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Matching Energy of Graphs with Maximum Degree at Most 3

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Abstract

The matching energy of a graph G, denoted by $ME(G)$, is defined as the sum of absolute values of the zeros of the matching polynomial of G. In this paper, we prove that if G is a connected graph of order n with maximum degree at most 3, then $ME(G) > n$ with only six exceptions. In particular, we show that there are only two connected graphs with maximum degree at most three, whose matching energies are equal to the number of vertices.

1 Introduction

All graphs we consider are finite, simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. By *order* and *size* of G, we mean the number of vertices and the number of edges of G, respectively. We denote the complete graph, the path and the cycle of order n, by K_n, P_n and C_n, respectively. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. For vertex disjoint graphs H and K, we use $H \cup K$ to denote their union. By mG we mean the graph consisting of m pairwise disjoint copies of G. The maximum degree of G is denoted by $\Delta(G)$ (or

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by Δ if G is clear from the context). For $S \subseteq V(G)$, $\langle S \rangle$ is the subgraph of G induced by S. A *traceable graph*, is a graph with a Hamilton path. A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. An r*-matching* in a graph G is a set of r pairwise non-incident edges. The number of r-matchings in G is denoted by $m(G, r)$. The *matching number* of G, $\mu(G)$, is the number of edges in a maximum matching of G.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a graph G, i.e the eigenvalues of its adjacency matrix. The energy of the graph G denoted by $\mathcal{E}(G)$, is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The theory of graph energy is well developed nowadays, for details see [2, 3, 9, 11, 15]. The Coulson integral formula [8] plays an important role in the study on graph energy, its version for an acyclic graph T is as follows:

$$\mathcal{E}(T) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left(\sum_{r \geq 0} m(T, r) x^{2r} \right) dx. \tag{1}$$

Motivated by formula (1), Gutman and Wagner in 2012 defined the matching energy of a graph G as

$$ME(G) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left(\sum_{r \geq 0} m(G, r) x^{2r} \right) dx, \tag{2}$$

see [12]. Energy and matching energy of graphs are closely related, and they are two quantities of relevance for chemical applications, [12]. Recall that the *matching polynomial* of G is defined by

$$\alpha(G, x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r m(G, r) x^{n-2r},$$

where n is the order of G and $m(G, 0)$ is considered to be 1, see [1,4–7,10]. For any graph G, all zeros of $\alpha(G, x)$ are real [13]. Furthermore, if μ is a matching zero of G, then so is $-\mu$. The following result gives an equivalent definition of matching energy:

Theorem 1. *[12] Let G be a graph and let $\mu_1, \ldots, \mu_n$ be the zeros of its matching polynomial. Then*

$$ME(G) = \sum_{i=1}^{n} |\mu_i|.$$

Since 2012 matching energy of graphs has been studied by several authors and a series of results concerning the extremal matching energy of graphs have been obtained. For details, we refer to [12, 16–18]. Recently, in [14] the authors presented some lower bounds for matching energy of graphs. They proved that for a connected graph G, $ME(G) \geq 2\mu(G)$. Also it was shown that if G has no perfect matching, then $ME(G) \geq 2\mu(G)+1$, except for $K_{1,2}$. Furthermore, a lower bound for $ME(G)$ in terms of the minimum degree of G was given. Among other results, they characterized some class of graphs whose matching energy exceeds the number of vertices. They proved that if G is a connected graph of order n such that the multiplicity of 0 as a matching root is 1, then $ME(G) > n$, except for $K_{1,2}$. Also it was shown that in a connected graph G, if the multiplicity of 0 as a matching root is 2, then except four graphs, the matching energy of G exceeds the number of vertices. In particular, all connected traceable graphs and all connected claw-free graphs whose matching energies are greater than the number of vertices were described. In this paper, we characterize all connected graphs with maximum degree at most 3, whose matching energies are equal to the number of vertices. In fact, we show that if G is a connected graph of order n with maximum degree at most 3, then $ME(G) > n$ with only six exceptions. The following lemmas are needed in the sequel.

Lemma 1. *[18] If H is a subgraph of G, then $ME(H) \leq ME(G)$, with equality if H and G are the same except possibly for isolated vertices.*

Lemma 2. *Let G be a connected graph and $H_1, \ldots, H_t$ be its t vertex-disjoint subgraphs. Then*

$$ME(G) > \sum_{i=1}^{t} ME(H_i).$$

Proof. Let $K = \cup_{i=1}^{t} H_i$. Now the assertion follows from Lemma 1 and [14, Lemma 16]. ■

Lemma 3. *[14] Let G be a connected graph of order n which has a perfect matching. Then $ME(G) \geq n$ and the equality holds only if $G = K_2$.*

Lemma 4. *[14] Let G be a connected traceable graph of order $n > 1$. Then $ME(G) \geq n$, except for $K_{1,2}$. The equality holds only if $G = K_2$.*

Lemma 5. *[14] Let $n \geq 3$. Then $ME(C_n) > n$. In particular, if n is even, then $ME(C_n) > n + 1$.*

2 Graphs with maximum degree at most 3 whose matching energies exceed the number of vertices

In this section, we characterize all connected graphs with maximum degree at most 3, whose matching energies are equal to the number of vertices. It is shown that if G is a connected graph of order n with maximum degree at most 3, then $ME(G) > n$, with only six exceptions.

Lemma 6. *Let G be a connected graph of order n with $\Delta \leq 2$. Then $ME(G) > n$, except for K_1, K_2 and $K_{1,2}$. The equality holds only if $G = K_2$.*

Proof. Since $\Delta \leq 2$, G is either a path or a cycle. Now, the assertion follows from Lemmas 4 and 5. ■

Lemma 7. *Let G be a connected graph of order $n > 2$. If G has a perfect matching, then $ME(G) \geq n + 0.47$.*

Proof. Let M be a perfect matching of G and $e = uv$ be a P_2-component of M. Since G is connected, there exists some P_2-component of M, say $f = wz$ such that e is connected to f. Let $H = \langle u, v, w, z \rangle$. A computer search shows that $ME(H) \geq 4.47$. Now, if $G = H$, then we are done. Otherwise, let $W = G \setminus V(H)$ and assume that $W_1, \ldots, W_t$, $t \geq 1$ are the components of W. Now, Lemma 3, implies that for each i, $1 \leq i \leq t$, $ME(W_i) \geq |V(W_i)|$. Consequently, by Lemma 2 we obtain:

$$ME(G) > ME(H) + \sum_{i=1}^{t} ME(W_i) \geq (|V(H)| + 0.47) + \sum_{i=1}^{t} |V(W_i)|$$
$$= n + 0.47\,,$$

so we are done. ■

Lemma 8. *Let G be a connected graph with $\Delta = 3$. Then there are vertex disjoint subgraphs $H_1, \ldots, H_l$ of G such that $V(G) = \cup_{i=1}^{l} V(H_i)$ and each H_i is isomorphic to one of the graphs K_2, $K_{1,2}$, C_3 or $K_{1,3}$.*

Proof. Let M be a matching of maximum size in G and $|M| = l$. Let $e_1, \ldots, e_l$ be the P_2-components of M. Assume that S is the set of vertices of G missed by M and $|S| = k$. Obviously, S is an independent set of G. If $k = 0$, then G has a perfect matching and we are done. Hence we may assume that $k > 0$. Since G is connected each $x \in S$ is connected to some P_2-component of M. Let S_1 be the set of vertices of S which are connected to e_1. For each $i > 1$, let S_i be the set of vertices of $S \setminus \cup_{j=1}^{i-1} S_j$ which are connected to e_i. Since $\Delta = 3$, it is easily seen that $|S_i| \leq 2$. Otherwise G has a matching of size at least $l + 1$, a contradiction. Now, let $H_i = \langle S_i, V(e_i) \rangle$, for $i = 1, \ldots, l$. This implies the statement. ■

By the above lemma, the following result is obvious.

Corollary 1. *If G is a connected graph with $\Delta = 3$, then there are vertex disjoint subgraphs $H_1, \ldots, H_l$ of G, not necessarily induced, such that $V(G) = \cup_{i=1}^{l} V(H_i)$ and each H_i is isomorphic to one of the graphs K_2, $K_{1,2}$ or $K_{1,3}$. Moreover, each subgraph H_i contains one P_2-component of a maximum matching M of G.*

Lemma 9. *Let G be a connected graph of order $n > 7$ with $\Delta = 3$. Then we can always assume that at least one of the subgraphs H_i, $i = 1, \ldots, l$ in Corollary 1, is K_2.*

Proof. Let M be a matching of maximum size in G and $|M| = l$. Let $H_1, \ldots, H_l$ be the subgraphs described in Corollary 1. If for some i, $1 \leq$

$i \leq l$, $H_i = K_2$, then there is nothing to prove. Hence assume that for each i, $H_i \in \{K_{1,2}, K_{1,3}\}$. Note that each H_i contains one P_2-component of M. Furthermore, since G is connected, for each i, H_i is joined to some H_j, $i \neq j$, by an edge. Let $e = xy$ such that $x \in V(H_i)$, $y \in V(H_j)$ and H_i is joined to H_j by e. Then, it is easily seen that either $d_{H_i}(x) \neq 1$ or $d_{H_j}(y) \neq 1$. Otherwise, G has a matching of size $l + 1$, a contradiction. Now, since $n > 7$ and $\Delta = 3$, we find that G contains one of the graphs (a) or (b) depicted in Fig. 1, as a subgraph. As it is shown in Fig. 1, in each case subgraphs H_1, H_2 and H_3 can be replaced by subgraphs H_1', H_2' and H_3' such that H_i' contains one P_2-component of M, for $i = 1, 2, 3$. This completes the proof.

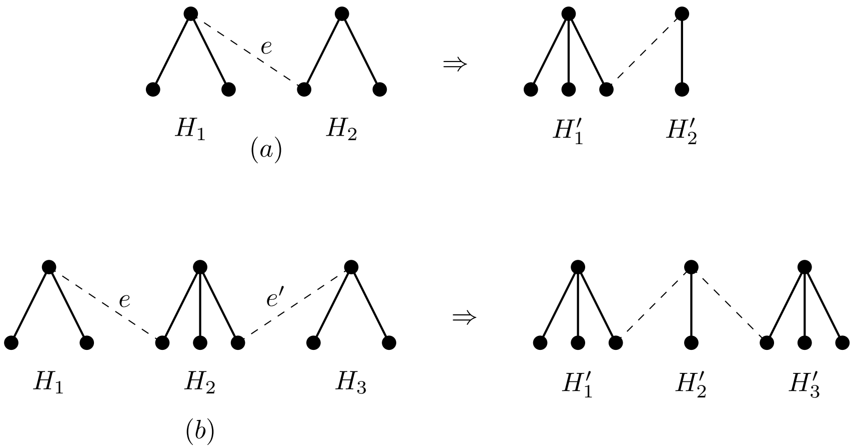
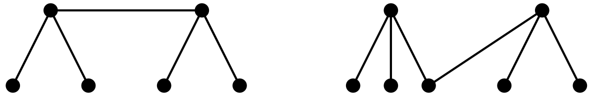


Figure 1. Graphs (a) and (b).

■

Theorem 2. *Let G be a connected graph of order n with $\Delta = 3$. If $G \notin \{K_{1,3}, G_1, G_2\}$, then $ME(G) > n$. In particular, G_1 is the only connected graph with $\Delta = 3$ whose matching energy is equal to its order.*



Proof. Let G be a connected graph of order n with $\Delta \leq 3$. Let $\mathcal{A} = \{K_1, K_2, K_{1,2}, K_{1,3}, G_1, G_2\}$. A computer search shows that if $n \leq 15$ and $G \notin \mathcal{A}$, then $ME(G) > n$. The matching energies of graphs in $\mathcal{A}$ is given in Table 1. In Table 2, the minimum matching energy of graphs $G \notin \mathcal{A}$ for $3 \leq n \leq 15$ and $1 < \Delta \leq 3$ is given.

Table 1. Matching energies of the graphs $G \in \mathcal{A}$.

G	$ME(G)$	G	$ME(G)$
K_1	0	$K_{1,3}$	3.46
K_2	2	G_1	6
$K_{1,2}$	2.82	G_2	6.82

Table 2. Matching energies of the graphs $G \notin \mathcal{A}$ for $3 \leq n \leq 15$ and $1 < \Delta \leq 3$.

n	Min $ME(G)$	n	Min $ME(G)$	n	Min $ME(G)$
3	3.46	8	8.42	13	13.48
4	4.47	9	9.33	14	14.91
5	5.22	10	10.12	15	15.92
6	6.15	11	11.68		
7	7.66	12	12.61		

In what follows, we assume that $\Delta = 3$ and prove the theorem by induction on n. Let $n > 15$ and M be a matching of maximum size in G with $|M| = l$. By Corollary 1, there are vertex disjoint subgraphs $H_1, \ldots, H_l$ of G such that $H_i \in \{K_2, K_{1,2}, K_{1,3}\}$ for $i = 1, \ldots, l$, each H_i contains one P_2-component of M and $V(G) = \cup_{i=1}^{l} V(H_i)$. Now, using Lemma 9, we can assume that $H_1 = K_2$. Let $W = G \setminus V(H_1)$. We consider two cases:

Case 1. $V(H_1)$ is not a vertex cut. Then since $n > 15$, we find that

$W \notin \mathcal{A}$. Now, if $\Delta(W) \leq 2$, then Lemma 6, yields that $ME(W) > |V(W)|$ and if $\Delta(W) = 3$, then by the induction hypothesis we have $ME(W) > |V(W)|$. Note that $ME(H_1) = |V(H_1)|$. Thus by Lemma 2 we obtain:

$$ME(G) > ME(H_1) + ME(W) \geq |V(H_1)| + |V(W)| = n,$$

so we are done.

Case 2. $V(H_1)$ is a vertex cut. Then since $\Delta = 3$, W has at most four components. Let $W_1, \ldots, W_t$, $1 < t \leq 4$, be the components of W. Note that none of the components of W is K_1. Now, if W has a component say W_1, such that $W_1 \notin \{K_{1,2}, K_{1,3}, G_2\}$, then either $\Delta(W_1) \leq 2$ in which case by Lemma 6, $ME(W_1) \geq |V(W_1)|$ or $\Delta(W_1) = 3$ in which case by the induction hypothesis we have $ME(W_1) \geq |V(W_1)|$, with equality only if $W_1 = G_1$. Let $K = \langle V(H_1) \cup (\cup_{i=2}^{t} V(W_i)) \rangle$. Again by the induction hypothesis, we get $ME(K) > |V(K)|$. Hence, Lemma 2 implies that

$$ME(G) > ME(W_1) + ME(K) \geq |V(W_1)| + |V(K)| = n.$$

It follows that we only need to assume that every component of W is either $K_{1,2}$, $K_{1,3}$ or G_2. Furthermore, it is easily seen that if W contains $3K_{1,3}$ as a subgraph, then G must have a matching of size $l + 1$, a contradiction. Now, since $n > 15$ we are reduced to the following three cases:

Case 2.1. W has two components. Then since $n > 15$, the only possibility is that $W = 2G_2$. Let $W_1 = W_2 = G_2$ and $L = \langle V(H_1) \cup V(W_2) \rangle$. Thus $|V(L)| = 9$ and from Tables 1 and 2, one can see that $ME(L) \geq 9.33$ and $ME(W_1) = 6.82$. Therefore

$$ME(G) > ME(W_1) + ME(L) \geq (|V(W_1)| - 0.18) + (|V(L)| + 0.33) > n.$$

Case 2.2. W has three components. Then since $n > 15$, we only need to consider the cases that at least one of the components of W is G_2. Recall that W should not contain $3K_{1,3}$ as a subgraph. Now, let $W_1 = G_2$ and $L = \langle V(H_1) \cup (\cup_{i=2}^{3} V(W_i)) \rangle$. Table 3, shows the possible graphs $W_2 \cup W_3$ and the minimum matching energy of the corresponding graph L.

Table 3. Matching energies of the graphs L.

$W_2 \cup W_3$	$\|V(L)\|$	Min $ME(L)$
$K_{1,2} \cup K_{1,3}$	9	9.33
$K_{1,2} \cup G_2$	12	12.61

From Table 3, one can see that $ME(L) \geq |V(L)| + 0.33$. It follows that

$$ME(G) > ME(W_1) + ME(L) \geq (|V(W_1)| - 0.18) + (|V(L)| + 0.33) > n.$$

Case 2.3. W has four components. Then since $n > 15$, we find that at least one of the components of W is not $K_{1,2}$. First assume that $W_1 = G_2$ and let $L = \langle V(H_1) \cup (\cup_{i=2}^{4} W_i)\rangle$. Table 4, shows the possible graphs $\cup_{i=2}^{4} W_i$ and the minimum matching energy of the corresponding graph L.

Table 4. Matching energies of the graphs L.

$\cup_{i=2}^{4} W_i$	$\|V(L)\|$	Min $ME(L)$
$3K_{1,2}$	11	11.68
$2K_{1,2} \cup K_{1,3}$	12	12.61
$2K_{1,2} \cup G_2$	15	15.92

As seen from Table 4, $ME(L) \geq |V(L)| + .0.61$. Therefore

$$ME(G) > ME(W_1) + ME(L) \geq (|V(W_1)| - 0.18) + (|V(L)| + 0.61) > n.$$

Next, suppose that $W_1 = K_{1,3}$ and non of the components of W is G_2. It is easy to check that $\cup_{i=2}^{4} W_i$ is either $3K_{1,2}$ or $2K_{1,2} \cup K_{1,3}$. Let $L = \langle V(H_1) \cup (\cup_{i=2}^{4} W_i)\rangle$. From Tables 1 and 4, one can see that $ME(W_1) = |V(W_1)| - 0.54$ and $ME(L) \geq |V(L)| + 0.61$. This implies that $ME(G) > n$. The proof is now complete. ■

The following result, is an immediate consequence of Lemma 6 and Theorem 2.

Corollary 2. *Let G be a connected graph of order n with $\Delta \leq 3$. If $G \notin \{K_1, K_2, K_{1,2}, K_{1,3}, G_1, G_2\}$, then $ME(G) > n$. In particular, K_2 and G_1 are the only connected graphs with $\Delta \leq 3$ whose matching energies are equal to their orders.*

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