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Improving a Lower Bound for Seidel Energy of Graphs

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Abstract

For a simple graph G, the Seidel energy, denoted by $\mathcal{E}(\mathcal{S}(G))$, is defined as the sum of absolute values of all eigenvalues of the Seidel matrix of G. Two graphs are called SC-equivalent if one of them is obtained from the other or its complement by a Seidel switching. In [3] Haemers conjectured that if G is a graph of order n, then $\mathcal{E}(\mathcal{S}(G)) \geq 2n-2$. Recently, in [S. Akbari, M. Einollahzadeh, M.M. Karkhaneei, M. A. Nematollahi, Proof of a conjecture on the Seidel energy of graphs, European J. Combin. 86 (2020): 103078] the authors proved this conjecture and showed that if G is a graph of order n, then $\mathcal{E}(\mathcal{S}(G)) \geq 2n-2$ and the inequality is strict provided that G is not SC-equivalent to K_n. In this paper, we improve this lower bound and show that if G is a graph of order $n \geq 7$ which is not SC-equivalent to K_n, then $\mathcal{E}(\mathcal{S}(G)) > 2n-1$.

1 Introduction and terminology

Throughout this paper all graphs we consider are simple and finite. For a graph G, we denote the set of vertices and edges of G by $V(G)$ and $E(G)$,

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respectively. The *order* of a graph is its number of vertices and the *size* of a graph is its number of edges. The complement of G is denoted by $\overline{G}$ and the complete graph and path graph of order n are denoted by K_n and P_n, respectively. In this paper, for $v \in V(G)$, $N_G(v)$ denotes the open neighborhood of v in G. Moreover, $\delta(G)$ represents the minimum degree of G.

For every Hermitian matrix A the *energy* of A, $\mathcal{E}(A)$, is defined to be sum of the absolute values of the eigenvalues of A. The well-known concept of energy of a graph G, denoted by $\mathcal{E}(G)$, is the energy of its adjacency matrix. Let G be a graph and $V(G) = \{v_1, \ldots, v_n\}$. The *Seidel matrix* of G, denoted by $\mathcal{S}(G)$, is an $n \times n$ matrix whose diagonal entries are zero, ij-th entry is -1 if v_i and v_j are adjacent and otherwise is 1 (It is noteworthy that at first, van Lint and Seidel introduced the concept of Seidel matrix for the study of equiangular lines in [8]). The *Seidel energy* of G is defined to be $\mathcal{E}(\mathcal{S}(G))$ (see for example [4, 5, 7] about some results on the Seidel energy of graphs). Moreover, the *Seidel switching* of G is defined as follows: Partition $V(G)$ into two subsets V_1 and V_2, delete the edges between V_1 and V_2 and join all vertices $v_1 \in V_1$ and $v_2 \in V_2$ which are not adjacent. Therefore, if we call the new graph by G', then we have $\mathcal{S}(G') = D\mathcal{S}(G)D$, where D is a diagonal matrix with entries 1 (resp. -1) corresponding to the vertices of V_1 (resp. V_2) ([3]). Hence, $\mathcal{S}(G)$ and $\mathcal{S}(G')$ are similar and they have the same Seidel energy. Note that if one of the V_1 or V_2 is empty, then G remains unchanged and also, for every $v \in V(G)$, using a Seidel switching on G, one can convert v to an isolated vertex. Two graphs G_1 and G_2 are called *SC-equivalent* if G_2 is obtained from G_1 or $\overline{G_1}$ by a Seidel switching and is denoted by $G_1 \cong G_2$. Note that in either cases, $\mathcal{S}(G_2)$ is similar to $\mathcal{S}(G_1)$ or $-\mathcal{S}(G_1)$, hence $\mathcal{E}(\mathcal{S}(G_1)) = \mathcal{E}(\mathcal{S}(G_2))$. If X and Y are two disjoint subsets of $V(G)$, the set of edges of G with one endpoint in X and another in Y is denoted by $E(X, Y)$. An ordered pair (X, Y) of disjoint subsets of $V(G)$ with $|X| = |Y| = 2$, is called an *odd pair* if $|E(X, Y)|$ is an odd number (which is either 1 or 3). One can easily see that applying a Seidel switching on an arbitrary graph G does not change its odd pair(s). We denote the number of odd pairs in G by $s(G)$. Also, if H is a graph of order r and $n > r$ be a positive integer, by $H^{\uparrow n}$, we mean the graph $H \cup (n-r)K_1$. One can easily check the following equalities:

$$s(K_n) = s(\overline{K_n}) = 0,$$
$$s(K_2{}^{\uparrow n}) = 2(n-2)(n-3).$$

A subset $\{u, v\} \subseteq V(G)$ is called an *odd set* if $\{u, v\}$ is the first component of an odd pair of G.

From a graph G, we construct a graph denoted by $\Lambda(G)$, as follows: $V(\Lambda(G)) = V(G)$ and $E(\Lambda(G))$ consists of all the edges $e = uv$ such that $\{u, v\}$ is an odd set of G. By $\lambda(v)$ we denote the degree of vertex $v \in V(G)$ in the graph $\Lambda(G)$.

In 2012 Haemers conjectured that if G is a graph of order n, then $\mathcal{E}(\mathcal{S}(G)) \geq 2n-2$ [3]. Recently, by means of the concept of odd pairs Akbari et al. in [1] proved this conjecture and showed that if G is a graph of order n, then $\mathcal{E}(\mathcal{S}(G)) \geq 2n-2$ and the inequality is strict provided that G is not SC-equivalent to K_n. More precisely, they show that for every graph G of order n,

$$\mathcal{E}(\mathcal{S}(G)) \geq \mathcal{E}(\mathcal{S}(K_n)) = 2n-2. \tag{1}$$

In this paper, we improve the lower bound (1) by proving that if G is a graph of order $n \geq 7$ which is not SC-equivalent to K_n, then $\mathcal{E}(\mathcal{S}(G)) > 2n-1$. Our result implies that there is no graph of order $n \geq 7$, say G, such that

$$2n-2 < \mathcal{E}(\mathcal{S}(G)) \leq 2n-1.$$

It is worthy to say that in [1], one of the main steps in the proof of inequality (1) is the fact that if G is a graph of order n which is not SC-equivalent to K_n, then $s(G) \geq 2(n-3)^2$. In this study, by proving several lemmas, we show that for every graph G of order n ($n \geq 4$), which is not SC-equivalent to K_n and $K_2{}^{\uparrow n}$, we have

$$s(G) \geq 4(n-3)^2. \tag{2}$$

Then, we show that $\mathcal{E}(\mathcal{S}(K_2{}^{\uparrow n})) = n - 2 + \sqrt{n^2 + 4n - 12}$ which implies

$$\mathcal{E}(\mathcal{S}(K_2{}^{\uparrow n})) > 2n - 1,$$

provided that $n \geq 7$. Finally, with the aid of Inequality (2), we prove that if G is a graph of order $n \geq 6$ which is not SC-equivalent to K_n and $K_2{}^{\uparrow n}$, then $\mathcal{E}(\mathcal{S}(G)) > 2n - 1$. According to the two latter facts, we conclude that if G is a graph of order $n \geq 7$ which is not SC-equivalent to K_n, then $\mathcal{E}(\mathcal{S}(G)) > 2n - 1$.

2 Improving the lower bound of Seidel energy

In this section, we begin with proving several lemmas to obtain the preliminaries for the proof of Theorem 1, which is a crucial step in the procedure of concluding our main result. Following this, Theorem 2 is devoted to calculate the Seidel energy of graph $K_2{}^{\uparrow n}$. At the end, by use of the obtained theorems and computer search, we express the principal result in Theorem 4.

Lemma 1. *[2, Lemma 5] Let G be a graph of order n and $e = uv$ be an edge in $E(\Lambda(G))$. Then, there exist at least $n - 3$ odd pairs in G such that their first component is $X = \{u, v\}$.*

Corollary. *Let G be a graph of order n such that $\delta(\Lambda(G)) \geq 8$. Then $s(G) \geq 4n(n - 3)$.*

Proof. In this case, the size of $\Lambda(G)$ is at least $4n$ and by Lemma 1 the proof is complete. ■

Lemma 2. *Let G be a graph of order n and $v \in V(G)$ be an arbitrary vertex. Also, assume that X is the set of all vertices of G which are adjacent to v in $\Lambda(G)$. Then, G is SC-equivalent to a graph H such that the isolated vertices of H are exactly $V(G) \setminus X$.*

Proof. Using a Seidel switching one can assume that v is an isolated vertex of G. Define $Y = V(G)\setminus(X\cup\{v\})$. Therefore, for every $w \in Y$, $\{v, w\}$ isn't an odd set in G. Hence, for every vertices $x, y \in V(G) \setminus \{v, w\}$, the parity of $|E(v, \{x, y\})|$ and $|E(w, \{x, y\})|$ are the same. But $E(v, \{x, y\}) = \emptyset$, so $E(w, \{x, y\})$ is of even order. Therefore, w is either connected to all vertices of $V(G) \setminus \{v, w\}$ or is not connected to any vertices in $V(G) \setminus \{v, w\}$. This is equivalent to say that if $\widehat{G}$ be the induced subgraph $G[V(G) \setminus \{v\}]$, then every vertex of Y is either isolated in $\widehat{G}$ or is connected to all other vertices of $\widehat{G}$. Note that the above two cases cannot occur simultaneously.

In the case where all vertices of Y are isolated in $\widehat{G}$, then all vertices of $Y \cup \{v\}$ are isolated in G. Now, define H to be the induced subgraph $G[X]$ which implies that $G \cong H^{\uparrow n}$.

On the other hand, in the second case, by considering the graph $\overline{G}$, one can assume that every vertex of Y is isolated in $\widehat{G}$. Now, using a Seidel switching with respect to $(v, V(G) \setminus v)$, we obtain a graph in which every vertex of $Y \cup \{v\}$ is isolated. Therefore, we have

$$G \cong G[X]^{\uparrow n},$$

as desired. ■

Corollary. *Let G be a graph of order n which is not SC-equivalent to K_n. Then, $\delta(\Lambda(G)) \geq 2$.*

Proof. Assume that $v \in V(G)$ be an arbitrary vertex of G. By Lemma 2, there exists a graph H of order $|\lambda(v)|$ such that $G \cong H^{\uparrow n}$. On the contrary, if $\lambda(v) = 0$ or 1, then $H^{\uparrow n}$ is $\overline{K_n}$ which implies that $G \cong K_n$, a contradiction. ■

Lemma 3. *Let H be a graph of order at most $n-2$ with no isolated vertex. Suppose that X is the set of all vertices of H in $H^{\uparrow n}$ and Y is $V(H^{\uparrow n}) \setminus X$. Then, every vertex of X is adjacent to every vertex of Y in the graph $\Lambda(H^{\uparrow n})$.*

Proof. Suppose that $y' \in Y \setminus \{y\}$ and $x' \in X$ is adjacent to x (Note that since $|H| \leq n-2$, such a vertex y' exists). Now, the only edge of $H^{\uparrow n}$ with one endpoint in $\{x, y\}$ and the other in $\{x', y'\}$ is $e = xx'$. Therefore, $(\{x, y\}, \{x', y'\})$ is an odd pair in $H^{\uparrow n}$ and hence, $\{x, y\}$ is an edge of $\Lambda(H^{\uparrow n})$. ■

Lemma 4. *Let G be a graph of order $n \geq 4$ which is not SC-equivalent to K_n. Then, the size of $\Lambda(G)$ is at least $2(n-2)$ which implies that $s(G) \geq 2(n-2)(n-3)$.*

Proof. Let m be the size of $\Lambda(G)$. By Lemma 1, $s(G) \geq m(n-3)$. If $\Lambda(G)$ is the complete graph K_n, then

$$m = \frac{n(n-1)}{2} > 2(n-2) \Rightarrow s(G) > 2(n-2)(n-3),$$

as desired.

Now, assume that $\Lambda(G) \neq K_n$. Therefore, there exists $v \in V(G)$ such that $\lambda(v) < n-1$. On the other hand, by Corollary 2, $\lambda(v) \geq 2$. Hence, by Lemma 2, G is SC-equivalent to $H^{\uparrow n}$ where $|H| < n-1$. One can assume that H has minimum order with this property. Since G is not SC-equivalent to K_n, then $|H| \geq 2$. Note that H has no isolated vertex (otherwise by deleting the isolated vertices of H the graph $\widehat{H}$ is obtained where $|V(\widehat{H})| < |V(H)|$ and $G = \widehat{H}^{\uparrow n}$, a contradiction). So, by Lemma 3, we have

$$m \geq |H|(n-|H|) \geq 2(n-2).$$

Therefore,

$$s(G) \geq m(n-3) \geq 2(n-2)(n-3),$$

which completes the proof. ■

Lemma 5. *Let G be a graph of order $n \geq 4$ which is not SC-equivalent to K_n or ${K_2}^{\uparrow n}$ and, $G \cong H^{\uparrow n}$, where H is either P_3 or K_3. Then $s(G) \geq 4(n-3)^2$.*

Proof. One can assume that H has no isolated vertex; otherwise by deleting the isolated vertices of H, a graph of less order is obtained.

First assume that $H = P_3$. In this case, one can easily check that we have

$$s(H^{\uparrow n}) = 4(n-3) + 4(n-3)(n-4) = 4(n-3)^2.$$

Hence, suppose that $H = K_3$. It is easy to see that we have

$$s(H^{\uparrow n}) = 6(n-3)(n-4).$$

If $n \geq 6$, then $6(n-3)(n-4) \geq 4(n-3)^2$. Also, if $n = 5$, then as sketched in Figure 1, we have $G = H^{\uparrow 5} \cong {K_2}^{\uparrow 5}$, a contradiction. Finally if $n = 4$, then using a Seidel switching with respect to the isolated vertex and the other three vertices of G, we obtain $G \cong K_4$, a contradiction. Now, the proof is complete. ■

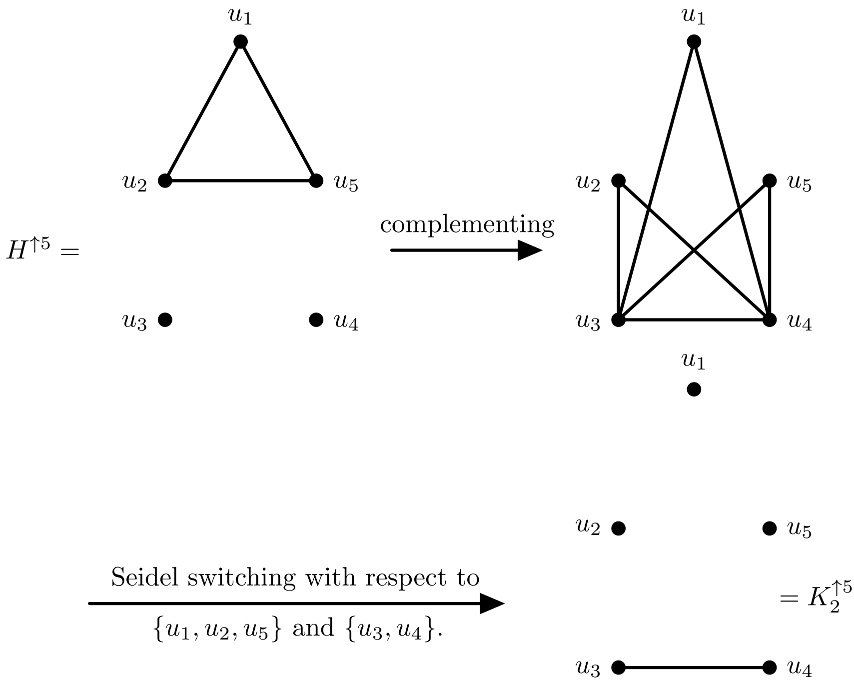

Figure 1. A visual proof that shows ${K_3}^{\uparrow 5} = H^{\uparrow 5} \cong {K_2}^{\uparrow 5}$.

Lemma 6. *Let G be a graph of order n and $u \in V(G)$ be an arbitrary vertex. Also, define $X = V(G) \setminus N_{\Lambda(G)}(u)$. Then, for every $v \in N_{\Lambda(G)}(u)$ and $w \in X$ we have $e = vw \in E(\Lambda(G))$.*

Proof. If $\lambda(u) = n - 1$, then $X = \{u\}$ and obviously the lemma holds. Therefore, assume that $|N_{\Lambda(G)}(u)| \leq n - 2$. Now, by Lemma 2, G is SC-equivalent to a graph H such that the non-isolated vertices of H are exactly $N_{\Lambda(G)}(u)$. Finally, by Lemma 3, the proof is complete. ■

Theorem 1. *Let G be a graph of order $n \geq 4$ which is not SC-equivalent to K_n or ${K_2}^{\uparrow n}$. Then,*

$$s(G) \geq 4(n-3)^2.$$

Proof. Let $d = \delta(\Lambda(G)) = \deg_{\Lambda(G)}(v)$. By Lemma 2, we can assume that the non-isolated vertices of G are exactly $N_{\Lambda(G)}(v)$. Therefore, if we define H to be the induced subgraph of G over $V(N_{\Lambda(G)}(v))$, then $G \cong H^{\uparrow n}$ and $|V(H)| = d$.

By Lemma 5, if $d \leq 3$, the theorem holds. On the other hand, if $d = n-1$, then $\Lambda(G)$ is complete graph and hence, $|E(\Lambda(G))| = \frac{n(n-1)}{2}$. So, $s(G) \geq \frac{n(n-1)}{2}(n-3) > 4(n-3)^2$. Moreover, if $d \geq 8$, then $|E(A(G))| \geq \frac{8n}{2} > 4(n-3)$; therefore $s(G) > 4(n-3)^2$. Hence, in the rest of the proof, we turn into the cases where $4 \leq d \leq 7$ and $d \leq n-2$. We consider two cases:

Case 1. $s(H) > 0$. Then H is not SC-equivalent to K_d and by Lemma 3, the size of $\Lambda(H)$ is at least $2(d-2)$ and obviously, $E(\Lambda(H)) \subseteq E(\Lambda(G))$. Moreover, by Lemma 6, there exist at least $d(n-d)$ edges in $E(\Lambda(G))$ different from those in $E(\Lambda(H))$. Hence,

$$|E(\Lambda(G))| \geq d(n-d) + 2(d-2).$$

Note that $d(n-d)+2(d-2) = d(n-d+2)-4$ and since $d, n-d+2 \geq 4$, we have

$$d(n-d+2) \geq 4(n-2) \Rightarrow E(\Lambda(G)) \geq 4(n-2) - 4 = 4(n-3),$$

which Lemma 1 implies that $s(G) \geq 4(n-3)^2$.

Case 2. $s(H) = 0$. Then, by Lemma 4, $H \cong K_d$. Hence, there exist non-negative integers a and b such that $a+b = d$ and H is either $K_{a,b}$ or $K_a \sqcup K_b$. Also, since H has no isolated vertex, the cases $K_{0,d} = \overline{K_d}$ and$K_1 \sqcup K_{d-1}$ cannot occur.

Since $s(H) = 0$, all odd pairs of G appear as one of the followings:

$$(\{x,u\},\{y,v\}), \text{ where } xy \in E(H) \text{ and } v, w \notin V(H). \qquad (3)$$

or

$$(\{x,u\},\{y,v\}) \text{ or } (\{y,v\},\{x,u\}), \text{ where } u \notin V(H), xy \in E(H) \\ \text{and } v \in V(H) \text{ and } xv \notin E(H). \qquad (4)$$

First we calculate the odd pairs of $H = K_{a,b}$. Here, G has $2ab(n-d)(n-d-1)$ odd pairs of the form (3) and $2(n-d)(a(a-1)b+b(b-1)a)$ odd pairs of the form (4). Therefore,

$$\begin{aligned} s(G) &\geq 2(n-d)(ab(n-d-1)+ab(a-1+b-1)) \\ &= 2(n-d)(ab(n-d-1)+ab(d-2)) \\ &= 2ab(n-d)(n-3). \end{aligned}$$

Note that $ab \geq d-1 \geq 3$ and $n-d \geq 2$. So

$$ab(n-d) \geq (d-1)(n-d) \geq 2(n-3),$$

which implies that $s(G) \geq 4(n-3)^2$, as desired.

Now we turn to the case $H = K_a \sqcup K_b$, where $0 \leq a \leq b \leq d$ and $a+b=d$. As we mentioned before, the case $a=1$ don't occur.

Here G has $(n-d)(n-d-1)(a(a-1)+b(b-1))$ odd pairs of the form (3) and
$2(n-d)(a(a-1)b+b(b-1)a)$ odd pairs of the form (4). Therefore,

$$s(G) = (n-d)((a^2+b^2-d)(n-d-1)+2ab(d-2)). \qquad (5)$$

All possible cases for (a,b) are as follows:
I: $a=0$. By (5), we have $s(G) = (n-d)(d-1)d(n-d-1)$. Note that if $n-d \geq 3$, then

$$\begin{aligned} &n = d+(n-d) \geq 7, \\ &n-d \geq 3, \quad d-1 \geq 3 \Rightarrow (d-1)(n-d) \geq 3(n-4) > 2(n-3) \quad (n>6) \\ &d \geq 4, \quad n-d-1 \geq 2 \Rightarrow d(n-d-1) \geq 2(n-3), \end{aligned}$$

which together yield $s(G) > 4(n-3)^2$. Also, if $n-d=2$, then G is

SC-equivalent to ${K_d}^{\uparrow n}$ which can be easily seen that is SC-equivalent to ${K_2}^{\uparrow n}$ (by complementing and a Seidel switching), a contradiction.

II: $(a, b) = (2, 2)$. So, $d = 4$ and hence $n \geq 6$. By (5), we have

$$s(G) = (n-4)(16 + 4(n-5)) = 4(n-4)(n-1).$$

Now, $n \geq 6$ implies that

$$s(G) = 4(n-4)(n-1) > 4(n-3)^2.$$

III: $(a, b) = (2, 3)$. So, $d = 5$ and hence $n \geq 7$. By (5), we have

$$s(G) = (n-5)(36 + 8(n-6)) = 4(n-5)(2n-3).$$

Now, $n \geq 7$ implies that

$$s(G) = 4(n-5)(2n-3) > 4(n-3)^2.$$

IV: $(a, b) = (2, 4)$. So, $d = 6$ and hence $n \geq 8$. By (5), we have

$$s(G) = (n-6)(64 + 14(n-7)) = 2(n-6)(7n-17) > 2(n-6) \cdot 7(n-3).$$

Now, $n \geq 8$ implies that

$$s(G) > 14(n-6)(n-3) > 4(n-3)^2.$$

V: $(a, b) = (2, 5)$. So, $d = 7$ and hence $n \geq 9$. By (5), we have

$$s(G) = (n-7)(100 + 22(n-8)) = 2(n-7)(11n-38) > 22(n-7)(n-4).$$

Now, $n \geq 9$ implies that

$$s(G) > 22(n-7)(n-4) > 4(n-3)^2.$$

VI: $(a, b) = (3, 3)$. So, $d = 6$ and hence $n \geq 8$. By (5), we have

$$s(G) = (n-6)(72 + 12(n-7)) = 12(n-6)(n-1) > 12(n-6)(n-3).$$

Now, $n \geq 8$ implies that

$$s(G) > 12(n-6)(n-3) > 4(n-3)^2.$$

VII: $(a,b) = (3,4)$. So, $d = 7$ and hence $n \geq 9$. By (5), we have

$$s(G) = (n-7)(120 + 18(n-8)) = 6(n-7)(3n-4) > 18(n-7)(n-2).$$

Now, $n \geq 9$ implies that

$$s(G) > 18(n-7)(n-2) > 4(n-3)^2,$$

which completes the proof. ■

The next theorem deals with the Seidel energy of ${K_2}^{\uparrow n}$:

Theorem 2. *For every integer $n \geq 7$, we have $\mathcal{E}(\mathcal{S}({K_2}^{\uparrow n})) > 2n-1$.*

Proof. First, we calculate $\mathcal{E}(\mathcal{S}({K_2}^{\uparrow n}))$, where $n \geq 4$ is an integer. Note that $S({K_2}^{\uparrow n})$ has the following form:

$$\left(\begin{array}{cc|ccccc} 0 & -1 & 1 & 1 & \cdots & 1 & \\ -1 & 0 & 1 & 1 & \cdots & 1 & \\ \hline 1 & 1 & 0 & & & & \\ 1 & 1 & & 0 & & \text{\huge 1} & \\ \vdots & \vdots & & & \ddots & & \\ 1 & 1 & & \text{\huge 1} & & 0 & \\ & & & & & & 0 \end{array}\right).$$

Hence, $S({K_2}^{\uparrow n}) + I$ has $n-2$ equal rows which implies that $S({K_2}^{\uparrow n})$ has eigenvalue -1 of multiplicity $n-3$. Similarly, $S({K_2}^{\uparrow n}) - I$ has two equal rows implying that 1 is an eigenvalue of $S({K_2}^{\uparrow n})$ with multiplicity 1. Denote the other two eigenvalues of $S({K_2}^{\uparrow n})$ by λ and μ. Therefore:

$$\lambda + \mu - (n-3) + 1 = 0.$$

$$\lambda^2+\mu^2+\underbrace{(-1)^2+\cdots+(-1)^2}_{n-3}+1^2=n(n-1).$$

Solving the above two equations imply that

$$\lambda=\frac{n-4}{2}+\sqrt{\frac{n^2}{4}+n-3}\quad,\quad \mu=\frac{n-4}{2}-\sqrt{\frac{n^2}{4}+n-3}.$$

Putting all the above together and the assumption that $n\geq 4$, we obtain

$$\mathcal{E}(\mathcal{S}({K_2}^{\uparrow n}))=n-2+\sqrt{n^2+4n-12}. \tag{6}$$

Substituting $n=4,5,6$ in (6), we obtain that

$$\mathcal{E}(\mathcal{S}({K_2}^{\uparrow 4}))\approx 6.47213595<7,\quad \mathcal{E}(\mathcal{S}({K_2}^{\uparrow 5}))\approx 8.74456265<9$$

and

$$\mathcal{E}(\mathcal{S}({K_2}^{\uparrow 6}))\approx 10.9282032<11,$$

in addition, one can easily check that for $n\geq 7$

$$\mathcal{E}(\mathcal{S}({K_2}^{\uparrow n}))=n-2+\sqrt{n^2+4n-12}>2n-1,$$

which completes the proof. ■

Note that in the procedure of the proof of Theorem 2 of [1], for a graph G of order $n\geq 4$, the inequality

$$\mathcal{E}(\mathcal{S}(G))\geq n-4+\sqrt{n^2-2n+4+4\sqrt{\frac{3}{4}n^2+s(G)}}. \tag{7}$$

was obtained. With this inequality in hand and Theorem 1, we are ready to express the following theorem:

Theorem 3. *Let G be a graph of order $n\geq 6$ which is not SC-equivalent to K_n or ${K_2}^{\uparrow n}$. Then,*

$$\mathcal{E}(\mathcal{S}(G))>2n-1. \tag{8}$$

Proof. By Theorem 1, $s(G)\geq 4(n-3)^2$. Plugging this into the Inequality (7), we show that for $n\geq 38$, the resulting value of $\mathcal{E}(\mathcal{S}(G))$ is greater than $2n-1$. Hence, it suffices to prove that

$$n-4+\sqrt{n^2-2n+4+4\sqrt{\frac{3}{4}n^2+4(n-3)^2}}>2n-1 \tag{9}$$

holds for $n \geq 38$. Squaring twice the latter expressions and simplifying both sides, one can easily obtain that the inequality (9) is equivalent to

$$12n^2-464n+551>0.$$

Solving the above inequality implies that $n<\dfrac{116-\sqrt{11803}}{6}$ or $n>\dfrac{116+\sqrt{11803}}{6}$ which yields the validity of theorem for $n \geq 38$.

Now, by computer searches, using the software SageMath [6] and Mathematica [9], we obtain Table 1 which clearly shows that for the graphs of order n, where $6 \leq n \leq 37$, we have

$$\mathcal{E}(\mathcal{S}(G))>2n-1,$$

which completes the proof.

n	Minimum of $\mathcal{E}(\mathcal{S}(G))$	n	Minimum of $\mathcal{E}(\mathcal{S}(G))$
6	11.21110255	22	44.98248915
7	13.54400375	23	47.00933586
8	15.79795897	24	49.03410215
9	18.0	25	51.0570234
10	20.16552506	26	53.07830041
11	22.3041347	27	55.09810561
12	24.4222051	28	57.11658792
13	26.5241747	29	59.13387668
14	28.61324773	30	61.15008476
15	30.69180601	31	63.16531114
16	32.75760169	32	65.17964300
17	34.80517472	33	67.19315742
18	36.84786941	34	69.20592281
19	38.88641242	35	71.21800012
20	40.92139078	36	73.22944384
21	42.95328426	37	75.24030280

Table 1. The graphs G are among those that are not SC-equivalent to K_n and $K_2{}^{\uparrow n}$.

$\blacksquare$

We close the paper by combining Theorem 2 and 3 which yields our main result:

Theorem 4. *Let G be a graph of order $n \geq 7$ which is not SC-equivalent to K_n. Then,*

$$\mathcal{E}(\mathcal{S}(G)) > 2n - 1.$$

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