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Several Methods for Generating Families of Orderenergetic, Integral and Equienergetic Graphs

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Abstract

We define a general unary graph operation and give several applications of these operation in this paper. The adjacency matrix and the complete spectrum of the derived graphs are determined. Different methods for generating sequences of orderenergetic graphs from known orderenergetic graphs are described. Several methods are described for generating orderenergetic graphs from non-orderenergetic graphs. Methods for generating new families of integral graphs using this new operation are also discussed. It is also possible to generate infinite sequences of pair of equienergetic and non-cospectral graphs using this graph operation.

1 Introduction

Let $G = (V, E)$ be a graph on n vertices and e edges and A_G denotes the adjacency matrix of this graph. The characteristic polynomial of the graph G is the characterestic polynomial of the adjacency matrix A_G and is denoted by $f_G(\lambda)$. Suppose that the eigenvalues of A_G are $\lambda_1, \lambda_2, \cdots, \lambda_n$.

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Then the energy of this graph [8] is defined to be the sum of the absolute value of the eigenvalues and is denoted by

$$\mathcal{E}_G = \sum_{i=1}^{n} |\lambda_i| \,. \tag{1}$$

The energy of graphs are having applications in chemical graph theory and it can be used to approximate the total π-electron energy of a molecule [9, 12, 15].

Comparing the order of the graphs and their energies, graphs can be grouped into different classes. An orderenergetic graph is recently defined in [1]. A graph G with order n is said to be an orderenergetic graph if $\mathcal{E}_G = n$. If $\mathcal{E}_G < n$, then such a graph is called hypoenergetic graph [9]. A graph is said to be hyperenergetic graph [7] if $\mathcal{E}_G > 2n-2$. The class of graphs for which $\mathcal{E}_G = 2n - 2$ are called borderenergetic graphs [4]. If two graphs G and G' are having equal energy, then such graphs are called equienergetic graphs [14]. Non-cospectral equienergetic graphs are graphs with same energy but with different spectra. A graph is said to be an integral graph if all of its eigenvalues are integers [10]. Construction and classification of all these kinds of graphs and their applications can be found in [2,3,6,9,12] and references therein. But, the study of orderenergetic graphs are in their initial stage and certain methods for constructing such graphs can be found in [1, 11]. An integral energy graph is defined as a graph whose energy is an integer. All integral graphs are clearly integral energy graphs, but not conversely. Orderenergetic graphs and borderenergetic graphs are always integral energy graphs.

It is shown that the families of complete bipartite graphs $K_{p,p}$ and complete tripartite graphs $K_{p,p,6p}$ are orderenergetic graphs [1], where p is any positive integer. They have also shown that the connected graphs obtained by taking the direct product of any two orderenergetic graphs with orders m and n, one of which is a non-bipartite graph, will be also a connected orderenergetic graph with order mn. Methods for constructing two different sequences of orderenergetic graphs from a given orderenergetic graph are given in [11]. Another method for generating orderenergetic graphs from non-orderenergetic graphs is also given in [11]. In this paper we present several new methods for generating orderenergetic graphs, integral graphs, integral energy graphs and equienergetic graphs.

In the next section a general unary graph operation is defined. This is used to generate families of new graphs from a given seed graph. The adjacency spectrum of the generated graph is computed in terms of the seed graph and is given in the third section. Several special cases of this graph operation is discussed in the fourth section. In each of these cases it s possible to generate new sequences of integral graphs and integral energy graphs. The main application of this graph operation is the construction of new sequences of orderenergetic graphs form given orderenergetic graphs and non-orderenergetic graphs.

2 A generalized unary graph operation

Let $G = (V_G, E_G)$ be a graph with order n and number of edges e. Let the vertex set $V_G = \{1, 2, 3, \cdots, n\}$. Given any two positive integers $p = p_1 + p_2 + p_3$ and $q = q_1 + q_2 + q_3$, where p_i's and q_i's non-negative integers, a new graph $H = (V_H, E_H)$ is constructed from the given graph G as follows.

1. The vertex set of H is
   $V_H = \{u_{ij}/1 \leq i \leq p, 1 \leq j \leq n\} \cup \{v_{kl}/1 \leq k \leq q, 1 \leq l \leq n\}$.
2. The edges in H are obtained as follows. Let (j, k) is an edge in G, then,
   - the edges $(u_{ij}, u_{ik}) \in E_H$ for all $1 \leq i \leq p_1 + p_2$,
   - the edges $(v_{ij}, v_{ik}) \in E_H$ for all $1 \leq i \leq q_1 + q_2$,
   - the edges $(u_{ij}, u_{lk}) \in E_H$ for all $1 \leq i \leq p_1$, $1 \leq l \leq p_1$, $l \neq i$,
   - the edges $(v_{ij}, v_{lk}) \in E_H$ ffor all $1 \leq i \leq q_1$, $1 \leq l \leq q_1$, $l \neq i$,
   - the edges $(u_{ij}, v_{lk}) \in E_H$ for all $1 \leq l \leq q$, $1 \leq i \leq p$,
   - the edges $(u_{ik}, v_{lj}) \in E_H$ for all $1 \leq l \leq q$, $1 \leq i \leq p$.

The total number of vertices in the derived graph H will be $n(p + q)$ and the total number of edges will be $e(2pq + p_1^2 + q_1^2 + p_2 + q_2)$. This graph constructed from the graph G is denoted by the symbol $H = G^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$.

The above unary graph operation is illustrated in some examples. The graph H constructed from the complete graph on two vertices K_2 with $p_1 = p_2 = q_1 = q_2 = 1$ and $p_3 = q_3 = 0$ is the first graph in figure 1. The graph H constructed from K_2 with $p_1 = q_2 = 1, p_2 = 3, p_3 = q_1 = q_3 = 0$ is the second graph shown in figure 1. The third and fourth graphs given in figure 1 are obtained from K_2 with $p_1 = p_2 = p_3 = q_1 = q_2 = q_3 = 1$ and $p_1 = 1, p_2 = q_3 = 2, p_3 = q_1 = q_2 = 0$ respectively. Graphs constructed from the path graph P_3 with edges $(1, 2)$ and $(1, 3)$ are shown in figure 2. First graph is obtained by taking $p_1 = p_2 = q_1 = 1, p_3 = q_2 = q_3 = 0$ and second graph is obtained by taking $p_1 = q_1 = 1, p_2 = 2, p_3 = q_2 = q_3 = 0$.

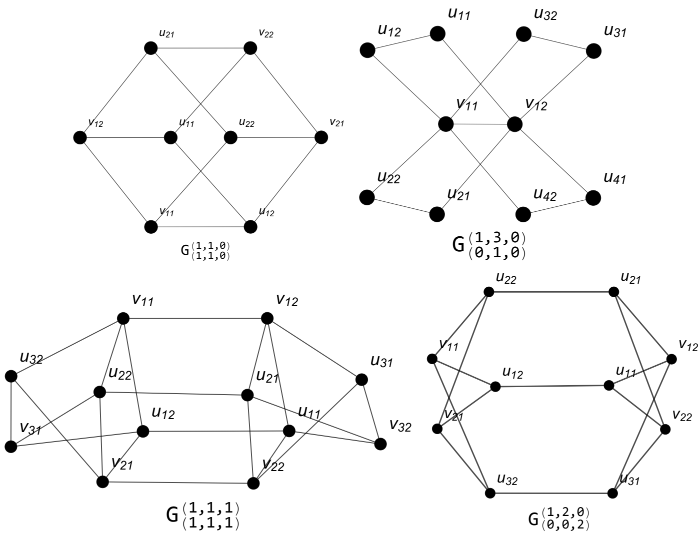

Figure 1. Graphs constructed from the graph $G = K_2$, using the graph operation defined in section 2. The first graph is obtained from G with $p_1 = p_2 = q_1 = q_2 = 1$ and $p_3 = q_3 = 0$, second graph is obtained from G with $p_1 = q_2 = 1, p_2 = 3, p_3 = q_1 = q_3 = 0$, third graph is obtained from G with $p_1 = p_2 = p_3 = q_1 = q_2 = q_3 = 1$ and the fourth graph is obtained form G with $p_1 = 1, p_2 = q_3 = 2, p_3 = q_1 = q_2 = 0$.

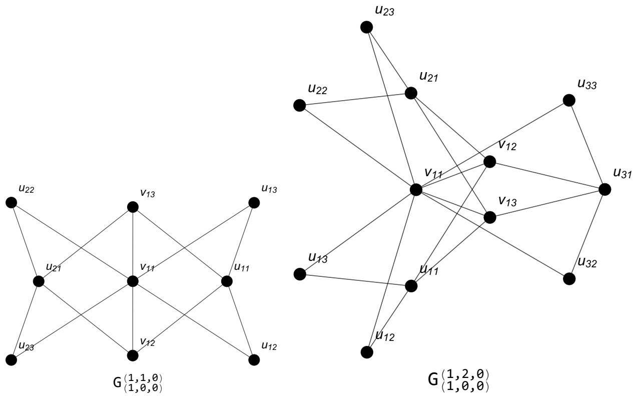

Figure 2. Graphs constructed from the path graph $G = P_3$, using the graph operation defined in section 2. The first graph is obtained from P_3 with $p_1 = p_2 = q_1 = 1, p_3 = q_2 = q_3 = 0$ and the second graph is obtained from P_3 with $p_1 = q_1 = 1, p_2 = 2, p_3 = q_2 = q_3 = 0$.

3 Spectrum of the graph $H = G^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$

We find out the spectrum of the graph $H = G^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$ constructed from the graph G using the operation defined in the previous section. The spectrum of the derived graph is to be expressed in terms of the spectrum of the given graph G. After a careful examination of the graph operation defined, we find that the adjacency matrix of the graph $H = G^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$ is given by the following Kronecker product

$$A_H = J^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)} \otimes A_G, \tag{2}$$

where A_G is the adjacency matrix of the graph G and $J^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$ is a square matrix of order $p + q$, whose block matrix representation is given by

$$J^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)} = \begin{bmatrix} C & J_{pq} \\ J_{qp} & D \end{bmatrix}. \tag{3}$$

Here C and D are again block matrices of order p and q respectively and are given by given by

$$C = \begin{bmatrix} J_{p_1} & 0 & 0 \\ 0 & I_{p_2} & 0 \\ 0 & 0 & 0_{p_3} \end{bmatrix} \tag{4}$$

and

$$D = \begin{bmatrix} J_{q_1} & 0 & 0 \\ 0 & I_{q_2} & 0 \\ 0 & 0 & 0_{q_3} \end{bmatrix}. \tag{5}$$

Here, J_{mn} represents the all one matrix of order $m \times n$, J_m represents the all one square matrix of order m, I_m represents the identity matrix of order m, 0_m represents a square zero matrix of order m and all other 0's represents zero matrices of compatible orders.

We state some known results from theory of matrices as lemmas, which are needed to prove the main theorem [5, 13].

Lemma 1. *If A and D are square matrices(need not be same order) and B and C are matrices with compatible orders, then the determinant of the following block matrix is given by*

$$\begin{aligned} Det \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= Det\,(D)\, Det\left(A - BD^{-1}C\right) \\ &= Det\,(A)\, Det\left(D - CA^{-1}B\right). \end{aligned} \tag{6}$$

provided D^{-1} or A^{-1} exists.

Lemma 2. *A block diagonal matrix*

$$A = \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ . & . & . & . & \\ . & . & . & . & \\ . & . & . & . & \\ 0 & 0 & 0 & \dots & A_k \end{bmatrix} \tag{7}$$

is invertible if and only if each of its main-diagonal blocks A_i are invertible, and in this case its inverse is given by the block diagonal matrix

$$A^{-1} = \begin{bmatrix} A_1^{-1} & 0 & 0 & \dots & 0 \\ 0 & A_2^{-1} & 0 & \dots & 0 \\ . & . & . & . & \\ . & . & . & . & \\ . & . & . & . & \\ 0 & 0 & 0 & \dots & A_k^{-1} \end{bmatrix}. \tag{8}$$

Also we have

$$det(A) = det(A_1)det(A_2)\cdots det(A_k). \tag{9}$$

Lemma 3. *Let A be an r^{th} order square matrix with eigenvalues $\{\alpha_i\}, 1 \leq i \leq r$ and B be an s^{th} order square matrix with eigenvalues $\{\beta_i\}, 1 \leq i \leq s$, then the eigenvalues of the square matrix $A \otimes B$ of order rs is given by all possible products $\{\alpha_i \beta_j\}$, for $1 \leq i \leq r$ and $1 \leq j \leq s$, which is rs in number.*

Lemma 4. *If $B = [b_{ij}]$ is a n^{th} order square matrix, then*

$$J_{mn} B J_{nm} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \right) J_m. \tag{10}$$

Lemma 5.

$$Det\left(\lambda I_m - J_m\right) = \lambda^{m-1}\left(\lambda - m\right). \tag{11}$$

Lemma 6 (Sherman–Morrison formula)**.** *Let A be an n^{th} order invertible square matrix and u and v be column vectors of length n. If $1 + v^T A^{-1} u \neq 0$, then $A + uv^T$ is invertible and the inverse is given by*

$$\left(A + uv^T\right)^{-1} = A^{-1} - \frac{A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u}. \tag{12}$$

In addition to above lemmas we need the following new lemmas to prove the next theorem.

Lemma 7.

$$
\begin{aligned}
&(i)\ (\lambda I_n - J_n)^{-1} = \frac{1}{\lambda} I_n + \frac{1}{\lambda(\lambda - n)} J_n. \\
&(ii)\ J_{mn} (\lambda I_n - J_n)^{-1} J_{nm} = \frac{n}{\lambda - n} J_m.
\end{aligned}
\tag{13}
$$

Proof. Letting $A = \lambda I_n$ and $u = -v = -\mathbf{1}_n$ in lemma 6, where $\mathbf{1}_n$ is the length n column vector of 1's, we get,

$$
\begin{aligned}
(\lambda I_n - J_n)^{-1} &= \left(\lambda I_n + (-\mathbf{1}_n)\mathbf{1}_n^T\right)^{-1} \\
&= (\lambda I_n)^{-1} - \frac{(\lambda I_n)^{-1}(-\mathbf{1}_n)\mathbf{1}_n^T(\lambda I_n)^{-1}}{1 + \mathbf{1}_n^T(\lambda I_n)^{-1}(-\mathbf{1}_n)} \\
&= \frac{1}{\lambda} I_n + \frac{\frac{1}{\lambda^2} J_n}{1 - \frac{1}{\lambda} n} \\
&= \frac{1}{\lambda} I_n + \frac{1}{\lambda(\lambda - n)} J_n.
\end{aligned}
\tag{14}
$$

Then,

$$
\begin{aligned}
J_{mn} (\lambda I_n - J_n)^{-1} J_{nm} &= J_{mn} \Gamma J_{nm},\ \text{where } \Gamma = \frac{1}{\lambda} I_n + \frac{1}{\lambda(\lambda - n)} J_n \\
&= \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij}\right) J_m,\ \text{from lemma 4 and } \Gamma = [\gamma_{ij}] \\
&= \left(\frac{n}{\lambda} + \frac{n^2}{\lambda(\lambda - n)}\right) J_m \\
&= \frac{n}{\lambda - n} J_n.
\end{aligned}
\tag{15}
$$

■

Lemma 8.

$$
\begin{aligned}
det &\left(\begin{bmatrix} (a+c) I_s & 0 \\ 0 & (a+b) I_{p-s} \end{bmatrix} - a\, J_p \right) \\
&= (a+b)^{p-s-1} (a+c)^{s-1} \\
&\quad \times \left((a+c)(a+b-ap) + as(c-b)\right)
\end{aligned}
\tag{16}
$$

Proof.

$$det\left(\begin{bmatrix}(a+c)I_s & 0\\ 0 & (a+b)I_{p-s}\end{bmatrix} - a\,J_p\right)$$

$$= det\begin{bmatrix}(a+c)I_s - a\,J_s & -a\,J_{s\,p-s}\\ -a\,J_{p-s\,s} & (a+b)I_{p-s} - a\,J_{p-s}\end{bmatrix}$$

$$= det\Big[(a+c)I_s - a\,J_s\Big]det\Big[\,((a+b)I_{p-s} - a\,J_{p-s}) - a^2 J_{p-s\,s}\left\{(a+c)I_s - a\,J_s\right\}^{-1} J_{s\,p-s}\Big]$$

$$= det\Big[(a+c)I_s - a\,J_s\Big]det\Big[\,((a+b)I_{p-s} - a\,J_{p-s}) - a\left(\frac{s}{\frac{a+c}{a} - s}\right)J_{p-s}\Big], \text{ by applying lemma 7}$$

$$= det\Big[(a+c)I_s - a\,J_s\Big]det\Big[(a+b)I_{p-s} - \sigma J_{p-s}\Big],$$

$$\text{where } \sigma = a\left(1 + \frac{as}{a+c-as}\right)$$

$$= a^s\left(\frac{a+c}{a}\right)^{s-1}\left(\frac{a+c}{a} - s\right)\sigma^{p-s}\frac{a+b}{\sigma}^{p-s-1} \times \left(\frac{a+b}{\sigma} - (p-s)\right), \text{ by applying lemma 5}$$

$$= (a+b)^{p-s-1}(a+c)^{s-1}\left((a+c)(a+b-ap) + as\,(c-b)\right),$$

by applying the value of σ and on straightforward simplification.

■

Now, we prove the following theorem which gives the complete spectrum of the matrix $J_{(p_1,p_2,p_3)}^{(q_1,q_2,q_3)}$.

Theorem 1. *The eigenvalues of the matrix* $J_{(p_1,p_2,p_3)}^{(q_1,q_2,q_3)}$ *are zero with multiplicity* $p_1+p_3+q_1+q_3-4$*, one with multiplicity* p_2+q_2-2 *and* $\alpha_i, 1 \le i \le 6$*, where* α_i*'s are the root of the polynomial* $p(x) = x^6 - a_5x^5 + a_4x^4 + a_3x^3 +$

$a_2x^2 + a_1x - a_0$ *with*

$$
\begin{aligned}
a_0 &= p_1p_3q_1q_3, \\
a_1 &= p_1(p_2+1)q_1q_3 + p_3q_1q_3 + p_1p_3(q_3 + q_1(q_2+2q_3+1)), \\
a_2 &= -p_2q_1q_3 - p_3(q_3 + q_1(q_2+2q_3+1)) - p_1(p_2(q_3 + q_1(q_2+q_3+1)) \\
&\qquad + p_3(q_2 + 2q_3 + q_1(q_2+q_3+2)) + q_1q_2 + 2q_1q_3 + q_3), \\
a_3 &= p_3q_2 + p_2q_3 + 2p_3q_3 + q_1(p_2(q_2+q_3+1) + p_3(q_2+q_3+2) - 1) \\
&\qquad + p_1(p_2(q_1+q_2+q_3) + p_3(q_1+q_2+q_3) \\
&\qquad\qquad + q_1q_2 + q_2 + q_1q_3 + 2q_3 - 1), \\
a_4 &= p_1(-q_2) - p_1q_3 - p_2(q_1+q_2+q_3) - p_3(q_1+q_2+q_3) \\
&\qquad + 2p_1 + 2q_1 + 1, \\
a_5 &= p_1 + q_1 + 2.
\end{aligned}
$$

Proof. The characteristic polynomial of $J^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$ from equation (3) is given by

$$
\begin{aligned}
f_J(\lambda) &= det\left(\lambda I_{p+q} - J^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}\right) \\
&= det\begin{bmatrix} \lambda I_p - C & -J_{pq} \\ -J_{qp} & \lambda I_q - D \end{bmatrix} \\
&= det\Big(\lambda I_p - C\Big) det\Big((\lambda I_q - D) - J_{qp}(\lambda I_p - C)^{-1} J_{pq}\Big).
\end{aligned} \tag{17}
$$

Now consider

$$
\begin{aligned}
&(\lambda I_p - C)^{-1} \\
&\quad = \begin{bmatrix} \lambda I_{p_1} - J_{p_1} & 0 & 0 \\ 0 & (\lambda-1)I_{p_2} & 0 \\ 0 & 0 & \lambda I_{p_3} \end{bmatrix}^{-1}, \text{ by equation (4)} \\
&\quad = \begin{bmatrix} (\lambda I_{p_1} - J_{p_1})^{-1} & 0 & 0 \\ 0 & ((\lambda-1)I_{p_2})^{-1} & 0 \\ 0 & 0 & (\lambda I_{p_3})^{-1} \end{bmatrix}, \text{using lemma 2}
\end{aligned} \tag{18}
$$

$$= \begin{bmatrix} \frac{1}{\lambda} I_{p_1} + \left(\frac{1}{\lambda(\lambda - p_1)} J_{p_1} \right) & 0 & 0 \\ 0 & \frac{1}{\lambda - 1} I_{p_2} & 0 \\ 0 & 0 & \frac{1}{\lambda} I_{p_3} \end{bmatrix}, \text{ using lemma 7(i).}$$

Hence, by lemma 4,

$$J_{qp} \left(\lambda I_p - C \right)^{-1} J_{pq} = \Gamma J_q, \tag{19}$$

where

$$\Gamma = \frac{p_1}{\lambda} + \frac{p_1^2}{\lambda(\lambda - p_1)} + \frac{p_2}{\lambda - 1} + \frac{p_3}{\lambda}. \tag{20}$$

Then from equation (17),

$$f_J(\lambda) = det\Big(\lambda I_p - C \Big) det\Big(\left(\lambda I_q - D \right) - \Gamma J_q \Big). \tag{21}$$

But, from equation (5),

$$\begin{aligned} & det\Big(\left(\lambda I_q - D \right) - \Gamma J_q \Big) \\ & \quad = det \left(\begin{bmatrix} \lambda I_{q_1} - J_{q_1} & 0 & 0 \\ 0 & \lambda - 1 I_{q_2} & 0 \\ 0 & 0 & \lambda I_{q_3} \end{bmatrix} - \Gamma J_q \right) \\ & \quad = det \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}, \\ & \qquad \text{where } B_{11} = \lambda I_{q_1} - \left(\Gamma + 1 \right) J_{q_1}, \ B_{12} = -\Gamma J_{q_1 q_2} = B_{21}^T \\ & \qquad B_{13} = -\Gamma J_{q_1 q_3} = B_{31}^T, \ B_{22} = \left(\lambda - 1 \right) I_{q_2} - \Gamma J_{q_2} \\ & \qquad B_{23} = -\Gamma J_{q_2 q_3} = B_{32}^T, \ B_{33} = \lambda I_{q_3} - \Gamma J_{q_3}. \\ & \quad = det\left(B_{11} \right) det \left(\begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix} - \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix} B_{11}^{-1} \begin{bmatrix} B_{12} & B_{13} \end{bmatrix} \right), \\ & \qquad \qquad \text{using lemma 1.} \end{aligned} \tag{22}$$

But,

$$
\begin{aligned}
&det\left(\begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix} - \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix} B_{11}^{-1} \begin{bmatrix} B_{12} & B_{13} \end{bmatrix}\right) \\
&\quad = det\left(\begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix} - \Gamma J_{q_2+q_3\; q_1} B_{11}^{-1} \Gamma J_{q_1\; q_2+q_3}\right) \\
&\quad = det\left(\begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix} - \Gamma^2 \left(\frac{1}{\Gamma+1}\right) \frac{q_1}{\frac{\lambda}{\Gamma+1} - q_1} J_{q_2+q_3}\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{using lemma 7(ii)} \\
&\quad = det\left(\begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix} - \Delta J_{q_2+q_3}\right), \text{where } \Delta = \Gamma^2 \frac{q_1}{\lambda - (\Gamma+1)q_1} \\
&\quad = det\begin{bmatrix} (\lambda-1)I_{q_2} - (\Gamma+\Delta)J_{q_2} & -(\Gamma+\Delta)J_{q_2\; q_3} \\ -(\Gamma+\Delta)J_{q_3\; q_2} & \lambda I_{q_3} - (\Gamma+\Delta)J_{q_3} \end{bmatrix} \\
&\quad = det\left(\begin{bmatrix} (\lambda-1)I_{q_2} & 0 \\ 0 & \lambda I_{q_3} \end{bmatrix} - (\Gamma+\Delta)J_{q_2+q_3}\right) \\
&\quad = (a+b)^{q_3-1}(a+c)^{q_2-1}\left((a+c)\left(a+b-a(q_2+q_3)\right) + aq_2(c-b)\right),
\end{aligned}
$$

$$
\text{using lemma 8, where } a = \Gamma+\Delta,\ b = \lambda - (\Gamma+\Delta),\ c = (\lambda-1) - (\Gamma+\Delta) \tag{23}
$$

Substituting this in equation (22) we get,

$$
\begin{aligned}
&det\Big((\lambda I_q - D) - \Gamma J_q\Big) \\
&\quad = det\,(B_{11})\,(a+b)^{q_3-1}(a+c)^{q_2-1} \\
&\qquad\qquad \times \left((a+c)\left(a+b-a(q_2+q_3)\right) + aq_2(c-b)\right) \\
&\quad = (\Gamma+1)^{q_1}\left(\frac{\lambda}{\Gamma+1}\right)^{q_1-1}\left(\frac{\lambda}{\Gamma+1} - q_1\right)(a+b)^{q_3-1}(a+c)^{q_2-1} \\
&\qquad \times \left((a+c)\left(a+b-a(q_2+q_3)\right) + aq_2(c-b)\right), \\
&\qquad\qquad \text{using lemma 5 for the matrix } B_{11}.
\end{aligned} \tag{24}
$$

Now,

$$\begin{aligned}
&det\left(\lambda I_p - C\right)\\
&\quad = det \begin{bmatrix} \lambda I_{p_1} - J_{p_1} & 0 & 0 \\ 0 & (\lambda - 1)\, I_{p_2} & 0 \\ 0 & 0 & \lambda I_{p_3} \end{bmatrix}, \text{ by equation (4)} \\
&\quad = det\left(\lambda I_{p_1} - J_{p_1}\right) det\left(\left(\lambda - 1\right) I_{p_2}\right) det\left(\lambda I_{p_3}\right), \text{ using lemma 2} \\
&\quad = \lambda^{p_1 + p_3 - 1}\left(\lambda - p_1\right)\left(\lambda - 1\right)^{p_2}, \text{ using lemma 5.}
\end{aligned} \tag{25}$$

Substituting the values from equations (24) and (25) in equation (21) and after a lengthy but straight forward simplification, we get,

$$\begin{aligned}
f_J(\lambda) = {} & \lambda^{p_1+p_3+q_1+q_3-4}(\lambda - 1)^{p_2+q_2-2} \\
& \times \left(\lambda^6 - a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda - a_0\right)
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
a_0 &= p_1 p_3 q_1 q_3, \quad a_5 = p_1 + q_1 + 2, \\
a_1 &= p_1\left(p_2 + 1\right) q_1 q_3 + p_3 q_1 q_3 + p_1 p_3\left(q_3 + q_1\left(q_2 + 2q_3 + 1\right)\right), \\
a_2 &= -p_2 q_1 q_3 - p_3\left(q_3 + q_1\left(q_2 + 2q_3 + 1\right)\right) - p_1\left(p_2\left(q_3 + q_1\left(q_2 + q_3 + 1\right)\right)\right. \\
&\qquad \left. + p_3\left(q_2 + 2q_3 + q_1\left(q_2 + q_3 + 2\right)\right) + q_1 q_2 + 2q_1 q_3 + q_3\right), \\
a_3 &= p_3 q_2 + p_2 q_3 + 2p_3 q_3 + q_1\left(p_2\left(q_2 + q_3 + 1\right) + p_3\left(q_2 + q_3 + 2\right) - 1\right) \\
&\qquad + p_1\left(p_2\left(q_1 + q_2 + q_3\right) + p_3\left(q_1 + q_2 + q_3\right)\right. \\
&\qquad\qquad \left. + q_1 q_2 + q_2 + q_1 q_3 + 2q_3 - 1\right), \\
a_4 &= p_1\left(-q_2\right) - p_1 q_3 - p_2\left(q_1 + q_2 + q_3\right) - p_3\left(q_1 + q_2 + q_3\right) \\
&\qquad + 2p_1 + 2q_1 + 1.
\end{aligned}$$

So, the spectrum of $J_{(p_1,p_2,p_3)}^{(q_1,q_2,q_3)}$ is obtained as stated in the theorem. ■

Theorem 2. *If G is a graph of order n whose non-zero eigenvalues are given by $\{\lambda_i\}, 1 \leq i \leq r$, for some $r \leq n$, then the non-zero eigenvalues of the graph $H = G_{(p_1,p_2,p_3)}^{(q_1,q_2,q_3)}$ are given by $\{\lambda_i\}, 1 \leq i \leq r$, each with multiplicity $p_2 + q_2 - 2$ and $\{\lambda_i \alpha_j\}, 1 \leq i \leq r, 1 \leq j \leq 6$, which are $6r$ in number and α_j are the roots of the sixth degree polynomial given in the* *statement of theorem 1.*

Proof. The adjacency matrix of the graph $H = G^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$ is

$$A_H = J^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)} \otimes A_G.$$

So, the theorem follows from lemma 3 and theorem 1. ■

4 Applications of the new graph operation

We discuss several simple but significant special cases of the generalized unary graph operation $G^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$ and their properties and applications in this section. We prove different methods for generating new orderenergetic graphs from known orderenergetic graphs and non-orderenergetic graphs. We also give different ways of generating integral graphs from known integral graphs.

4.0.1 Case 1

Let $p_2 = s$, $q_3 = t$ and $p_1 = p_3 = q_1 = q_2 = 0$. Then the characteristic polynomial of $J^{(0,0,t)}_{(0,s,0)}$ is obtained from equation (26) as

$$f_J(\lambda) = (\lambda - 1)^{s-1}\lambda^{t-1}((\lambda - 1)\lambda - st) \tag{27}$$

Then the eigenvalues of $J^{(0,0,t)}_{(0,s,0)}$ are 1 with multiplicity $s-1$, 0 with multiplicity $t-1$ and $\frac{1}{2}\left(1 \pm \sqrt{4st+1}\right)$ with multiplicity one. Using these eigenvalues, we can easily find eigenvalues of the graph $H = G^{(0,0,t)}_{(0,s,0)}$ from theorem 2. This gives an efficient way for generating integral graphs from known integral graphs and a new method for generating orderenergetic graphs. In what follows we define x and y are complementary factors of a positive integer z if $z = xy$.

Theorem 3. *Let G is an integral graph of order n. If r is positive integer such that s and t are any pair of complementary factors of $r^2 + 3r + 2$, then the graph $H = G^{(0,0,t)}_{(0,s,0)}$ is also an integral graph. Moreover, if the given graph G is an orderenergetic graph and r is an even integer, then the graph H is also orderenergetic.*

Proof. Let G is a graph of order n. If s and t are as stated in the theorem, then the eigenvalues of the matrix $J^{(0,0,t)}_{(0,s,0)}$ are 1 with multiplicity $s-1$, 0 with multiplicity $t-1$ and $-(1+r)$ and $2+r$ each with multiplicity one. Then by theorem 2, the eigenvalues of the graph H are $-(r+1)\lambda_i, (2+r)\lambda_i$, each λ_i with multiplicity $s-1$ and 0 with multiplicity $n(t-1)$, where λ_i are the n eigenvalues of the graph G. Hence the graph H is an integral graph.

The energy of the graph H can be computed as

$$\begin{aligned} \mathcal{E}_H &= (1+r)\mathcal{E}_G + (2+r)\mathcal{E}_G + (s-1)\mathcal{E}_G \\ &= (s+2r+2)\mathcal{E}_G. \end{aligned} \tag{28}$$

Suppose the given graph G is an orderenergetic graph so that $\mathcal{E}_G = n$. Now we derive the condition for the graph H to be orderenergetic. The order of this graph is $n(s+t)$. If H is orderenergetic, then form equation (28) we get

$$n(s+t) = (s+2r+2)\mathcal{E}_G = (s+2r+2)n. \tag{29}$$

So, the graph H is orderenergetic if $s+t = (s+2r+2)$. This gives the condition $t = 2r+2$. Then $s = (r+2)/2$, as s and t are complementary factors of r^2+3r+2. Since s is always an integer, it follows that $r = 2k$ for some positive integer k. So the necessary condition for the graph H to be orderenergetic graph is that $s = k+1$ and $t = 4k+2$ for some positive integer k. ∎

The following result easily follows from the above theorem.

Corollary 1. *Let G is an orderenergetic graph. Then the graph $H = G^{(0,0,4k+2)}_{(0,k+1,0)}$ is always an orderenergetic graph for any positive integer k.*

It is also possible to generate several new orderenergetic graphs from non-orderenergetic graphs in certain special cases using the unary operation.

Corollary 2. *Let G is a graph of order n with energy $\mathcal{E}_G = m$ is an integer. If there exist positive integers s and t such that $(s+2r+2)m = n(t+s)$*

where $st = r^2 + 3r + 2$ *, then the graph* $H = G_{(0,s,0)}^{(0,0,t)}$ *is an orderenergetic graph.*

Proof. The proof follows from the equation (28) and the fact that the order of the graph H is $n(s+t)$. ■

We illustrate this type of construction of orderenergetic graphs from non-orderenergetic graphs. Consider the graph G on five vertices given in figure 3. The energy of this graph is 6. Let $s = 2$, $t = 10$ and $r = 3$. Then the conditions in corollary 2 are satisfied and the generated graph $H = G_{(0,2,0)}^{(0,0,10)}$ is orderenergetic. For further examples, consider the six graphs on 10 vertices given in figure 4, each with energy 12. From each of these non-orderenergetic graphs we can generate orderenergetic graphs using corollary 2 with $s = 2$, $t = 10$ and $r = 3$.

It is also possible to generate sequences of orderenergetic graphs using a non-orderenergetic graph as a seed graph. Let G_k be the graph whose adjacent matrix is $J_k \otimes A_G$, for any positive integer k and G is as given in figure 3. Then, it is easy to see that G_k is a graph on $5k$ vertices with energy $6k$. Then the conditions in corollary 2 are satisfied and it follows that the graphs $H = G_{k(0,2,0)}^{(0,0,10)}$ are all orderenergetic graphs for any positive integer k. Similarly, Let G'_k be the graph whose adjacent matrix is $J_{(k,0,0)}^{(0,0,2k)} \otimes A_G$, for any positive integer k and G is as given in figure 3. Then, G'_k is also a graph on $5k$ vertices with energy $6k$ [11]. So, it follows that the graphs $H = {G'}_{k(0,2,0)}^{(0,0,10)}$ are all orderenergetic graphs for any positive integer k. We can construct other families of orderenergetic graphs using suitable seed graphs such as given in figure 4.

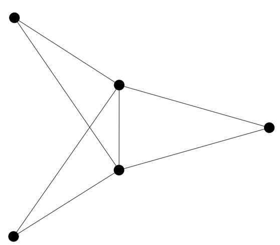

Figure 3. Graph on 5 vertices with energy $\mathcal{E}_G = 6$.

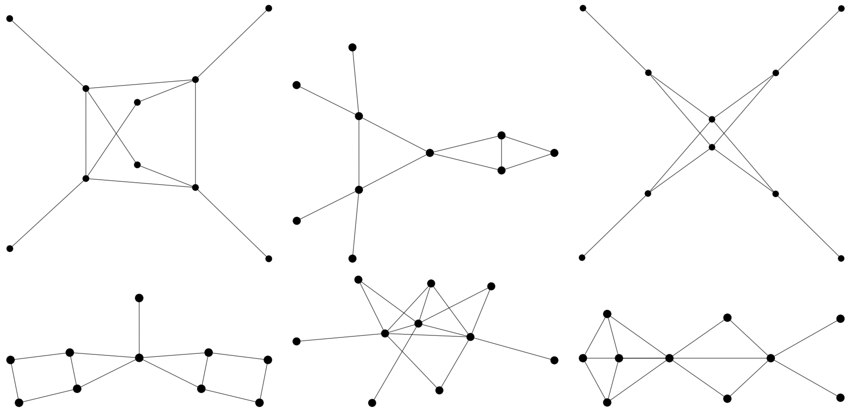

Figure 4. Six Graphs on 10 vertices with energy $\mathcal{E}_G = 12$.

4.0.2 Case 2

Let $p_1 = s$, $q_3 = t$ and $p_2 = p_3 = q_1 = q_2 = 0$. Then the characteristic polynomial of $J^{(0,0,t)}_{(s,0,0)}$ is obtained from equation (26) as

$$f_J(\lambda) = \lambda^{s+t-2} \left(\lambda^2 - s(\lambda + t)\right) \tag{30}$$

Then the eigenvalues of $J^{(0,0,t)}_{(s,0,0)}$ are 0 with multiplicity $s + t - 2$ and $\frac{1}{2}\left(s \pm \sqrt{s(s + 4t)}\right)$ with multiplicity one each. Using these eigenvalues, we can easily find eigenvalues of the graph $H = G^{(0,0,t)}_{(s,0,0)}$ from theorem 2. This gives an efficient way for generating integral graphs from known integral graphs and another method for generating orderenergetic graphs. Some results in this case are given in [11].

Theorem 4. *[11] Let G is an integral graph of order n. If $s = k^2$ is a perfect square, then the graph $H = G^{(0,0,(k+r)r)}_{(s,0,0)}$ is also an integral graph for any non-negative integer r. Otherwise, if s is not a perfect square, then the graph $H = G^{(0,0,sr(r+1))}_{(s,0,0)}$ is an integral graph for any non-negative integer r. Moreover, if the given graph G is an integral energy graph with energy m and r is a non-negative integer satisfying $m(2r + 1) = n(r^2 + r + 1)$, then the graph $H = G^{(0,0,sr(r+1))}_{(s,0,0)}$ is always orderenergetic graph for any positive integer s.*

This theorem gives two different methods to generate orderenergetic graphs from known orderenergetic graphs.

Corollary 3. *If G is an orderenergetic graph on n vertices, then the sequence of graphs $H = G_{(s,0,0)}^{(0,0,0)}$ and $H' = G_{(s,0,0)}^{(0,0,2s)}$ are orderenergetic graphs for any positive integer s with order sn and $3sn$ respectively.*

Proof. Corollary follows from theorem 4 by putting $r = 0$ and $r = 1$ respectively. ■

4.0.3 Case 3

Let $p_2 = s$, $q_2 = t$ and $p_1 = p_3 = q_1 = q_3 = 0$. Then the characteristic polynomial of $J_{(s,0,0)}^{(0,0,t)}$ is obtained from equation (26) as

$$f_J(\lambda) = (\lambda - 1)^{p+q-2}\left((\lambda - 1)^2 - pq\right) \tag{31}$$

Then the eigenvalues of $J_{(0,s,0)}^{(0,t,0)}$ are 1 with multiplicity $s+t-2$ and $1 \pm \sqrt{pq}$ with multiplicity one each. Using these eigenvalues, we can easily find eigenvalues of the graph $H = G_{(0,s,0)}^{(0,t,0)}$ from theorem 2. This gives another efficient way for generating integral graphs from known integral graphs and another method for generating orderenergetic graphs.

Theorem 5. *Let G is an integral graph of order n. If r is positive integer and s and t are any pair of complementary factors of r^2, then the graph $H = G_{(0,s,0)}^{(0,t,0)}$ is also an integral graph. Moreover, if the given graph G is an integral energy graph with $\mathcal{E}_G = m$ and $(s + t + 2r - 2)m = n(s + t)$, then the graph H is orderenergetic.*

Proof. Let G is a graph of order n. If s and t are as stated in the theorem, then the eigenvalues of the matrix $J_{(0,s,0)}^{(0,t,0)}$ are 1 with multiplicity $s+t-12$ and $(1 - r)$ and $1 + r$ each with multiplicity one. Then by theorem 2, the eigenvalues of the graph H are $(1-r)\lambda_i, (1+r)\lambda_i$, each λ_i with multiplicity $s + t - 2$, where λ_i are the n eigenvalues of the graph G. Hence the graph H is an integral graph.

The energy of the graph H can be computed as

$$\begin{aligned} \mathcal{E}_H &= (1-r)\mathcal{E}_G + (1+r)\mathcal{E}_G + (s+t-2)\mathcal{E}_G \\ &= (s+t+2r-2)\mathcal{E}_G. \end{aligned} \tag{32}$$

Suppose the given graph G is an integral energy graph so that $\mathcal{E}_G = m$. Now we derive the condition for the graph H to be orderenergetic. The order of the graph H is clearly $n(s+t)$. But it is given that $n(s+t) = (s+t+2r-2)m$. So it follows that $n(s+t) = (s+t+2r-2)\mathcal{E}_G = \mathcal{E}_H$, from equations (32). Hence H is an orderenergetic graph. ■

It is clear from the above theorem that we cannot generate orderenergetic graphs from a given orderenergetic graph in this case. But we can generate orderenergetic graphs from non-orderenergetic graphs in certain cases. We illustrate this type of construction of orderenergetic graphs from non-orderenergetic graphs by using a seed graph. Consider the star graph G on 10 vertices given in figure 5. The energy of this graph is 6 an it is not orderenergetic. Let s = 3, t = 3 and r = 3. Then the conditions in theorem 5 are satisfied and the generated graph $H = G^{(0,3,0)}_{(0,3,0)}$ is orderenergetic. Let G_k be the graph whose adjacent matrix is $J_k \otimes A_G$, for any positive integer k and G is as given in figure 5. Then, it is easy to see that G_k is a graph on $10k$ vertices with energy $6k$. It follows that the graphs $H = G_{k(0,3,0)}^{\ (0,3,0)}$ are all orderenergetic graphs for any positive integer k. Similarly, let G'_k be the graph whose adjacent matrix is $J^{(0,0,2k)}_{(k,0,0)} \otimes A_G$, for any positive integer k and G is as given in figure 5. Then, G'_k is also a graph on $10k$ vertices with energy $6k$ [11]. So, it follows that the graphs $H = G'^{\ (0,3,0)}_{k(0,3,0)}$ are all orderenergetic graphs for any positive integer k. We can construct other families of orderenergetic graphs using suitable seed graphs other than the star graph on 10 vertices.

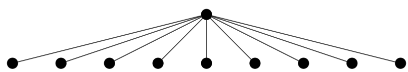

Figure 5. A graph on 10 vertices with energy $\mathcal{E}_G = 6$.

4.0.4 Case 4

Let $p_1 = 2s$, $q_1 = s$, $q_3 = s$ and $p_2 = p_3 = q_2 = 0$. Then the characteristic polynomial of $J_{(2s,0,0)}^{(s,0,s)}$ is obtained from equation (26) as

$$f_J(\lambda) = \lambda^{4s-3}\left(\lambda^3 + 2s^3 - 2\lambda s^2 - 3\lambda^2 s\right). \tag{33}$$

Then the eigenvalues of $J_{(2s,0,0)}^{(s,0,s)}$ are 0 with multiplicity $4s-3$ and $-s$, and $\left(2 \pm \sqrt{2}\right)s$ with multiplicity one each. Using these eigenvalues, we can easily find eigenvalues of the graph $H = G_{(2s,0,0)}^{(s,0,s)}$ from theorem 2. This gives another efficient way for generating integral graphs from known integral graphs and another method for generating orderenergetic graphs.

Theorem 6. *Let G is an integral graph of order n.Then the graph $H = G_{(2s,0,0)}^{(s,0,s)}$ is an integral energy graph for any positive integer s. Moreover, if the given graph G is an integral energy graph with $\mathcal{E}_G = m$ and $4n = 5m$, then the graph H is orderenergetic for any positive integer s.*

Proof. Let G is a graph of order n. Then the eigenvalues of the matrix $J_{(2s,0,0)}^{(s,0,s)}$ are 0 with multiplicity $4s-3$ and $-s$, $\left(2 \pm \sqrt{2}\right)s$ with multiplicity one each. Then by theorem 2, the eigenvalues of the graph H are $-s\lambda_i$, $\left(2+\sqrt{2}\right)\lambda_i$, $\left(2-\sqrt{2}\right)\lambda_i$, 0 with multiplicity $n(4s-3)$, where λ_i are the n eigenvalues of the graph G. So, the graph H is not an integral graph. The energy of the graph H can be computed as

$$\mathcal{E}_H = s\mathcal{E}_G + \left(2+\sqrt{2}\right)\mathcal{E}_G + \left(2-\sqrt{2}\right)\mathcal{E}_G = 5s\mathcal{E}_G. \tag{34}$$

Hence, the energy of the graph H is an integer and hence it is an integral energy graph for any positive integer s.

Suppose the given graph G is an integral energy graph so that $\mathcal{E}_G = m$ for some positive integer m. Now we derive the condition for the graph H to be orderenergetic. The order of the graph H is clearly $4ns$. But it is given that $4n = 5m$. So, it follows from equation (34) that $4ns = 5sm = 5s\mathcal{E}_G = \mathcal{E}_H$. Hence H is an orderenergetic graph. ∎

As an example, consider the star graph G on five vertices. Clearly energy of this graph is 4 and hence $4n = 5m$ is satisfied. Hence $H =$

$G^{(s,0,s)}_{(2s,0,0)}$ is always an orderenegetic graph for any positive integer s. We can generate one more sequence of orderenergetic graphs from this star graph as follows. Let G'_k be the graph whose adjacent matrix is $J^{(0,0,2k)}_{(k,0,0)} \otimes A_G$, for any positive integer k and G is star graph on five vertices. Then G'_k is a graph on $5k$ vertices with energy $4k$ [11] and the condition in the above theorem is clearly satisfied. So, it follows that the graphs $H = {G'}^{(s,0,s)}_{k(2s,0,0)}$ are all orderenergetic graphs for any positive integer k. We can construct other families of orderenergetic graphs using suitable seed graphs other than the star graph on 5 vertices.

So far we have discussed four different special cases of the new unary graph operation defined in this paper. In all these cases it is possible to generate new orderenergetic graphs from known orderenergetic graphs and non-orderenergetic graphs. In a similar fashion it may be possible to find other special cases using which it is possible to generate further orderenergetic graphs. Finally, we prove a general method for generating families of equienergetic graphs using the new graph operation.

Corollary 4. *Let G and $\hat{G}$ be a pair of equienergetic graphs, then the graph $G^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$ and the graph $\hat{G}^{(q_1,q_2,q_3)}_{(p_1,p_2,p_3)}$ are also pair of equienergetic graphs for any positive integers p_i and q_i for all $i = 1, 2, 3$. Moreover, the graphs $G^{(0,0,0)}_{(3s,0,0)}$ and $G^{(0,0,2s)}_{(s,0,0)}$ are non-cospectral equienergetic graphs of same order for any positive integer s.*

Proof. First part follows from theorem 1 and 2. Second part follows from corollary 3 and noting that the non-zero eigenvalue of $J^{(0,0,0)}_{(3s,0,0)}$ is $3s$ only and the non-zero eigenvalues of $J^{(0,0,2s)}_{(s,0,0)}$ are $-s$ and $2s$ only. ■

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