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On Sombor Index and Graph Energy

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Abstract

Sombor index is a recently introduced degree based graph topological index. For a graph G, it is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

where d_u denotes the degree of the vertex u in G. Within a short period of time after introduction of this index by Gutman, many aspects of it have been studied by many researchers. Relating the Sombor index $SO(G)$ of a graph G with the energy $\varepsilon(G)$ of G is one such instance among the others. In this article, we aim to provide some improved results relating $SO(G)$ and $\varepsilon(G)$.

1 Introduction

Let G be a simple, undirected graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G)$ are the vertex set and edge set of G respectively. We denote the edge between the vertices u and v by uv. Degree of a vertex u is the number of edges adjacent to the vertex u, denoted by d_u. We denote the maximum and minimum vertex degree of G by Δ and δ respectively.

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Let $A(G) = (a_{ij})_{n\times n}$ be the 0-1 adjacency matrix of the graph G, where $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$. Energy of G is defined as

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Finding lower and upper bounds of graph energy has been a hot topic of research in the recent years [10, 11, 17].

A topological index is a numerical quantity involving different graph parameters. Various topological indices which have some correlation with physico-chemical properties of the underlying graphs of different molecules are extensively studied by the researchers in the field of mathematical chemistry. Numerous graph degree based topological indices are found in the literature. In general, the bond incident degree (BID) graph invariants are given by the form

$$BID(G) = \sum_{uv\in E(G)} \Gamma(d_u, d_v)$$

where, Γ is a symmetric function of its arguments. First Zagreb index is one of the oldest topological index of this kind. It was introduced by Gutman and Trinajstic in 1972 [8] and was defined as

$$M_1(G) = \sum_{uv\in E(G)} (d_u + d_v) \,.$$

Very recently, as a manifestation of a geometric approach, Gutman introduced a new topological index called Sombor index [9] by taking $\Gamma(x, y) = \sqrt{x^2 + y^2}$. So the Sombor index of a graph G is defined as

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}.$$

After the introduction of Sombor index, within a short span of time, a lot of works on it have been done by many researchers [3–5,13–16]. Relating the Sombor index $SO(G)$ of a graph G with the energy $\varepsilon(G)$ of G is one such instance among the others. It has been studied by Ulker et al. in [15] and [16]. In this article, we have provided relationships between $SO(G)$ and $\varepsilon(G)$ which are improvements of the results presented in [15, 16].

2 Energy of a vertex and some useful results

In 2018, Arizmendi et al. [1] introduced the concept of energy of a vertex of a graph as the corresponding diagonal entry of the absolute value of the adjacency matrix. The trace of a matrix B is denoted by $Tr(B)$ and its absolute value $(BB^*)^{1/2}$, is denoted by $|B|$. Then the energy of G is

$$\varepsilon(G) = Tr(|A(G)|) = \sum_{i=1}^{n}(|A(G)|_{ii}).$$

Definition 1. Let G be a graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$. The energy of a vertex $u_i \in V(G)$ is denoted by $\varepsilon(u_i)$ and is defined by $\varepsilon(u_i) = |A(G)|_{ii}$, where $A(G)$ is the adjacency matrix of G and $|A(G)| = (A(G)A(G)^*)^{1/2}$.

Thus the energy of G can be viewed as the sum of the energies of its vertices, i.e.,

$$\varepsilon(G) = \varepsilon(u_1) + \varepsilon(u_2) + \cdots + \varepsilon(u_n).$$

Then following [6], one can have the energy of an edge $e = uv \in E(G)$ as,

$$\varepsilon(e) = \frac{\varepsilon(u)}{d_u} + \frac{\varepsilon(v)}{d_v}.$$

Then the energy of G can also be written as

$$\varepsilon(G) = \sum_{e\in E(G)} \varepsilon(e) = \sum_{e=uv\in E(G)} \left(\frac{\varepsilon(u)}{d_u} + \frac{\varepsilon(v)}{d_v}\right). \tag{1}$$

Following useful results on the bounds of vertex energy were obtained by Arizmendi et al. [1].

Theorem 1. *[1] Let G be a graph and $u \in V(G)$ be any vertex of G.*

Then

$$\varepsilon(u) \leq \sqrt{d_u},$$

with equality holds if and only if $G \cong K_{1,n}$ *and* u *is the central vertex of* $K_{1,n}$.

Theorem 2. *[1] Let* G *be a graph with at least one edge. Then for all* $u \in V(G)$, $\varepsilon(u) \geq \frac{d_u}{\Delta}$, *and hence* $\varepsilon(G) \geq \frac{2m}{\Delta}$ *with equality holds if and only if* $G \cong K_{\Delta,\Delta}$.

Several upper bounds of energy of a graph are found in the literature. McClleland bound [12] is one among the mostly cited bounds.

Theorem 3. *[12] Let* G *be a graph with* n *vertices and* m *edges. Then*

$$\varepsilon(G) \leq \sqrt{2mn}.$$

In [7], Gutman et al. obtained the following lower bounds for energy of regular graphs.

Theorem 4. *[7] Let* G *be a* Δ*-regular graph on* n *vertices. Then*

$$\varepsilon(G) \geq n. \tag{2}$$

Equality is attained if and only if every component of G *is isomorphic to the complete bipartite graph* $K_{\Delta,\Delta}$.

Corollary. *[7] If* G *is a triangle– and quadrangle–free* Δ*-regular graph on* n *vertices, then*

$$\varepsilon(G) \geq \frac{n\Delta}{\sqrt{2\Delta - 1}}. \tag{3}$$

An upper bound of Sombor index in terms of first Zagreb index is obtained by Das et al. [4] as follows.

Theorem 5. *[4] Let* G *be a graph with* m *edges and minimum degree* δ. *Then*

$$SO(G) \leq M_1(G) - (2 - \sqrt{2})\delta m,$$

where $M_1(G)$ *is the first Zagreb index of* G. *Moreover, the equality holds if and only if* G *is a regular graph.*

3 Upper bounds of Sombor index in terms of graph energy

Ulker et al. [15], obtained the following result.

Theorem 6. *[15] Let G be a graph with maximum degree Δ. If $\varepsilon(G)$ is its graph energy and $SO(G)$ is its Sombor index, then $SO(G) \leq \varepsilon(G)\Delta^3$.*

In this section, we present an improvement of the above result. For that purpose, we shall use the following lemma.

Lemma 1. *If $x_i \geq 0$ for $i = 1, 2, \ldots, n$ is a list of n non-negative real numbers, then*

$$\frac{1}{n}\left(\sum_{i=1}^{n}\sqrt{x_i}\right)^2 \leq \sum_{i=1}^{n} x_i, \tag{4}$$

with equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. Using Arithmetic Mean - Quadratic Mean inequality [2] on the positive square roots of $x_i, i = 1, 2, \ldots, n$, we have

$$\frac{\sum_{i=1}^{n}\sqrt{x_i}}{n} \leq \sqrt{\frac{\sum_{i=1}^{n} x_i}{n}}.$$

Squaring both sides, the lemma follows. ■

Theorem 7. *Let G be a graph with m edges and maximum degree Δ. If $\varepsilon(G)$ is its graph energy and $SO(G)$ is its Sombor index, then $SO(G) \leq \sqrt{\varepsilon(G) m \Delta^5}$.*

Proof. From (1), we have

$$\begin{aligned}
\varepsilon(G) = \sum_{e \in E(G)} \varepsilon(e) &= \sum_{e=uv \in E(G)} \left(\frac{\varepsilon(u)}{d_u} + \frac{\varepsilon(v)}{d_v}\right) \\
&\geq \sum_{uv \in E(G)} \left(\frac{\varepsilon(u)}{d_u^2} + \frac{\varepsilon(v)}{d_v^2}\right) \\
&= \sum_{uv \in E(G)} \frac{d_v^2\varepsilon(u) + d_u^2\varepsilon(v)}{d_u^2 d_v^2}
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{uv\in E(G)} \frac{d_v^2\frac{d_u}{\Delta}+d_u^2\frac{d_v}{\Delta}}{d_u^2d_v^2} \quad \text{(by Theorem 2)}\\
&\geq \frac{1}{m}\left(\sum_{uv\in E(G)} \sqrt{\frac{d_v^2\frac{d_u}{\Delta}+d_u^2\frac{d_v}{\Delta}}{d_u^2d_v^2}}\right)^2 \quad \text{(by Lemma 1)}\\
&\geq \frac{1}{m}\left(\sum_{uv\in E(G)} \frac{\sqrt{d_v^2\frac{d_u}{\Delta}+d_u^2\frac{d_v}{\Delta}}}{\Delta^2}\right)^2\\
&\geq \frac{1}{m}\left(\sum_{uv\in E(G)} \frac{\sqrt{d_v^2+d_u^2}}{\Delta^{\frac{5}{2}}}\right)^2\\
&= \frac{1}{m\Delta^5}\left(\sum_{uv\in E(G)} \sqrt{d_u^2+d_v^2}\right)^2\\
&= \frac{1}{m\Delta^5}\cdot SO(G)^2.
\end{aligned}
$$

Thus we get the inequality $SO(G) \leq \sqrt{\varepsilon(G)m\Delta^5}$. ■

Remark. It can be shown that the upper bound of $SO(G)$ given in the above theorem is better than that given by Theorem 6. Because, otherwise

$$
\begin{aligned}
\sqrt{\varepsilon(G)m\Delta^5} &\geq \varepsilon(G)\Delta^3\\
\Rightarrow \varepsilon(G)m\Delta^5 &\geq \varepsilon(G)^2\Delta^6\\
\Rightarrow m &\geq \varepsilon(G)\Delta\\
\Rightarrow \varepsilon(G) &\leq \frac{m}{\Delta}
\end{aligned}
$$

which contradicts Theorem 2.

Using Theorem 5, we can have another improvement of Theorem 6 as given below.

Theorem 8. *Let G be a graph with m edges, maximum and minimum degree Δ and δ respectively. If $\varepsilon(G)$ is its graph energy and $SO(G)$ is its Sombor index, then $SO(G) \leq \varepsilon(G)\Delta^3 - (2-\sqrt{2})\delta m$.*

Proof. From (1), we have

$$
\begin{aligned}
\varepsilon(G) = \sum_{e\in E(G)} \varepsilon(e) = & \sum_{e=uv\in E(G)} \left(\frac{\varepsilon(u)}{d_u} + \frac{\varepsilon(v)}{d_v}\right) \\
& \geq \sum_{uv\in E(G)} \left(\frac{\varepsilon(u)}{d_u^2} + \frac{\varepsilon(v)}{d_v^2}\right) \\
& = \sum_{e=uv\in E(G)} \frac{d_v^2\varepsilon(u) + d_u^2\varepsilon(v)}{d_u^2 d_v^2} \\
& \geq \sum_{e=uv\in E(G)} \frac{d_v^2\frac{d_u}{\Delta} + d_u^2\frac{d_v}{\Delta}}{d_u^2 d_v^2} \quad \text{(By Theorem 2)} \\
& \geq \sum_{e=uv\in E(G)} \frac{\frac{1}{\Delta}(d_v + d_u)}{d_u d_v} \\
& \geq \sum_{e=uv\in E(G)} \frac{(d_v + d_u)}{\Delta^3} \\
& = \frac{1}{\Delta^3} M_1 \\
& \geq \frac{1}{\Delta^3}\left\{SO(G) + (2-\sqrt{2})\delta m\right\} \quad \text{(By Theorem 5)} \\
\Rightarrow SO(G) \leq \Delta^3\varepsilon(G) - (2-\sqrt{2})\delta m. &
\end{aligned}
$$

■

Remark. Since $(2-\sqrt{2})\delta m$ is always a positive quantity, the bound given in the above theorem is always better than that given in Theorem 6.

For regular graph, Ulker et al. [15] obtained the following result.

Theorem 9. *[15] Let G be a Δ-regular graph. If $\varepsilon(G)$ is its graph energy and $SO(G)$ is its Sombor index, then $SO(G) \leq \varepsilon(G)\Delta^2$.*

This can also be improved as described in the following theorem.

Theorem 10. *Let G be a Δ-regular graph. If $\varepsilon(G)$ is its graph energy and $SO(G)$ is its Sombor index, then $SO(G) \leq \frac{\varepsilon(G)\Delta^2}{\sqrt{2}}$.*

Proof. Let G has n vertices and m edges. Since G is Δ-regular,

$$
SO(G) = \sum_{e=uv\in E(G)} \sqrt{d_u^2 + d_v^2} = \sum_{e=uv\in E(G)} \sqrt{2\Delta^2} = \sqrt{2}m\Delta = \frac{n\Delta^2}{\sqrt{2}}.
$$

Thus from Theorem 4 we have,

$$\varepsilon(G) \geq n = \frac{\sqrt{2}SO(G)}{\Delta^2} \Rightarrow SO(G) \leq \frac{\Delta^2 \varepsilon(G)}{\sqrt{2}}\cdot \qquad \blacksquare$$

Remark. Similar bound was obtained by Ulker et al. in [16] for connected graphs only. Here we have obtained it for regular graphs which are not necessarily connected. It is illustrated with the graph given in Figure 1.

Figure 1. 2-regular disconnected graph.

For this graph, $\Delta = 2, SO(G) = 12\sqrt{2}, \varepsilon(G) = 8$ and as such the inequaity $SO(G) \leq \frac{\varepsilon(G)\Delta^2}{\sqrt{2}}$ holds true.

Theorem 11. *Let G be a triangle– and quadrangle– free Δ-regular graph. If $\varepsilon(G)$ is its graph energy and $SO(G)$ is its Sombor index, then $SO(G) \leq \frac{\Delta\sqrt{2\Delta-1}}{\sqrt{2}}\varepsilon(G)$.*

Proof. As earlier,

$$SO(G) = \frac{n\Delta^2}{\sqrt{2}}\cdot$$

Thus from Corollary 2 we have,

$$\varepsilon(G) \geq \frac{n\Delta}{\sqrt{2\Delta - 1}} = \frac{\sqrt{2}SO(G)}{\Delta\sqrt{2\Delta - 1}} \Rightarrow SO(G) \leq \frac{\Delta\sqrt{2\Delta - 1}}{\sqrt{2}}\varepsilon(G)\cdot \qquad \blacksquare$$

4 Lower bound of Sombor index in terms of graph energy

Using Theorem 1, we find below a new bound of graph energy which is better than the McClleland bound.

Theorem 12. *Let G be a graph with n vertices, m edges and minimum degree δ. Then*

$$\varepsilon(G) \leq \sqrt{(n-1)(2m-\delta)} + \sqrt{\delta}. \tag{5}$$

Proof. Let $V = \{v_1., v_2, \ldots, v_n\}$ be the vertex set of G and d_i be the degree of the vertex v_i. Let us assume without loss of generality that $d_n = \delta$. Then, by Theorem 1, we have

$$\begin{aligned}
\varepsilon(G) = \sum_{v \in V(G)} \varepsilon(v) &\leq \sum_{i=1}^{n} \sqrt{d_i} \\
&= \sum_{i=1}^{n-1} \sqrt{d_i} + \sqrt{d_n} \\
&\leq \sqrt{(n-1) \sum_{i=1}^{n-1} d_i} + \sqrt{\delta} \text{ (by Lemma 1)} \\
&= \sqrt{(n-1)(2m-\delta)} + \sqrt{\delta}.
\end{aligned}$$

■

Remark. The above bound is better than the McClleland's bound. Because otherwise,

$$\begin{aligned}
\sqrt{(n-1)(2m-\delta)} + \sqrt{\delta} &> \sqrt{2mn} \\
\Rightarrow (n-1)(2m-\delta) + 2\sqrt{\delta(n-1)(2m-\delta)} + \delta &> 2mn \\
\Rightarrow 2\sqrt{\delta(n-1)(2m-\delta)} &> 2m + (n-2)\delta \\
\Rightarrow 4\delta(n-1)(2m-\delta) &> \Big(m + (n-2)\delta\Big)^2 \\
\Rightarrow 4mn\delta &> 4m^2 + n^2\delta^2 \\
\Rightarrow (2m - n\delta)^2 &< 0,
\end{aligned}$$

which is impossible.

In [9], Gutman has shown that the Sombor index among all connected graphs is minimum for the path graph P_n. In the following, we obtain a lower bound of the Sombor index of a connected graph in terms of the number of edges. To establish that, we define the following transformations on connected graphs and show that application of those transformations reduce the Sombor index of the graph.

Transformation A

In the graph G, let $u_1 - u_2 - \cdots - u_p, p \geq 2$ be an induced path and u_1v_1

be another pendant edge attached to the same vertex u_1 with $d_{u_1} \geq 3$. Let G' be the graph obtained from G by removing the edge u_1v_1 and by inserting the edge u_pv_1, as shown in Figure 2. Then G' is said to be the graph obtained from graph G by Transformation A.

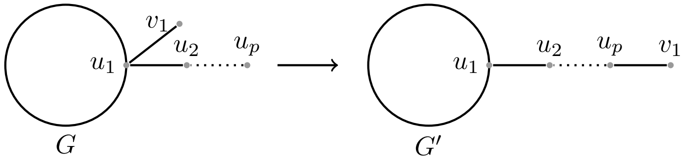

Figure 2. Transformation A.

Lemma 2. *Let G be a connected graph with $n \geq 3$ vertices at least two of which are pendant vertices and G' be the graph obtained from G by Transformation A. Then $SO(G) > SO(G')$.*

Proof. We consider two different cases as described below.

Case 1. ($p = 2$) In this case, both u_2 and v_1 are pendant vertices in G. In G', degree of u_2 becomes 2, degree of u_1 decreases by 1, and degrees of all other vertices remain same.

So,

$$\begin{aligned}
& SO(G) - SO(G') \\
\geq\ & \sum_{\substack{v\sim u_1 \\ v\neq u_2,v_1}} \sqrt{d_{u_1}^2 + d_v^2} + 2\sqrt{d_{u_1}^2 + 1} \\
& - \sum_{\substack{v\sim u_1 \\ v\neq u_2,v_1}} \sqrt{(d_{u_1}-1)^2 + d_v^2} - \sqrt{(d_{u_1}-1)^2 + 4} - \sqrt{5} \\
\geq\ & 2\sqrt{d_{u_1}^2 + 1} - \sqrt{d_{u_1}^2 + 4} - \sqrt{5}.
\end{aligned}$$

Let $f(x) = 2\sqrt{x^2+1} - \sqrt{x^2+4} - \sqrt{5}$.
Then, $f'(x) = \frac{2x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+4}} \geq 0$ for $x \geq 0$.
Hence, $f(x) \geq f(3) = 2\sqrt{10} - \sqrt{13} - \sqrt{5} > 0$ for $x \geq 3$.
Thus, $SO(G) - SO(G') \geq f(d_{u_1}) > 0$ since $d_{u_1} \geq 3$.
Therefore, $SO(G) - SO(G') > 0$, i.e., $SO(G) > SO(G')$.

Case 2. ($p \geq 3$) Here, u_p is a pendant vertex in G and it becomes a two degree vertex in G'. Thus in G', degree of u_p changes to 2, degree of u_1 decreases by 1, and degrees of all other vertices remain same. So,

$$\begin{aligned}
SO(G) - SO(G') \quad &\geq \quad \sum_{\substack{v \sim u_1 \\ v \neq u_2}} \sqrt{d_{u_1}^2 + d_v^2} + \sqrt{d_{u_1}^2 + 4} + \sqrt{5} + \sqrt{d_{u_1}^2 + 1} \\
&\quad - \sum_{\substack{v \sim u_1 \\ v \neq u_2}} \sqrt{(d_{u_1} - 1)^2 + d_v^2} - \sqrt{(d_{u_1} - 1)^2 + 4} \\
&\quad -2\sqrt{2} - \sqrt{5} \\
&\geq \quad \sqrt{d_{u_1}^2 + 1} - 2\sqrt{2}.
\end{aligned}$$

Let $g(x) = \sqrt{x^2 + 1} - 2\sqrt{2}$.
Then, $g'(x) = \frac{x}{\sqrt{x^2+1}} \geq 0$ for $x \geq 0$.
Hence, $g(x) \geq g(3) = \sqrt{10} - 2\sqrt{2} > 0$ for $x \geq 3$.
Thus, $SO(G) - SO(G') \geq g(d_{u_1}) > 0$ since $d_{u_1} \geq 3$.
Therefore, $SO(G) - SO(G') > 0$, i.e., $SO(G) > SO(G')$. ■

Transformation B

In a graph G, let $u_1 - u_2 - \cdots - u_p, p \geq 2$ and $w_1 - w_2 - \cdots - w_q, q \geq 2$ be two induced paths attached to the vertices u_1 and w_1 respectively, where $d_{w_1} \geq d_{u_1} \geq 3$. Also let G' be the graph obtained from G by removing the edge w_1w_2 and by inserting the edge u_pw_2, as shown in Figure 3. Then G' is said to be the graph obtained from graph G by Transformation B.

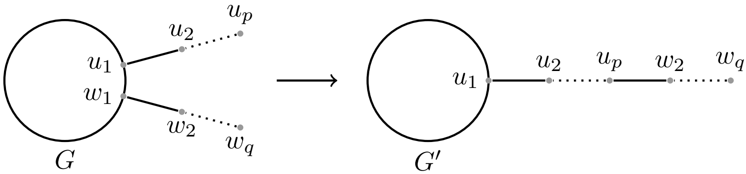

Figure 3. Transformation B.

Lemma 3. *Let G be a connected graph with $n \geq 3$ vertices at least two of which are pendant vertices and G' be the graph obtained from G by Transformation B. Then $SO(G) > SO(G')$.*

Proof. We consider different cases as described below.

Case 1. $(p = q = 2)$ In this case, both u_2 and w_2 are pendant vertices in G. In G', degree of u_2 becomes 2, degree of w_1 decreases by 1, and degrees of all other vertices remain same. So,

$$\begin{aligned} SO(G) - SO(G') &\geq \sum_{\substack{v\sim w_1 \\ v\neq w_2}} \sqrt{d_{w_1}^2 + d_v^2} + \sqrt{d_{w_1}^2 + 1} \\ &\quad + \sqrt{d_{u_1}^2 + 1} - \sum_{\substack{v\sim w_1 \\ v\neq w_2}} \sqrt{(d_{w_1} - 1)^2 + d_v^2} \\ &\quad - \sqrt{d_{u_1}^2 + 4} - \sqrt{5} \\ &\geq 2\sqrt{d_{u_1}^2 + 1} - \sqrt{d_{u_1}^2 + 4} - \sqrt{5}. \end{aligned}$$

Taking $f(x) = 2\sqrt{x^2 + 1} - \sqrt{x^2 + 4} - \sqrt{5}$ as before, we have $SO(G) > SO(G')$.

Case 2. $(p = 2, q \geq 3)$ In this case, u_2 is a pendant vertex, but w_2 is a two degree vertex in G. As in Case – 1, in G', degree of u_2 becomes 2, degree of w_1 decreases by 1, and degrees of all other vertices remain same. So,

$$\begin{aligned} SO(G) - SO(G') &\geq \sum_{\substack{v\sim w_1 \\ v\neq w_2}} \sqrt{d_{w_1}^2 + d_v^2} + \sqrt{d_{w_1}^2 + 4} + \sqrt{d_{u_1}^2 + 1} \\ &\quad - \sum_{\substack{v\sim w_1 \\ v\neq w_2}} \sqrt{(d_{w_1} - 1)^2 + d_v^2} - 2\sqrt{2} - \sqrt{d_{u_1}^2 + 4} \\ &\geq \sqrt{d_{u_1}^2 + 1} - 2\sqrt{2}. \end{aligned}$$

Taking $g(x) = \sqrt{x^2 + 1} - 2\sqrt{2}$ as earlier, we get $SO(G) > SO(G')$.

Case 3. $(p \geq 3, q \geq 3)$ Here, u_p is a pendant vertex in G and it becomes a two degree vertex in G'. Thus in G', degree of u_p changes to 2, degree of w_1 decreases by 1, and degrees of all other vertices remain same. So,

$$SO(G) - SO(G') \geq \sum_{\substack{v\sim w_1 \\ v\neq w_2}} \sqrt{d_{w_1}^2 + d_v^2} + \sqrt{d_{w_1}^2 + 4} + \sqrt{5}$$

$$- \sum_{\substack{v \sim w_1 \\ v \neq w_2}} \sqrt{(d_{w_1} - 1)^2 + d_v^2} - 2\sqrt{2} - \sqrt{5}$$

$$\geq \sqrt{d_{u_1}^2 + 1} - 2\sqrt{2}.$$

Thus, proceeding as before, we get $SO(G) > SO(G')$. ■

Lemma 4. *Let G be a connected graph with $n \geq 3$ vertices at least one of which is a pendant vertex and with m edges. Then*

$$SO(G) \geq 2\sqrt{5} + (m-2)2\sqrt{2}, \tag{6}$$

with equality holds if and only if $G \simeq P_n$.

Proof. If G does not contain a cycle, then it is a tree. In [9], it is shown that the Sombor index among the trees is minimum for the path graph P_n and so, $SO(G) \geq SO(P_n) = 2\sqrt{5} + (n-3)2\sqrt{2}$ and the lemma holds true since $m = n - 1$ for any tree.

If G contains a cycle and a single pendant edge, then there is a vertex u with degree $d_u \geq 3$ to which an induced path P is attached and degree of each non-pendant vertex is at least 2. If length of P is 1, then $SO(G) \geq \sqrt{10}+2\sqrt{13}+(m-3)2\sqrt{2} > 2\sqrt{5}+(m-2)2\sqrt{2}$. If length of P is greater than or equal to 2, then $SO(G) \geq \sqrt{5}+3\sqrt{13}+(m-4)2\sqrt{2} > 2\sqrt{5}+(m-2)2\sqrt{2}$.

If G contains a cycle and two or more pendant edges, then by repeated applications of Transformation A and/ or Transformation B, G can be reduced to a graph G' with an induced path attached to a vertex of degree 3 or more and at each step of application of either of the transformations, the Sombor index decreases (as shown in Lemma 2 and Lemma 3). Clearly, $SO(G) > SO(G') > 2\sqrt{5} + (m-2)2\sqrt{2}$ in this case also.

It is straightforward to see that the equality holds only for $G \simeq P_n$.

Hence the proof is complete. ■

Ulker et al. [16] found the following result.

Theorem 13. *[16] Let G be a connected graph with $n \geq 3$ vertices. Then*

$$SO(G) \geq \begin{cases} \frac{\delta}{\sqrt{2}}\left[\varepsilon(G)^2 - n(n-1)\Delta\right] & if\ \delta \geq 2, \\ \frac{\sqrt{5}}{2}\left[\varepsilon(G)^2 - n(n-1)\Delta\right] & if\ \delta = 1. \end{cases}$$

While proving the above theorem, Ulker et al. [16] have used the energy bound for the graph G with n vertices, m edges and maximum degree Δ as

$$\varepsilon(G) \leq \sqrt{2m + n(n-1)\Delta}. \tag{7}$$

But, it can be shown that for $n \geq 2$, the bound (7) is weaker than the McClleland's bound and hence weaker than the bound (5). Because, otherwise

$$\begin{aligned} 2mn &> 2m + n(n-1)\Delta \\ \Rightarrow 2m(n-1) &> n(n-1)\Delta \\ \Rightarrow \frac{2m}{n} &> \Delta. \end{aligned}$$

Since the average degree can not exceed the maximum degree, it is a contradiction.

In the following, we present a modification of the above theorem.

Theorem 14. *Let G be a connected graph with $n \geq 3$ vertices and minimum degree δ. Then*

$$SO(G) \geq \begin{cases} \frac{\delta}{\sqrt{2}}\left[\frac{\left(\varepsilon(G)-\sqrt{\delta}\right)^2}{n-1} + \delta\right] & if\ \delta \geq 2, \\ \frac{\sqrt{2}(\varepsilon(G)-1)^2}{n-1} + 2\sqrt{5} - 3\sqrt{2} & if\ \delta = 1. \end{cases}$$

Proof. Since $\delta \leq d_u$ for every vertex u, from the definition of $SO(G)$ it follows that

$$\begin{aligned} \sqrt{2}\delta m &\leq SO(G) \\ \Rightarrow 2m &\leq \frac{\sqrt{2}SO(G)}{\delta}. \end{aligned}$$

Substituting the value of $2m$ in the inequality (5), we get,

$$
\begin{aligned}
\left(\varepsilon(G)-\sqrt{\delta}\right)^2 &\leq (n-1)(2m-\delta)\\
\Rightarrow \left(\varepsilon(G)-\sqrt{\delta}\right)^2 &\leq (n-1)\left(\frac{\sqrt{2}SO(G)}{\delta}-\delta\right)\\
\Rightarrow SO(G) &\geq \frac{\delta}{\sqrt{2}}\left\{\frac{\left(\varepsilon(G)-\sqrt{\delta}\right)^2}{n-1}+\delta\right\}.
\end{aligned}
$$

For $\delta = 1$, substituting the value of $2m$ from inequalty (6) into the inequality (5), we get,

$$
\begin{aligned}
(\varepsilon(G)-1)^2 &\leq (n-1)(2m-1)\\
\Rightarrow (\varepsilon(G)-1)^2 &\leq (n-1)\left(\frac{SO(G)-2\sqrt{5}+3\sqrt{2}}{\sqrt{2}}\right)\\
\Rightarrow SO(G) &\geq \frac{\sqrt{2}\,(\varepsilon(G)-1)^2}{n-1}+2\sqrt{5}-3\sqrt{2}.
\end{aligned}
$$

Hence the theorem follows. ■

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