MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **89** (2023) 733–740
ISSN: 0340–6253
doi: 10.46793/match.89-3.733O

The Mean Value of Sombor Index of Graphs

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(Received November 29, 2022)

Abstract

The Sombor index of a simple graph G is defined as $SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$, where d_u is the degree of the vertex u. In this paper we study the mean value of the Sombor index of graphs. Let $\mathcal{F}_n$ be the set of all labeled graphs on n vertices $v_1, \ldots, v_n$. We obtain some explicit formulas for $\sum_{G \in \mathcal{F}_n} SO(G)$. As a consequence we find that for large enough n, $\sum_{G \in \mathcal{F}_n} SO(G) \simeq (\sqrt{2})^{n^2}$.

1 Introduction

Throughout this paper the graphs are simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of $V(G)$ is called the *order* of G. By $e = uv$ we mean the edge e with end points u and v. The *degree* of $v \in V(G)$, denoted by $d_v(G)$ (for short d_v), is the number of neighbors of v in G. By a *pendent vertex* we mean a vertex with only one neighboring, and a *pendent edge* is an edge with at least one pendent end points. If every vertex of a graph has degree r, then is called r-*regular*. By $\delta(G)$ and $\Delta(G)$ we mean the minimum and the maximum degree of vertices of G, respectively. The *chromatic number* of G is denoted by $\chi(G)$. A *unicyclic* graph is a graph that has exactly one cycle. The *complete graph*, the *empty*

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graph, the *path* and the *cycle* of order n are indicated by K_n, $\overline{K_n}$, P_n and C_n, respectively. The *complete bipartite graph* with section sizes r and s is indicated by $K_{r,s}$. By the *star* of order n we mean the complete bipartite graph $K_{1,n-1}$.

In chemical graph theory, numerous topological indices have been introduced. Some of well known of them are Zagreb indices. For a simple graph G the first Zagreb index and the second Zagreb index of G, denoted by $Z_1(G)$ and $Z_2(G)$, are defined respectively as

$$Z_1(G) = \sum_{uv\in E(G)} (d_u + d_v) = \sum_{v\in V(G)} d_v^2,$$

and

$$Z_2(G) = \sum_{uv\in E(G)} d_u d_v.$$

One of the newest and interesting of graph indices is nominated *Sombor index*. This index recommended in [9] by Gutman recently. The Sombor index of a graph G, denoted by $SO(G)$, is

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}\,.$$

For example, since K_n is an $(n-1)$-regular graph, we find that for every n, $SO(K_n) = \frac{n(n-1)^2}{\sqrt{2}}$.

There are numerous papers regarding the Sombor index and characterizing the graphs with special Sombor index. Studying the Sombor index of trees, unicyclic and bicyclic graphs is one of the attractive problem. Computing the Sombor index of special families of graphs also has done. There are aslo some results about the integrality of Sombor index of graphs, see [7], [14] and [15] for instance. To see other achievements on Sombor index we refer to [1]– [19] and the references therein.

Another interesting problem about graph indices is studying the mean value (also called average, arithmetic mean or arithmetic average) of an index. In this paper we study the mean value of the Sombor index. We find some explicit formulas for $\sum SO(G)$ where the summation is taken over all labeled graphs on fixed number of vertices. As a consequence we find that for large enough n, $\sum SO(G) \simeq (\sqrt{2})^{n^2}$, where the summation is taken over all labeled graphs on n vertices $v_1, \ldots, v_n$.

2 Sombor index

For every n, by $\mathcal{F}_n$ we mean the set of all labeled graphs on n vertices $v_1, \ldots, v_n$. In particular, by $\mathcal{F'}_n$ we mean the set of all labeled graphs on n vertices $v_1, \ldots, v_n$ that have at least one edge. We note that the cardinality of $\mathcal{F}_n$ is $2^{\binom{n}{2}}$ and the cardinality of $\mathcal{F'}_n$ is $2^{\binom{n}{2}} - 1$. In this section we study the average of Sombor index among all graphs of $\mathcal{F}_n$ and $\mathcal{F'}_n$. More precisely we investigate the values of

$$\Theta_n := \frac{\sum_{G \in \mathcal{F}_n} SO(G)}{2^{\binom{n}{2}}}, \ \textit{and} \ \Theta'_n := \frac{\sum_{G \in \mathcal{F'}_n} SO(G)}{2^{\binom{n}{2}} - 1}. \tag{1}$$

We note that $SO(\overline{K_n}) = 0$, thus $\sum_{G \in \mathcal{F}_n} SO(G) = \sum_{G \in \mathcal{F'}_n} SO(G)$. For example

$$\Theta_1 = 0, \ \Theta_2 = \frac{\sqrt{2}}{2}, \ \Theta_3 = \frac{9\sqrt{2} + 6\sqrt{5}}{8}$$

and

$$\Theta_4 = \frac{144\sqrt{2} + 48\sqrt{5} + 24\sqrt{10} + 48\sqrt{13}}{64}.$$

Remark 1. *Let $m \geq 0$ be an integer. Since $\sum_{i=0}^{m} \binom{m}{i} = 2^m$ (by binomial expansion), we obtain that*

$$\sum_{r,s=0}^{m} \binom{m}{r}\binom{m}{s} = \Big(\sum_{r=0}^{m} \binom{m}{r}\Big)\Big(\sum_{s=0}^{m} \binom{m}{s}\Big) = 2^m 2^m = 2^{2m}. \tag{2}$$

Lemma 1. *For every integer $m \geq 0$,*

$$\sum_{k=0}^{m} \binom{m}{k} k = m2^{m-1}, \tag{3}$$

and

$$\sum_{r,s=0}^{m} \binom{m}{r}\binom{m}{s} r = m2^{2m-1}. \tag{4}$$

Proof. Let $f_m(x) = \binom{m}{1}x + 2\binom{m}{2}x^2 + \cdots + m\binom{m}{m}x^m$. Hence

$$f_m(x) = x\Big(\binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m\Big)',$$

where $'$ denotes the derivative with respect to x. On the other hand by binomial expansion $\binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m = (x+1)^m - 1$. Therefore $f_m(x) = mx(x+1)^{m-1}$. Now by considering $f_m(1)$ the Equation (3) follows. For the second part, note that

$$\sum_{r,s=0}^{m} \binom{m}{r}\binom{m}{s} r = \Big(\sum_{s=0}^{m}\binom{m}{s}\Big)\Big(\sum_{r=0}^{m}\binom{m}{r} r\Big).$$

Using Equation (3) and the binomial expansion, the Equation (4) follows. ■

Theorem 1. *For every integer $n \geq 2$,*

$$\sum_{G\in\mathcal{F}_n} SO(G) = \binom{n}{2} 2^{\binom{n-2}{2}} \sum_{1\leq i,j\leq n-1} \binom{n-2}{i-1}\binom{n-2}{j-1}\sqrt{i^2+j^2}. \tag{5}$$

Proof. Since $SO(\overline{K_n}) = 0$, it is enough to find $\sum_{G\in\mathcal{F}'_n} SO(G)$. The number of graphs, say H, in $\mathcal{F}'_n$ such that $v_r v_s$ $(r \neq s)$ is an edge of H such that $deg_H(v_r) = i$ and $deg_H(v_s) = j$ is

$$2^{\binom{n-2}{2}}\binom{n-2}{i-1}\binom{n-2}{j-1}. \tag{6}$$

It it worthy to mention that in (6), the number of ways for selecting the neighbors of v_r except v_s is $\binom{n-2}{i-1}$. Similarly the number of ways for selecting the neighbors of v_s except v_r is $\binom{n-2}{j-1}$. In addition, we have $2^{\binom{n-2}{2}}$ cases for other edges of H. Thus $\sqrt{i^2+j^2}$ appears $2^{\binom{n-2}{2}}\binom{n-2}{i-1}\binom{n-2}{j-1}$ times in $\sum_{G\in\mathcal{F}'_n} SO(G)$. On the other hand the cases to consider the edge $v_r v_s$

is $\binom{n}{2}$, in addition i and j can be any number between 1 and $n-1$. This completes the proof. ■

Now we find some bounds for $\sum_{G\in\mathcal{F}_n} SO(G)$ in terms of n.

Theorem 2. *For every integer $n \geq 2$,*

$$\frac{n}{2\sqrt{2}}\binom{n}{2}2^{\binom{n}{2}} \leq \sum_{G\in\mathcal{F}_n} SO(G) \leq \frac{n}{2}\binom{n}{2}2^{\binom{n}{2}}. \tag{7}$$

Moreover, in the left hand side the equality holds if and only if $n = 2$, and in the other side the equality does not happen.

Proof. One can see that for every two non-negative integers i and j,

$$\frac{i+j}{\sqrt{2}} \leq \sqrt{i^2+j^2} \leq i+j. \tag{8}$$

Moreover, in the left hand side the equality holds if and only if $i = j$ while in the other side the equality holds if and only if $i = 0$ or $j = 0$. Let

$$\Omega_n = \frac{\sum_{G\in\mathcal{F}_n} SO(G)}{\binom{n}{2}2^{\binom{n-2}{2}}}.$$

Thus by Theorem 1 we find that

$$\sum_{1\leq i,j\leq n-1}\binom{n-2}{i-1}\binom{n-2}{j-1}\frac{i+j}{\sqrt{2}} \leq \Omega_n \leq \sum_{1\leq i,j\leq n-1}\binom{n-2}{i-1}\binom{n-2}{j-1}(i+j). \tag{9}$$

In addition, in the left hand side the equality holds if and only if $n = 2$ while in the other side the equality does not happen. Using Equations (2) and (4) we can compute the both sides of Equation (9) and obtain that

$$\frac{1}{\sqrt{2}}n2^{2n-4} \leq \Omega_n \leq n2^{2n-4}. \tag{10}$$

In other words,

$$\frac{1}{\sqrt{2}} n 2^{2n-4} \leq \frac{\sum_{G \in \mathcal{F}_n} SO(G)}{\binom{n}{2} 2^{\binom{n-2}{2}}} \leq n 2^{2n-4}. \tag{11}$$

This completes the proof. ■

Now as a consequence of the previous results, we obtain one of the main results of the paper.

Theorem 3. *For large enough n,*

$$\sum_{G \in \mathcal{F}_n} SO(G) \simeq (\sqrt{2})^{n^2}. \tag{12}$$

More precisely,

$$\sqrt[n^2]{\sum_{G \in \mathcal{F}_n} SO(G)} \to \sqrt{2}, \text{ as } n \to +\infty. \tag{13}$$

Proof. Using Theorem 2 we find that

$$n^2(n-1) 2^{\frac{n^2-n-5}{2}} \leq \sum_{G \in \mathcal{F}_n} SO(G) \leq n^2(n-1) 2^{\frac{n^2-n-4}{2}}. \tag{14}$$

Hence

$$a_n \leq \sqrt[n^2]{\sum_{G \in \mathcal{F}_n} SO(G)} \leq b_n, \tag{15}$$

where

$$a_n = \sqrt[n^2]{n^2(n-1)} 2^{\frac{n^2-n-5}{2n^2}}$$

and

$$b_n = \sqrt[n^2]{n^2(n-1)} 2^{\frac{n^2-n-4}{2n^2}}.$$

It is not hard to check that

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} b_n = \sqrt{2}.$$

Thus by the sandwich theorem we find that

$$\lim_{n\to+\infty} \sqrt[n^2]{\sum_{G\in\mathcal{F}_n} SO(G)} = \sqrt{2}.$$

The proof is complete. ■

Using Theorem 2 we obtain some bounds for the mean value (arithmetic average) of the Sombor index of graphs in terms of the number of vertices, see (1).

Theorem 4. *For every integer $n \geq 3$,*

$$\frac{n}{2\sqrt{2}}\binom{n}{2} < \frac{\sum_{G\in\mathcal{F}_n} SO(G)}{2^{\binom{n}{2}}} < \frac{n}{2}\binom{n}{2}. \tag{16}$$

In other words,

$$\frac{n^3-n^2}{4\sqrt{2}} < \Theta_n = \frac{\sum_{G\in\mathcal{F}_n} SO(G)}{2^{\binom{n}{2}}} < \frac{n^3-n^2}{4}. \tag{17}$$

***Acknowledgment*:** The author is grateful to the referee for her/his comments.

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