}\end{bmatrix}$$

Let $W=\begin{bmatrix}X\\Y\end{bmatrix}$ be an eigenvector such that X consists of σ components and Y is of $(n-\sigma)$ tupled vector. Let $\mu(G_S)$ be a eigenvalue of $L(G_S)$. Then,

$$[\mu(G_S)I-L(G_S)]\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}[(\mu(G_S)-n-1)I+J]\,X_{\sigma\times1}+JY_{\sigma\times1}\\ JX_{(n-\sigma)\times1}+[(\mu(G_S)-n)I+J]Y_{(n-\sigma)\times1}\end{bmatrix}\tag{12}$$

Case 1: Let $X=X_j=e_1-e_j$, $j=2,3,\ldots,\sigma.$ and $Y=O_{n-\sigma\times1}$

From expression 12,

$$(\mu(G_S) - n - 1)X_j + JX_j + JO_{n-\sigma} = (\mu(G_S) - (n+1))X_j$$

It implies that, $\mu(G_S) = (n+1)$ is an eigenvalue with multiplicity of at least $(\sigma - 1)$ since there are $(\sigma - 1)$ linearly independent eigenvectors of the form X_j.

Case 2: Suppose X is a zero vector of order σ and $Y = Y_j = e_1 - e_j$, $j = 2, 3, \ldots, n - \sigma$
Then from expression 12,

$$JO_\sigma + (\mu(G_S) - n)Y_j + JY_j = (\mu(G_S) - n)Y_j$$

So, $\mu(G_S) = n$ is an eigenvalue with multiplicity at least $(n - \sigma - 1)$ since there are $n - \sigma - 1$ linearly independent eigenvectors of the form Y_j.

Case 3: Let $X = 1_\sigma$ and $Y = \frac{-\sigma}{(\mu(G_S)-\sigma)} 1_{n-\sigma}$ where, $\mu(G_S)$ is root of the equation $\mu(G_S)^2 - (n+1)\mu(G_S) + \sigma = 0$
From Eq. 12, $((\mu(G_S) - n - 1)I + J)1_\sigma + J\left(\frac{-\sigma}{n-\sigma}\right)1_{n-\sigma}$

$$= \left((\mu(G_S) - n - 1) + \sigma + (n - \sigma)\left(\frac{-\sigma}{\mu(G_S) - \sigma}\right)\right)1_\sigma$$
$$= \left(\frac{(\mu(G_S) - n - 1)(\mu(G_S) - \sigma) + \sigma(\mu(G_S) - \sigma) - \sigma(\mu(G_S) - \sigma)}{\mu(G_S) - \sigma}\right)1_\sigma$$
$$= \left(\frac{\mu(G_S)^2 - (n+1)\mu(G_S) + \sigma}{\mu(G_S) - \sigma}\right)1_\sigma$$

Thus, $\mu(G_S) = \frac{(n+1)+\sqrt{(n+1)^2-4\sigma}}{2}$ and $\mu(G_S) = \frac{(n+1)-\sqrt{(n+1)^2-4\sigma}}{2}$ are the eigenvalues with multiplicity of at least one. Therefore Laplacian spectrum of complete graph with self-loops is given by,

$$\left\{\begin{matrix} n & n+1 & \frac{n+1+\sqrt{(n+1)^2-4\sigma}}{2} & \frac{(n+1)-\sqrt{(n+1)^2-4\sigma}}{2} \\ n-1-\sigma & \sigma-1 & 1 & 1 \end{matrix}\right\}.$$

The Lapalcian energy of complete graph with self-loops is given by,

$$
\begin{aligned}
LE((K_n)_S) =&(n-1-\sigma)\left|n-\frac{n^2-n+\sigma}{n}\right|+(\sigma-1)\left|(n+1)-\frac{n^2-n+\sigma}{n}\right| \\
&+\left|\frac{(n+1)+\sqrt{(n+1)^2-4\sigma}}{2}-\frac{n^2-n+\sigma}{n}\right| \\
&+\left|\frac{(n+1)-\sqrt{(n+1)^2-4\sigma}}{2}-\frac{n^2-n+\sigma}{n}\right| \\
&=(n-1-\sigma)\left(\frac{n-\sigma}{n}\right)+(\sigma-1)\left(\frac{2n-\sigma}{n}\right)+\sqrt{(n+1)^2-4\sigma}
\end{aligned}
$$

Further simplification will result in,

$$LE((K_n)_S)=\frac{(n^2-3n+2\sigma)}{n}+\sqrt{(n+1)^2-4\sigma}.$$

■

Theorem 4. *For complete bipartite graph $K_{m,n}$ with number of self-loops s and k in the partite m and n respectively. Then the characteristic polynomial $P(x)$ is* $P\left(\mu\left(K_{(m,n)_S}\right)\right)=\left(\mu\left(K_{(m,n)_S}\right)-(n+1)\right)^{s-1}\left(\mu\left(K_{(m,n)_S}\right)-(m+1)\right)^{k-1}\left(\mu\left(K_{(m,n)_S}\right)-n\right)^{m-s-1}\left(\mu\left(K_{(m,n)_S}\right)-m\right)^{n-k-1}$ $Q\left(\mu\left(K_{(m,n)_S}\right)\right)$, *where* $Q\left(\mu\left(K_{(m,n)_S}\right)\right)=\mu^4\left(K_{(m,n)_S}\right)-2(m+n+1)\mu^3\left(K_{(m,n)_S}\right)+(m(m+3)+n(n+3)+3mn+1)\mu^2\left(K_{(m,n)_S}\right)-(m(n^2+k+1)+n(m^2+s+1)+(m+n)^2)\mu\left(K_{(m,n)_S}\right)+km(1+n)+ns(m+1)-ks.$

Proof. Let $(K_{m,n})_S$ be complete bipartite graph with s, k number of self-loops in the two partites of order m and n respectively. Then the Laplacian matrix $L(K_{(m,n)_S})=$

$$
\begin{bmatrix}
(n+1)I_{s\times s} & 0_{s\times(m-s)} & -1_{s\times k} & -1_{s\times(n-k)} \\
0_{(m-s)\times s} & nI_{(m-s)\times(m-s)} & -1_{(m-s)\times k} & -1_{(m-s)\times(n-k)} \\
-1_{k\times k} & -1_{k\times(m-s)} & (m+1)I_{k\times k} & 0_{k\times(n-k)} \\
-1_{(n-k)\times k} & -1_{(n-k)\times(m-s)} & 0_{(n-k)\times k} & mI_{(n-k)\times(n-k)}
\end{bmatrix}
$$

Consider $\det\left(\mu\left(K_{(m,n)_S}\right)I-L\left(K_{(m,n)_S}\right)\right)$, where $\mu(K_{(m,n)_S})$ is an eigenvalue of $L\left(K_{(m,n)_S}\right)$.

Step 1: Replace R_i by $R'_i=R_i-R_{i-1}$, where $i=n,n-1,\dots,k+3,k+2,k,k-1,\dots,m+3,m+2,m,m-1,\dots,s+3,s+2,s,s-1,\dots,3,2.$

Then $\det\left(\mu\left(K_{(m,n)_S}\right)I - L\left(K_{(m,n)_S}\right)\right)$ will get reduced to new determinant say $\det(A)$.

Step 2: Replace C_i of $\det(A)$ by $C_i' = C_i + C_{i+1}$, where $i = n-1, n-2, \ldots, k+1, k-1, k-2, \ldots, m+1, m-1, \ldots, s+1, s-1, \ldots, 2, 1$. Let the new determinant be $\det(B)$.

Step 3: Expanding $\det(B)$ successively along the rows i, $i = 2, 3, \ldots, s, s+2, \ldots, m, m+2, \ldots, k, k+2, \ldots, n-1, n$. Then, $\det(B) = (\mu\left(K_{(m,n)_S}\right) - (n+1))^{s-1}(\mu\left(K_{(m,n)_S}\right) - (m+1))^{k-1}(\mu\left(K_{(m,n)_S}\right) - n)^{m-s-1}(\mu\left(K_{(m,n)_S}\right) - m)^{n-k-1}\det(C)$, where $\det(C) =$

$$\begin{vmatrix} \mu\left(K_{(m,n)_S}\right) - n - 1 & 0 & k & n-k \\ 0 & \mu\left(K_{(m,n)_S}\right) - n & k & n-k \\ s & m-s & \mu\left(K_{(m,n)_S}\right) - m - 1 & 0 \\ s & m-s & 0 & \mu\left(K_{(m,n)_S}\right) - m \end{vmatrix}.$$

Further simplifying the determinant we get, $\det(C) = \mu^4\left(K_{(m,n)_S}\right) - 2(m+n+1)\mu^3\left(K_{(m,n)_S}\right) + (m(m+3)+n(n+3)+3mn+1)\mu^2\left(K_{(m,n)_S}\right) - (m(n^2+k+1)+n(m^2+s+1)+(m+n)^2)\mu\left(K_{(m,n)_S}\right) + km(1+n) + ns(m+1) - ks$. Then the Laplacian characteristic polynomial of $(K_{m,n})_S$ is given by $P\left(\mu\left(K_{(m,n)_S}\right)\right) = \left(\mu\left(K_{(m,n)_S}\right) - (n+1)\right)^{s-1}\left(\mu\left(K_{(m,n)_S}\right) - (m+1)\right)^{k-1}\left(\mu\left(K_{(m,n)_S}\right) - n\right)^{m-s-1}\left(\mu\left(K_{(m,n)_S}\right) - m\right)^{n-k-1} Q\left(\mu\left(K_{(m,n)_S}\right)\right)$, where $Q\left(\mu\left(K_{(m,n)_S}\right)\right) = \mu^4\left(K_{(m,n)_S}\right) - 2(m+n+1)\mu^3\left(K_{(m,n)_S}\right) + (m(m+3)+n(n+3)+3mn+1)\mu^2\left(K_{(m,n)_S}\right) - (m(n^2+k+1)+n(m^2+s+1)+(m+n)^2)\mu\left(K_{(m,n)_S}\right) + km(1+n) + ns(m+1) - ks$.

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