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# On Sombor Index of Graphs with a Given Number of Cut-Vertices

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**Abstract**

Introduced by Gutman in 2021, the Sombor index is a novel graph-theoretic topological descriptor possessing potential applications in the modeling of thermodynamic properties of compounds. Let $\mathbb{H}_n^k$ be the family of graphs on order $n$ and $k$ number of cut-vertices having at least one cycle. In this paper, we present minimum Sombor indices of graphs in $\mathbb{H}_n^k$. The corresponding extremal graphs have been characterized as well.

# 1 Introduction

We consider finite, connected and simple graphs only.

In modern chemistry, it is a cornerstone idea that the structure (molecular) of a chemical compound comprises information on the physicochemical characteristics of the respective compound. Topological indices are numerical quantities which provide efficient tools to retrieve this structural information of compounds. They have diverse applications in materials

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science, pharmacology, chemistry. among other, [12,29]. Recently in 2021, Gutman [9] introduced a novel degree-based topological descriptors called the Sombor index. For a graph $\Gamma$, it is defined as follows:

$$SO(\Gamma) = \sum_{xy \in E(\Gamma)} \sqrt{\deg_\Gamma(x)^2 + \deg_\Gamma(y)^2},$$

where $\deg_\Gamma(x)$ (resp. $\deg_\Gamma(y)$) is the degree of the vertex $x$ (resp. $y$) in $\Gamma$.

Within a short span of time, the Sombor index has attracted a significant amount of attention among researchers from mathematics and theoretical chemistry. Redžepović [25] investigated predictive potential of the Sombor index for statistical modeling of enthalpy of vaporization and entropy for alkanes. Successful predictive potential of the Sombor index has also been employed for simulating the thermodynamic characteristics of organic compounds. The reader is referred to [1–11,13,16–24,26,27,30], for a detailed mathematical treatment of the Sombor index of graphs.

A graph $\Gamma$ is an ordered pair $\Gamma = (V, E)$, where $V$ is the set of point called vertices and $E \subseteq \binom{V}{2}$ is the set of lines called edges. The cardinality $n =\mid V \mid$ (resp. $\varepsilon =\mid E \mid$) of $\Gamma$ is called the order (resp. size) of $\Gamma$. Two vertices $y, z \in V(\Gamma)$ are said to be adjacent or neighbors if $yz \in E(\Gamma)$. The set $N_\Gamma(x) = \{y \in V(\Gamma) \mid xy \in E(\Gamma)\}$ or $N(x)$ for short, is called the neighborhood of the vertex $x$ in $\Gamma$. The number $\deg_\Gamma(x) =\mid N_\Gamma(x) \mid$ is said to be the degree of $x$. For an edge $xy \in E(\Gamma)$, the graph $\Gamma - xy$ is obtained by deleting $xy \in E(\Gamma)$ from $\Gamma$. On other hand, the subgraph $\Gamma - x$ for $x \in V(\Gamma)$ is obtained by deleting $x$ and its incident edges from $\Gamma$. We denote the $n$-vertex path by $P_n$. For further understanding of graphs and its notation see [31].

Investigating mathematical behavior of topological indices is a contemporary research topic these days. Determining extremal values of Sombor indices of graphs with a given property has been started recently. For instance, Sun & Du [28] studied extremal values of the Sombor index of graphs with given domination number. Zhou et al. [33, 34] investigated the extremal Sombor index of trees & unicyclic graphs with given matching number and maximum degree. On the other hand, studying extremal topological indices of graphs with given number of cut-vertices has also been studied frequently. For instance, Hua & Zhang [14] studied extremal Merrifield-Simmons index of graphs with given number of cut-vertices. Xu et al. [32] studied minimum Kirchhoff index of graphs with a given number of cut vertices. Ji & Wang recently studied minimum multiplicative Zagreb indices of graphs with given number of cut-vertices. In this paper, we study the minimum Sombor index of non-tree graphs with a given number of cut-vertices. Corresponding extremal graphs have also been characterized.

First, we introduced some auxiliary results which will be used in our characterization.

## 2 Auxiliary results

Regarding the Sombor index of a graph, we start with an elementary lemma below.

**Lemma 1.** *Let* $\Gamma$ *be a graph. If* $xy \in E(\Gamma)$*, then* $SO(\Gamma - xy) < SO(\Gamma)$*.*

For the Sombor index, the following lemma presents an important property of maximal 2-connected blocks.

**Lemma 2.** *Let* $B$ *be a maximal 2-connected block of a graph* $\Gamma \in \mathbb{H}_n^k$*, where* $\Gamma$ *has the smallest Sombor index. If* $| B | \geq 3$*, then* $B$ *is a cycle.*

*Proof.* The case of $| B | = 3$ follows immediately that $B \cong C_t$ for some $t \leq n$. Next, we assume that $| B | \geq 4$. On contrary, we suppose that the block $B$, comprising no cut-vertices, is not isomorphic to $B \cong C_t$ for some $t \leq n$. Since, the block $B$ is 2-connected, so by deleting an edge, say $xy$, the graph $\Gamma$ still contains $k$-cut vertices. Also, the assumption that $B$ is not isomorphic to $C_t$ for some $t \leq n$ implies that the deletion of of an edge $xy$ results in $\Gamma$ still containing at least one cycle. This implies that $\Gamma - xy \in \mathbb{H}_n^k$. By Lemma 1, we have $SO(\Gamma - xy) < SO(\Gamma)$, which arises a contradiction that $\Gamma$ has the smallest Sombor index. This shows the lemma. ■

Next, we present four crucial auxiliary operations playing a key role in our characterization.

Here we first explain the $\alpha$-operation on a graph $\Gamma \in \mathbb{H}_n^k$. Let $\Gamma \in \mathbb{H}_n^k$ be a graph with $y, y_1, x_1, x_2 \in \Gamma$ such that $\deg_\Gamma(y) \geq 3$, $\deg_\Gamma(y_1) = 1$, and $x_1x_2$ is an edge lying on a cycle of $\Gamma$. See Figure 1. Let $\Gamma_\alpha = \Gamma - \{x_1x_2, y_1y\} + \{x_1y_1, x_2y_1\}$. Then, we say that $\Gamma_\alpha$ is the $\alpha$-switched graph obtained from $\Gamma$ by $\alpha$-operation.

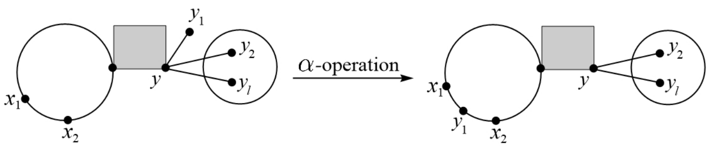


**Figure 1.** The $\alpha$-operation used in Lemma 3.

Based on the $\alpha$-operation, we obtain the following lemma.

**Lemma 3.** *Let $\Gamma_\alpha$ be the $\alpha$-switched graph of a graph $\Gamma \in \mathbb{H}_n^k$ described in Figure 1. Then, $SO(\Gamma_\alpha) < SO(\Gamma)$.*

*Proof.* Following the structure of $\Gamma$ in Figure 1, assume $y, x_1, x_2 \in \Gamma$ such that $\deg_\Gamma(y) \geq 3$ and $x_1x_2$ is an edge lying on a cycle of $\Gamma$, where $\deg_\Gamma(x_1), \deg_\Gamma(x_2) \geq 2$. Furthermore, let $\Gamma$ comprises at least one pendent vertex $y_1$. Let $N_\Gamma(y) = \{y_1, y_2, \ldots, y_l\}$ be the neighborhood set for $y$ with $l \geq 3$ (see Figure 1).

Let $C$ be an arbitrary cycle of $\Gamma$. If $y \notin C$, then the $\alpha$-switched graph $\Gamma_\alpha$ is constructed by removing the edges $yy_1, x_1x_2$ and adding $x_1y_1, x_2y_1$. By taking this into account, the following relation between Sombor indices of $\Gamma$ and $\Gamma_\alpha$ emerges.

$$
\begin{aligned}
& SO(\Gamma) - SO(\Gamma_\alpha) = \\
& \sum_{j=1}^{l} \sqrt{\deg_\Gamma(y)^2 + \deg_\Gamma(y_j)^2} + \sqrt{\deg_\Gamma(x_1)^2 + \deg_\Gamma(x_2)^2} - \\
& \sum_{j=2}^{l} \sqrt{\deg_{\Gamma_\alpha}(y)^2 + \deg_{\Gamma_\alpha}(y_j)^2} - \sqrt{\deg_{\Gamma_\alpha}(x_1)^2 + \deg_{\Gamma_\alpha}(y_1)^2} - \\
& \sqrt{\deg_{\Gamma_\alpha}(x_2)^2 + \deg_{\Gamma_\alpha}(y_1)^2}, \\
> \quad & \sqrt{\deg_\Gamma(y)^2 + \deg_\Gamma(y_1)^2} + \sqrt{\deg_\Gamma(x_1)^2 + \deg_\Gamma(x_2)^2} - \\
& \sqrt{\deg_{\Gamma_\alpha}(x_1)^2 + \deg_{\Gamma_\alpha}(y_1)^2} - \sqrt{\deg_{\Gamma_\alpha}(x_2)^2 + \deg_{\Gamma_\alpha}(y_1)^2}
\end{aligned}
$$

$$\geq \quad \sqrt{3^2+1^2}+\sqrt{2^2+2^2}-\sqrt{2^2+2^2}-\sqrt{2^2+2^2}.$$

This implies that $SO(\Gamma) - SO(\Gamma_\alpha) = \sqrt{10} - \sqrt{8} > 0$.

Thus, we are left with the remaining case, where $y \in C$ for some cycle $C$ of $\Gamma$. In that case, if the vertex $y_1$ is the unique pendent vertex of $\Gamma$, then the case is settled. Next, we assume $\Gamma$ contains another pendent vertex, say $z_1$, and then $\Gamma_\alpha = \Gamma - yy_1 + y_1z_1$. Thus, the conclusion is verified as well. Repeating this process would need similar arguments, thus, they have been omitted.

Therefore, the proof is finished. ■

Next, we provide the $\beta$-operation on a graph $\Gamma \in \mathbb{H}_n^k$. Let $\Gamma \in \mathbb{H}_n^k$ be a graph with $y, z_1, z_2 \in \Gamma$ such that $\deg_\Gamma(y) \geq 3$ and $z_1z_2$ is an edge lying on $C$, where $C$ is a cycle of $\Gamma$. See Figure 2. Let $\Gamma_\beta = \Gamma - \{yy_2, x_{21}y_2, z_1z_2\} + \{y_2z_1, y_2z_2, x_{21}x_{lt_l}\}$. Then, we say that $\Gamma_\beta$ is the $\beta$-switched graph obtained from $\Gamma$ by $\beta$-operation.

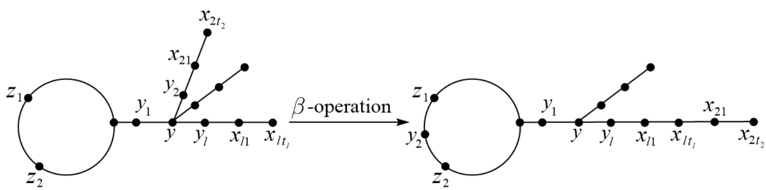


**Figure 2.** The $\alpha$-operation used in Lemma 4.

The following lemma compares the Sombor indices of $\Gamma$ and $\Gamma_\beta$.

**Lemma 4.** *Let $\Gamma_\beta$ be the $\beta$-switched graph of a graph $\Gamma \in \mathbb{H}_n^k$ described in Figure 2. Then*

$$SO(\Gamma_\beta) < SO(\Gamma).$$

*Proof.* Following the structure of $\Gamma$ in Figure 2, assume $y, z_1, z_2 \in \Gamma$ such that $\deg_\Gamma(y) \geq 3$ and $z_1z_2$ is an edge lying on $C$, where $C$ is an arbitrary cycle of $\Gamma$. Let $N_\Gamma(y) = \{y_1, y_2, \ldots, y_l\}$ be the neighborhood set for $y$ with $l \geq 3$ (see Figure 1). Moreover, if $\Gamma$ comprises at least one pendent vertex with $y$, then the $\beta$-operation can be reduced to $\alpha$-operation, and Lemma 3 is applicable. Thus, we assume that $y$ is associated with pendent paths only with length at least 2. Give that, if $y$ is associated to a unique pendent path, then the case is settled.

Next, assume the case in which $y$ is associated with at least two pendent paths, say, $P_2(y_2x_{21}\dots x_{2t_2})$ and $P_l(y_l\dots x_{l1}x_{lt_l})$, with $t_2, t_l \geq 1$. Let $\Gamma_\beta = \Gamma - \{yy_2, x_{21}y_2, z_1z_2\} + \{y_2z_1, y_2z_2, x_{21}x_{lt_l}\}$. By taking this into account, the following relation between Sombor indices of $\Gamma$ and $\Gamma_\beta$ emerges.

$$\begin{aligned}
& SO(\Gamma) - SO(\Gamma_\beta) = \\
& \sum_{j=1}^{l} \sqrt{\deg_\Gamma(y)^2 + \deg_\Gamma(y_j)^2} + \sqrt{\deg_\Gamma(z_1)^2 + \deg_\Gamma(z_2)^2} + \\
& \sqrt{\deg_\Gamma(x_{21})^2 + \deg_\Gamma(y_2)^2} + \sqrt{\deg_\Gamma(x_{lt_{l-1}})^2 + \deg_\Gamma(x_{lt_l})^2} - \\
& \sum_{j=1, j\neq 2}^{l} \sqrt{\deg_{\Gamma_\beta}(y)^2 + \deg_{\Gamma_\beta}(y_j)^2} - \sqrt{\deg_{\Gamma_\beta}(y_2)^2 + \deg_{\Gamma_\beta}(z_1)^2} - \\
& \sqrt{\deg_{\Gamma_\beta}(y_2)^2 + \deg_{\Gamma_\beta}(z_2)^2} - \sqrt{\deg_{\Gamma_\beta}(x_{lt_{l-1}})^2 + \deg_{\Gamma_\beta}(x_{lt_l})^2} - \\
& \sqrt{\deg_{\Gamma_\beta}(x_{21})^2 + \deg_{\Gamma_\beta}(x_{lt_l})^2} \\
> \; & \sqrt{\deg_\Gamma(y)^2 + \deg_\Gamma(y_2)^2} + \sqrt{8} + \sqrt{8} + \sqrt{5} - \sqrt{8} - \sqrt{8} - \\
& \sqrt{8} - \sqrt{8} \geq \sqrt{13} + \sqrt{5} - 2\sqrt{8} > 0.
\end{aligned}$$

This completes the proof. ■

Next, we provide the third operation which we call the $\gamma$-operation on a graph $\Gamma \in \mathbb{H}_n^k$. Following the structure of $\Gamma$ in Figure 3, let $\Gamma_0$ be a graph with $\mid \Gamma_0 \mid \geq 2$ and $x, z \in \Gamma_0$. Assume that $\Gamma_1$ is the graph comprising $C$, where $C$ is a cycle. Next, from $\Gamma_0$, we construct an $n$-vertex graph $\Gamma \in \mathbb{H}_n^k$ with $n \geq 6$ by identifying a vertex $c \in C$ (resp. $c' \in C'$) with $x$ (resp. $z$).

We construct $\Gamma_\gamma = \Gamma - \{y_1z, y_0y_2, x_1x_2\} + \{x_1y_0, x_2y_1\}$ and we say that $\Gamma_\gamma$ is the $\gamma$-switched graph obtained from $\Gamma$ by applying the $\gamma$-operation. The following lemma compares the Sombor indices of $\Gamma \in \mathbb{H}_n^k$ and $\Gamma_\gamma$.

**Lemma 5.** *Let $\Gamma_\gamma$ be the $\gamma$-switched graph of a graph $\Gamma \in \mathbb{H}_n^k$ described in Figure 3. Then $SO(\Gamma_\gamma) < SO(\Gamma)$.*

*Proof.* Following the structure of $\Gamma$ in Figure 3, assume that $x, z \in \Gamma$ (resp. $C, C'$) be two cut-vertices (resp. cycles) in $\Gamma$ such that $C'$ is an endblock.

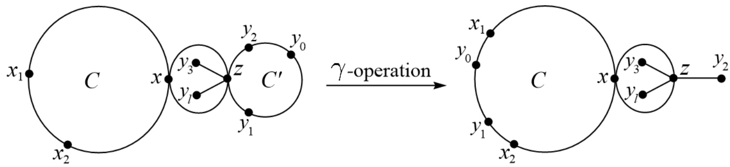


**Figure 3.** The $\gamma$-operation used in Lemma 5.

Let $x_1x_2 \in E(C)$ and $y_1z, y_0y_2, y_2z \in E(C')$ be the edges of $C$ and $C'$ such that $\deg_\Gamma(x_1), \deg_\Gamma(x_2) \geq 2$. Considering $\Gamma_\gamma = \Gamma - \{y_1z, y_0y_2, x_1x_2\} + \{x_1y_0, x_2y_1\}$, we obtain the following relation between Sombor indices of $\Gamma$ and $\Gamma_\gamma$:

$$\begin{aligned}
& SO(\Gamma) - SO(\Gamma_\gamma) = \\
& \sum_{j=1}^{l} \sqrt{\deg_\Gamma(z)^2 + \deg_\Gamma(y_j)^2} + \sqrt{\deg_\Gamma(y_2)^2 + \deg_\Gamma(y_0)^2} + \\
& \sqrt{\deg_\Gamma(x_1)^2 + \deg_\Gamma(x_2)^2} - \sum_{j=2}^{l} \sqrt{\deg_{\Gamma_\gamma}(z)^2 + \deg_{\Gamma_\gamma}(y_j)^2} - \\
& \sqrt{\deg_{\Gamma_\gamma}(x_1)^2 + \deg_{\Gamma_\gamma}(y_0)^2} - \sqrt{\deg_{\Gamma_\gamma}(x_2)^2 + \deg_{\Gamma_\gamma}(y_1)^2}, \\
> \; & \sqrt{\deg_\Gamma(z)^2 + \deg_\Gamma(y_1)^2} + \sqrt{8} + \sqrt{8} - \sqrt{8} - \sqrt{8}, \\
\geq \; & \sqrt{13} > 0.
\end{aligned}$$

This shows the lemma. ∎

Finally, we provide the fourth operation called the $\delta$-operation on a graph $\Gamma \in \mathbb{H}_n^k$. Following the structure of $\Gamma$ in Figure 4, let $\Gamma_0$ be a graph with $\mid \Gamma_0 \mid \geq 2$ and $y \in \Gamma_0$ is a vertex of $\Gamma_0$. Assume that $\Gamma_1$ is the graph comprising $C$, where $C$ is a cycle. Next, from $\Gamma_0$, we construct an $n$-vertex graph $\Gamma \in \mathbb{H}_n^k$ by attaching a vertex $c \in C$ and $c' \in C'$ with $y$. This implies that $C'$ is an endblock of $\Gamma$. Considering $\Gamma_\delta = \Gamma - \{yy_2, y_0y_1\} + y_0y_2$, we say that $\Gamma_\delta$ is constructed from $\Gamma$ by applying the $\delta$-operation.

The following lemma compares the Sombor indices of $\Gamma \in \mathbb{H}_n^k$ and $\Gamma_\delta$.

**Lemma 6.** *Let $\Gamma_\delta$ be the $\delta$-switched graph of a graph $\Gamma \in \mathbb{H}_n^k$ as described in Figure 4. Then $SO(\Gamma_\delta) < SO(\Gamma)$.*

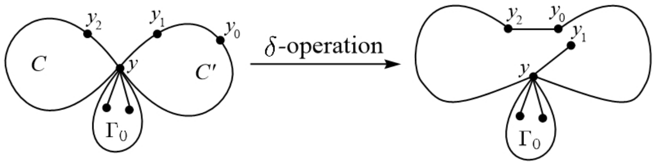


**Figure 4.** The $\delta$-operation used in Lemma 6.

*Proof.* Following the structure of $\Gamma$ in Figure 4, assume that $C, C'$ (resp. $y \in \Gamma$) be two cycles (resp. common vertex of $C, C'$) in $\Gamma$ such that $\deg_\Gamma(y) \geq 4$ and $C'$ is an endblock. Assume $N_\Gamma(y) = \{y_1, y_2, \ldots y_t\}$ with $t \geq 4$ be the set denoting neighbors of $y \in \Gamma$.

Considering $\Gamma_\delta$ to be constructed from $\Gamma$ by removing edges $yy_2, y_0y_1$, and joining $y_2$ to $y_0$, we obtain the following relation between Sombor indices of $\Gamma$ and $\Gamma_\delta$:

$$\begin{aligned}
& SO(\Gamma) - SO(\Gamma_\delta) = \\
& \sum_{j=1}^{t} \sqrt{\deg_\Gamma(y)^2 + \deg_\Gamma(y_j)^2} + \sqrt{\deg_\Gamma(y_0)^2 + \deg_\Gamma(y_1)^2} - \\
& \sum_{j=3}^{t} \sqrt{\deg_{\Gamma_\delta}(y)^2 + \deg_{\Gamma_\delta}(y_j)^2} - \sqrt{\deg_{\Gamma_\delta}(y)^2 + \deg_{\Gamma_\delta}(y_1)^2} - \\
& \sqrt{\deg_{\Gamma_\delta}(y_0)^2 + \deg_{\Gamma_\delta}(y_2)^2}, \\
= \quad & \sqrt{\deg_\Gamma(y)^2 + \deg_\Gamma(y_1)^2} + \sqrt{\deg_\Gamma(y)^2 + \deg_\Gamma(y_2)^2} + \\
& \sqrt{8} - \sqrt{\deg_{\Gamma_\delta}(y)^2 + \deg_{\Gamma_\delta}(y_1)^2} - \sqrt{8}, \\
\geq \quad & \sqrt{4^2 + 2^2} + \sqrt{4^2 + 2^2} - \sqrt{3^2 + 1^2} > 0.
\end{aligned}$$

This verifies the conclusion. ■

Next, we consider $\Omega$ to be a graph satisfying $\mid E(\Omega) \mid - \mid V(\Omega) \mid \geq 0$ and two vertices $x, y \in \Omega$ lying on a cycle of $\Omega$. By identifying a path $P_r$ (resp. $P_s$) to $x$ (resp. $y$) in $\Omega$, we construct the graph $\Omega(r, s)$.

**Lemma 7.** *For* $x, y \in \Omega(r, s)$ *such that* $\deg_{\Omega(r,s)}(x), \deg_{\Omega(r,s)}(y) \geq 3$, *we* *have*

$$SO(\Omega(r,s)) \geq SO(\Omega(1,r+s-1)).$$

*Proof.* We have that $\deg_{\Omega(r,s)}(x), \deg_{\Omega(r,s)}(y) \geq 3$, as $x, y \in \Omega$ lie on a cycle of $\Omega$. One may assume, without loss of generality, that $\deg_{\Omega(r,s)}(x) \geq \deg_{\Omega(r,s)}(y)$. Let $xx_1x_2 \dots x_{r-1}$ (resp. $yy_1y_2 \dots y_{s-1}$) be the vertices of the path $P_r$ (resp. $P_s$). Assume that $N_{\Omega(r,s)}(x) = \{x_1, z_1, z_2, \dots, z_t\}$, such that $t \geq 2$. Then, we construct the graph $\Omega(1, r+s-1)$ from $\Omega(r,s)$ by removing $xx_1 \in E\big(\Omega(r,s)\big)$ and adjoining $x_1$ with $y_{s-1}$. For the sake of simplicity, let $\Omega_0 = \Omega(r,s)$ and $\Omega_0' = \Omega(1, r+s-1)$. Then, we deduce the following relation between the Sombor indices of $\Omega_0$ and $\Omega_0'$.

$$\begin{aligned}
& SO(\Omega_0) - SO(\Omega_0') = \\
& \sum_{j=1}^{t} \sqrt{\deg_{\Omega_0}(x)^2 + \deg_{\Omega_0}(z_j)^2} + \sqrt{\deg_{\Omega_0}(x_0)^2 + \deg_{\Omega_0}(x_1)^2} + \\
& \sqrt{\deg_{\Omega_0}(y_{s-2})^2 + \deg_{\Omega_0}(y_{s-1})^2} - \sum_{j=1}^{t} \sqrt{\deg_{\Omega_0'}(x)^2 + \deg_{\Omega_0'}(z_j)^2} - \\
& \sqrt{\deg_{\Omega_0'}(y_{s-2})^2 + \deg_{\Omega_0'}(y_{s-1})^2} - \sqrt{\deg_{\Omega_0'}(y_{s-1})^2 + \deg_{\Omega_0'}(x_1)^2}, \\
\geq \quad & \sqrt{3^2+2^2} + \sqrt{3^2+2^2} + \sqrt{3^2+2^2} + \sqrt{2^2+1^2} - \\
& \sqrt{3^2+2^2} - \sqrt{3^2+2^2} - \sqrt{2^2+2^2} - \sqrt{2^2+2^2}, \\
= \quad & \sqrt{13} + \sqrt{5} - 2\sqrt{8} > 0.
\end{aligned}$$

Thus, this verifies the proof. ■

If we identify the two vertices $x, y$ in $\Omega(r,s)$, then using a similar way as we did in Lemma 7, we obtain a graph $\Omega(2, r+s-2)$ implying that $SO(\Omega(r,s)) \geq SO(\Omega(r',s'))$ such that $r' = 2, s' = r+s-2$. Therefore, this implies that $x_1$ is a pendant vertex and $P_{r'} = xx_1$. In that case, we employ Lemma 3, which suggests that there exists $\Omega'$ with $\mid \Omega' \mid = \mid \Omega \mid +1$ which is constructed from $\Omega$ by applying subdivision on edge $z_1z_2$ lying on some cycle and denoting this vertex as $x_1$ such that $SO(\Omega(r,s)) \geq SO(\Omega'(1, r+s-2))$ is verified. This suggests the following corollary:

**Corollary.** *The identification of two vertices $x, y$ in $\Omega(r,s)$ suggests the* *existence of a graph $\Omega'$ on $\mid \Omega \mid +1$ vertices for which $SO\big(\Omega(r,s)\big) \geq SO\big(\Omega'(1, r+s-2)\big)$ is satisfied.*

# 3 Main result

This section provides a sharp lower bound on the Sombor index of graphs in $\mathbb{H}_n^k$. The corresponding graphs have been characterized as well. For $n, k \geq 2$, let $C_{n-k}$ be a cycle of order $n-k$ and $x \in C_{n-k}$ be a vertex of $C_{n-k}$. Then, the family $C_{n,k}$ is obtained by adjoining a path $P_{k+2}$ of order $k+2$ with the vertex $x$ of $C_{n-k}$. This makes $C_{n-k}$ a graph of $\mathbb{H}_n^k$ on order $n$ and $k$ cut-vertices. See Figure 5 for the graph $C_{n,k}$.

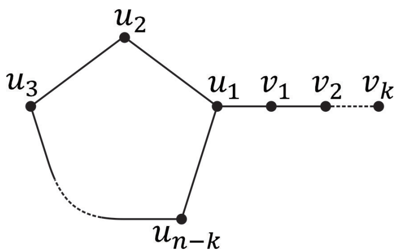


**Figure 5.** The graph $C_{n,k}$.

Next, we show the main result of this paper.

**Theorem 1.** *Let $\Gamma$ be a graph in $\mathbb{H}_n^k$. Then*

$$SO(\Gamma) \geq 2(n-4)\sqrt{2} + 3\sqrt{13} + \sqrt{5},$$

*where equality holds if and only if $\Gamma \cong C_{n,k}$.*

*Proof.* Let $\Gamma \in \mathbb{H}_n^k$ be a graph attaining the minimal Sombor index. Assume $S$ is a set of $k$ cut-vertices in $\Gamma$. Let $B_1, \ldots, B_r$ be the blocks of $\Gamma$ corresponding to $k$ cut-vertices. Notice that, for some $\ell$, either $\mid B_\ell \mid = 2$ or $\mid B_\ell \mid \geq 3$. First, we prove the following claim.

**Claim 1.** There exists exactly one pendent tree $P_t$ in $\Gamma$. Moreover, $P_t$ is a path.

*Proof of Claim 1.* Among graphs in $\mathbb{H}_n^k$, since $\Gamma$ is assumed to attain minimum value of $SO(\Gamma)$. We claim that $\Gamma$ comprises one or more pendent trees, as otherwise, we would have constructed $\Gamma'$ from $\Gamma$ by Lemmas 5 & 6 for which $SO(\Gamma') < SO(\Gamma)$ holds which then arises a contradiction to the choice of $\Gamma$. Furthermore, each of the pendant tree has to be a path, as otherwise, Lemma 4 ensures construction of $\Gamma''$ from $\Gamma$ which $SO(\Gamma'') < SO(\Gamma)$, which arises a contradiction again. Thus, we obtain that there exist at least one pendent tree in $\Gamma$ and, in fact, every pendent tree is a path.

Next, we show that $\Gamma$ comprises precisely one pendent tree i.e. a path. One contrary, assume that there exists at least two pendent trees in $\Gamma$. Then, Lemma 7 and Corollary 2 ensures existence of graph $\Gamma_1$ such that $SO(\Gamma_1) < SO(\Gamma)$ is satisfied. This contradict with the selection of $\Gamma$.

It is important to notice that the cardinality $\mid B \mid$ of $B$ is not effected under operations of Lemmas 4, 5, 6 & 7. This shows the claim. □

By employing Lemma 2, we obtain that $\Gamma$ is a block graph with $K_2$ or cycles as its blocks. Moreover, Claim 1 implies that we obtain that $\Gamma$ contains $P_t$ as its unique pendent path. If $\Gamma$ comprises only one block which is a cycle, then $\Gamma \cong C_{n,k}$ and the case is settled. Thus, we assume that $\Gamma$ contains two or more cycles.

Next, we show that, except for $K_2$ of $P_t$, all endblocks are cycles. As otherwise, $\Gamma$ would contain at least two pendent trees which contradicts Claim 1. Next, we divide our discussion into two possible cases.

**Case 1**. There exists precisely two endblocks in $\Gamma$.
By using our earlier claim, one of the endblocks is a cycle $C_1$ and the other is $K_2$. By assumption, $\Gamma$ comprises another cycle, say, $C_2$. Given that, Lemmas 5 & 6 ensure existence of a graph $\Gamma'$ such that $SO(\Gamma') < SO(\Gamma)$ holds. We arrive at a contradiction to the choice of $\Gamma$.

**Case 2**. The graph $\Gamma$ contains at least two endblocks.
Earlier discussion implies that $\Gamma$ comprises at least two endclocks which are cycles. Once again, by Lemmas 5 & 6, there exists a graph $\Gamma''$ satisfying $SO(\Gamma'') < SO(\Gamma)$, which is a contradiction.

Both Cases 1 & 2 arise contradiction to the minimality of $\Gamma$. Thus, $\Gamma$ contains exactly one cycle $C_r$. Since $\Gamma$ lies in $\mathbb{H}_n^k$, we conclude that $C_r \cong C_{n-k}$ and $P_t \cong P_{k+2}$. This implies that $\Gamma \cong C_{n,k}$. By some routine calculations, we arrive at $SO(C_{n,k}) = 2(n-4)\sqrt{2} + 3\sqrt{13} + \sqrt{5}$. This completes the proof and shows the theorem. ■

# 4 Concluding remarks

This paper studies the Sombor index of non-tree $n$-vertex graphs having $k$ cut-vertices i.e. $\mathbb{H}_n^k$. We find a sharp lower bound on the Sombor index of graphs in $\mathbb{H}_n^k$. Moreover, all graphs achieving the lower bound have been characterized.

**Problem 1.** Find maximum graphs with respect to the Sombor index in $\mathbb{H}_n^k$.

Let $\mathbb{V}_n^k$ (resp. $\mathbb{E}_n^k$) be the set of all $n$-vertex graphs with vertex connectivity (resp. edge connectivity) at most $k$. We propose the following open problems on the Sombor index of graphs.

**Problem 2.** Find extremal graphs with respect to the Sombor index among the families in $\mathbb{V}_n^k$ and $\mathbb{E}_n^k$.

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