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A Relation on Trees and the Topological Indices Based on Subgraph

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Abstract

A topological index reflects the physical, chemical and structural properties of a molecule, and its study has an important role in molecular topology, chemical graph theory and mathematical chemistry. It is a natural problem to characterize non-isomorphic graphs with the same topological index value. By introducing a relation on trees with respect to edge division vectors, denoted by $\langle \mathcal{T}_n, \preceq \rangle$, in this paper we give some results for the relation order in $\langle \mathcal{T}_n, \preceq \rangle$. It allows us to compare the size of the topological index value without relying on the specific forms of them, and naturally we can determine which trees have the same topological index value. Based on these results we characterize some classes of trees that are uniquely determined by their edge division vectors. Moreover we construct infinite classes of non-isomorphic trees with the same topological index value, particularly such trees of order no more than 10 are completely determined.

1 Introduction

A topological index is usually defined by some graph invariants, such as the number of vertices, the number of edges, vertex degree, degree sequence,

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matching number, etc [19]. In the fields of bioinformatics, molecular topology, chemical graph theory and mathematical chemistry, a topological index is a type of molecular descriptors that are calculated based on the molecular graph of a chemical compound.

In the field of chemistry, it is common to calculate topological indices of the molecular graph in order to figure out the physical or chemical properties of a molecule statistically. The inverse problem of topological indices is proposed by X. Li et al. in [13] as follows: given an index value, one wants to design chemical compounds (given as graphs or trees) having that index value, it is not necessary to obtain all the isomorphic graphs. For more results on this topic, readers may refer to [12]. As the research developed, some researchers tried to use some kind of topological index or a few topological indices for classification based on isomorphism [3–5]. The main problem of classification based on isomorphism is that the topological indices may be identical even for two or several non-isomorphic graphs and the situation becomes worse with the increment of vertices of graph [6]. There are some similar inverse problems such as finding and constructing of cospectral graphs [17] and equienergetic graphs [1, 2].

In [8], X. Guo and M. Randić characterized some classes of trees with the same JJ index. Recently, D. Vukičević and J. Sedlar in [20] introduced a relation order on trees with respect to edge division vector. They also gave the relationship between edge division vector and the topological index on the class of trees, which enables us to simply calculate the topological indices of trees by their edge division vectors. In [18], the authors gave a new criterion to determine the order of trees with respect to the edge division vector. Based on these results a large of extremal trees are determined including old and new, one can refer to Table 3 in [18].

In this paper, we focus on considering the problem of characterizing the non-isomorphic trees with the same topological index value without depending on the individual form of topological index. In Section 2, we introduce some notions and symbols and give two lemmas to determine a kind of relation order on trees by using edge division vector. In Section 3, we give a graph transformation to construct the trees that have the same edge division vector. Moreover we find some sufficient conditions to determine wether such trees are isomorphic or not. Based on these conditions, we can simply produce infinite families of non-isomorphic trees that have the same edge division vector. In Section 4, we give several classes of trees which are uniquely determined by edge division vectors. In Section 5, we characterize some classes of non-isomorphic trees with the same edge division vector. In particular, all the pairs of non-isomorphic trees of order no more than 10 with the same edge division vector are classified. Based on above results, in Section 6, without relying on the specific form of individual topological index we give some classes of non-isomorphic trees with the same topological index value.

2 Preliminaries

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of neighbors of v. The cardinality of $N_G(v)$ is called the degree of vertex v and is denoted by $d_G(v)$. The distance $d_G(u, v)$ between vertices u and v is the length of shortest path connecting them. Usually we will write only $N(v)$, $d(v)$ and $d(u, v)$ when it does not lead to confusion.

Let T be the tree of order n. A vertex v is a pendent vertex or leaf if $d_T(v) = 1$ and a branching vertex if $d_T(v) \geq 3$. The tree $T - v$ is defined by removing the vertex v and deleting all edges incident to v from T, and the tree $T - e$ is defined by removing the edge e of T. Let S_n and P_n denote the star and path with order n, respectively. For a tree T of order n, let $e = uv \in E(T)$. By $T_u(e)$ and $T_v(e)$ we will denote the two components of $T - e$ containing u and v, respectively. We denote $n_u(e) = |T_u(e)|$ and $n_v(e) = |T_v(e)|$. Furthermore, we define an edge function $\mu_T(e) = \min\{n_u(e), n_v(e)\}$ or simply $\mu(e)$. By definition we have $n_u(e) + n_v(e) = n$ and $\mu(e) \leq \lfloor \frac{n}{2} \rfloor$.

Let $\mathcal{T}_n$ denote the set of trees on n vertices. For a tree $T \in \mathcal{T}_n$, let $r_i(T)$ denote the number of edges in T for which $\mu(e) = i$, i.e., $r_i(T) = |\{e \in E(T) \mid \mu(e) = i\}|$. It is clear that $r_1(T)$ is just the number of pendent edges and $r_i(T) = 0$ for every $i > \lfloor \frac{n}{2} \rfloor$ due to $\mu(e) \leq \lfloor \frac{n}{2} \rfloor$. The edge division vector $\mathbf{r}(T)$ is defined as a vector $\mathbf{r}(T) = (r_1(T), r_2(T), \ldots, r_{\lfloor \frac{n}{2} \rfloor}(T))$. We will write only $\mathbf{r}$ and r_i when it does not lead to confusion. Recently, D. Vukičević and J. Sedlar in [20] defined the order of edge division vectors: two edge division vectors $\mathbf{r}$ and $\mathbf{r}'$ of trees T and T' in $\mathcal{T}_n$, respectively, have a relation, denoted by $\mathbf{r} \preceq \mathbf{r}'$, if the inequality $\sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} r_i \leq \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} r_i'$ holds for every $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. If the inequality is strict for at least one k, then we say that $\mathbf{r} \prec \mathbf{r}'$. Naturally, we define $T \preceq T'$ if $\mathbf{r} \preceq \mathbf{r}'$ ($T \prec T'$ if $\mathbf{r} \prec \mathbf{r}'$). Specially, we denote by $T \approx T'$ if $\mathbf{r} = \mathbf{r}'$. Thus the trees of $\mathcal{T}_n$ is a set defined with the relation $\preceq$, which is denoted by $\langle \mathcal{T}_n, \preceq \rangle$. Two trees T, T' are called EDV-equivalent trees if $T \approx T'$. However, $T \approx T'$ does not imply $T \cong T'$ (to see Figure 6 for example). A tree T is said to be determined by edge division vector (DEDV for short) if, for any $T' \in \mathcal{T}_n$, we have $T' \cong T$ whenever $T \approx T'$.

First we give two lemmas to determine the order on $\langle \mathcal{T}_n, \preceq \rangle$, from which we will characterize the EDV-equivalent trees and DEDV-trees.

For $T, T' \in \mathcal{T}_n$, let $\varphi : E(T) \longrightarrow E(T')$ be a bijection. T and T' are said to be (φ, μ)-similar with respect to $e_1 \in E(T)$ if $\mu_T(e) = \mu_{T'}(\varphi(e))$ for any $e \neq e_1$. We start with a lemma which is given in [18].

Lemma 2.1 ([18]). *Suppose that $T, T' \in \mathcal{T}_n$ are (φ, μ)-similar with respect to e_1, and $\varphi(e_1) = e_1'$. We have*
(1) If $\mu_T(e_1) < \mu_{T'}(e_1')$, then $T \prec T'$;
(2) If $\mu_T(e_1) > \mu_{T'}(e_1')$, then $T \succ T'$;
(3) If $\mu_T(e_1) = \mu_{T'}(e_1')$, then $T \approx T'$.

For $T \in \mathcal{T}_n$, let uv and xu be two edges of T. Denote by $T_x(ux)$ and $T_u(ux)$ the components in $T - ux$ containing vertex x and u, respectively. We denote $\hat{T} = T_u(ux)$. Let $T' = T - ux + xv$ be the tree obtained from T by moving the component $T_x(ux)$ from u to v (see Figure 1), such tree T' we call a branch-moving of $T_x(ux)$ from T.

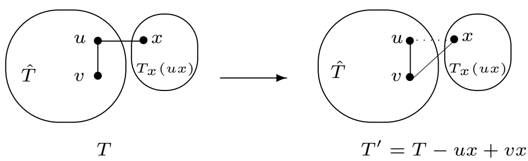


Figure 1. The branch-moving transformation

Lemma 2.2. *For $T \in \mathcal{T}_n$, let uv and xu be edges of T. Let $T' = T - ux + xv$. We have*
(1) If $n_u(uv) \leq n_v(uv)$ then $T \succ T'$;
(2) If $n_u(uv) > n_v(uv)$ then

$$\begin{cases} T \prec T' & \textit{whenever } n_u(uv) - n_v(uv) > |T_x(ux)|, \\ T \succ T' & \textit{whenever } n_u(uv) - n_v(uv) < |T_x(ux)|, \\ T \approx T' & \textit{whenever } n_u(uv) - n_v(uv) = |T_x(ux)|. \end{cases}$$

Proof. First we define bijection $\varphi : E(T) \longrightarrow E(T')$ such that

$$\varphi(e) = \begin{cases} e & \text{if } e \neq ux \text{ is an edge of } T, \\ vx & \text{if } e = ux. \end{cases}$$

It is clear that $\mu_T(e) = \mu_{T'}(\varphi(e))$ if $e \neq uv$. Therefore, T and T' are (φ, μ)-similar with respect to $e_1 = uv$.

If $n_u(uv) \leq n_v(uv)$, then $\mu_T(e_1) = \min\{n_u(uv), n_v(uv)\} = n_u(uv)$. We have

$$\mu_T(e_1) = n_u(uv) > n_u(uv) - |T_x(ux)| = n'_u(uv) = \mu_{T'}(uv) = \mu_{T'}(e_1).$$

Therefore, we have $T \succ T'$ by Lemma 2.1 (2), and (1) holds.

If $n_u(uv) > n_v(uv)$, then $\mu_T(e_1) = \min\{n_u(uv), n_v(uv)\} = n_v(uv)$. Notice that $\mu_{T'}(e_1) = \min\{n'_u(uv), n'_v(uv)\}$ and

$$\begin{cases} n'_u(uv) = n_u(uv) - |T_x(ux)|, \\ n'_v(uv) = n_v(uv) + |T_x(ux)|. \end{cases}$$

It is easy to verify that

$$\begin{cases} \mu_T(e_1) < \mu_{T'}(e_1) & \text{if } n_u(uv) - n_v(uv) > |T_x(ux)|, \\ \mu_T(e_1) > \mu_{T'}(e_1) & \text{if } n_u(uv) - n_v(uv) < |T_x(ux)|, \\ \mu_T(e_1) = \mu_{T'}(e_1) & \text{if } n_u(uv) - n_v(uv) = |T_x(ux)|. \end{cases}$$

It follows (2) by Lemma 2.1.

We complete this proof. ∎

Lemma 2.2 indicates the changes for the order $\preceq$ in $\langle \mathcal{T}_n, \preceq \rangle$ when we apply branch-moving along an edge, which can be used to find the maximum or minimum tree under the meaning of order $\preceq$.

3 Branch-exchange for trees

In this section we will introduce a graph transformation, called the branch-exchange, which will be used to construct the pairs of EDV-equivalent trees.

For a tree $T \in \mathcal{T}_n$, let u and v be two vertices of T and $P_{uv} = uu_2 \cdots u_{k-1}v$ be the path connecting them. Now let

$$T_u(P_{uv}) = \{T_x(ux) \mid x \in N_T(u) \setminus u_2\},$$

which is called the u-branch of T (with respect to P_{uv}). Each $T_x(ux) \in T_u(P_{uv})$ is a subtree in $T - u$ that has $x \in N_T(u) \setminus u_2$ as its root vertex, and v-branch $T_v(P_{uv}) = \{T_y(vy) \mid y \in N_T(v) \setminus u_{k-1}\}$ is similarly defined. We say that two subsets $S(u) = \{x_1, \ldots, x_s\} \subseteq N_T(u) \setminus u_2$ and $S(v) = \{y_1, \ldots, y_t\} \subseteq N_T(v) \setminus u_{k-1}$ are *balanced* if

$$\sum_{x_i \in S(u)} |T_{x_i}(ux_i)| = \sum_{y_j \in S(v)} |T_{y_j}(vy_j)|.$$

Further, we call $T_{S(u)} = \{T_{x_i}(ux_i) \mid x_i \in S(u)\}$ and $T_{S(v)} = \{T_{y_j}(vy_j) \mid y_j \in S(v)\}$ the *balanced components* with respect to u and v. By deleting the balanced components from T, we get

$$T^* = T - (\sum_{x_i \in S(u)} ux_i + T_{S(u)}) - (\sum_{y_j \in S(v)} vy_j + T_{S(v)}).$$

In addition, we have $T^*_u(P_{uv}) = T_u(P_{uv}) - T_{S(u)}$ and $T^*_v(P_{uv}) = T_v(P_{uv}) - T_{S(v)}$ (see Figure 2).

Definition 3.1. Let T' be the tree obtained from T by exchanging the balanced components $T_{S(u)}$ and $T_{S(v)}$ (see Figure 2), that is

$$\begin{cases} T = T^* + (\sum_{x_i\in S(u)} ux_i + T_{S(u)}) + (\sum_{y_j\in S(v)} vy_j + T_{S(v)}) \\ T' = T^* + (\sum_{y_j\in S(v)} uy_j + T_{S(v)}) + (\sum_{x_i\in S(v)} vx_i + T_{S(u)}). \end{cases} \tag{1}$$

We call T' the *branch-exchange* of T with $T_{S(u)}$ and $T_{S(v)}$.

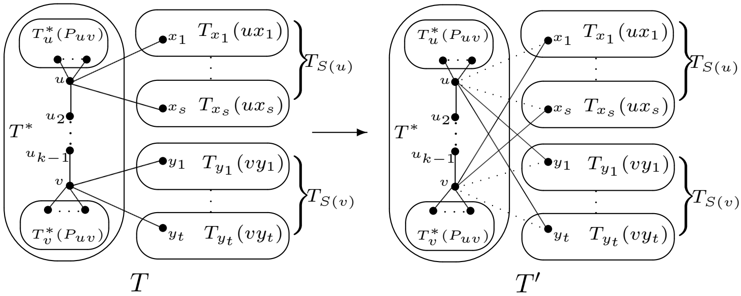


Figure 2. The branch-exchange transformation

For a tree F with vertex x, let F_x denote the tree with specified root vertex x. For two trees F and H, F_x and H_y are said to be *strongly isomorphic*, denoted by $F_x \simeq H_y$, if there exists an isomorphism φ from F to H such that $\varphi(x) = y$. It is clear that $F_x \simeq H_y$ implies $F_x \cong H_y$ but not vice versa. Let $F^f = \cup_{i=1}^{s} F_{x_i}$ and $H^f = \cup_{i=1}^{t} H_{y_i}$, where $F_{x_1}, \ldots, F_{x_s}$ (respectively, $H_{y_1}, \ldots, H_{y_t}$) are vertex disjoint trees. We say that $F^f \simeq H^f$ if $s = t$ and there exists a permutation ϕ on $\{1, 2, \ldots, s\}$ such that $F_{x_i} \simeq H_{y_{\phi(i)}}$ for $i = 1, \ldots, s$.

Let α be an automorphism of T, i.e., $\alpha \in Aut(T)$. The image of P_{uv} under α is also a path, say $\alpha(P_{uv}) = u'u_2' \cdots u_{k-1}'v' = P_{u'v'}$ where $\alpha(u_i) = u_i'$ for $i = 2, \ldots, k-1$ and $\alpha(u) = u'$, $\alpha(v) = v'$. By $T_u(P_{uv}) \simeq T_{\alpha(u)}(\alpha(P_{uv})) = T_{u'}(P_{u'v'})$ we mean that, for each $T_x(xu) \in T_u(P_{uv})$, there is some $T_z(zu') \in T_{u'}(P_{u'v'})$ such that $T_x(xu) \simeq T_z(zu')$ (i.e., $T_x(xu)$ and $T_z(zu')$ are strongly isomorphic). We call u and v *similar* if there exists $\alpha \in Aut(T)$ such that α contains the transposition $(u\ v)$ (i.e., $\alpha(u) = v$ and $\alpha(v) = u$). Obviously, if u and v are similar then $T_u(P_{uv}) \simeq T_v(P_{uv})$.

By using the above notions and symbols we can prove the following result.

Lemma 3.1. *Let T and T' be the trees described in (1) and shown in Figure 2, where $T_{S(u)}$ and $T_{S(v)}$ are balanced components. Then $T \approx T'$. Moreover, if $T \cong T'$, then either $T_{S(u)} \simeq T_{S(v)}$ or u and v are similar in T^*.*

Proof. We define bijection $\varphi : E(T) \longrightarrow E(T')$ such that

$$\varphi(e) = \begin{cases} e & \text{if } e \neq ux_i, vy_j \text{ for } i = 1, \ldots, s,\ j = 1, \ldots, t, \\ vx_i & \text{if } e = ux_i \text{ for } i = 1, \ldots, s, \\ uy_j & \text{if } e = vy_j \text{ for } j = 1, \ldots, t. \end{cases}$$

It is routine to verify that $\mu_T(e) = \mu_{T'}(\varphi(e))$ for any $e \in E(T)$ since $S(u)$ and $S(v)$ are balanced. It follows $T \approx T'$ by Lemma 2.1(3).

Let $B_u = \{T_{\alpha(u)}(\alpha(P_{uv})) \mid \alpha \in Aut(T)\}$ be the set of strongly isomorphic copies of u-branch that consists of an orbit of $Aut(T)$, and similarly we define $B_v = \{T_{\alpha(v)}(\alpha(P_{uv})) \mid \alpha \in Aut(T)\}$. Note that the u-branch $T_u(P_{uv}) \in B_u$, we have $b_u = |B_u| \geq 1$, and all the copies of the u-branch in B_u are vertex-disjoint due to T is a tree. Similarly we have $b_v = |B_v| \geq 1$.

Since $T \cong T'$, T' also contains b_u copies of u-branch and b_v copies of v-branch. From Figure 2 and the representation of T in (1), we see that besides of $T_u(P_{uv})$ the other $b_u - 1$ number of u-branches are included in T^*. Similarly besides of $T_v(P_{uv})$ the other $b_v - 1$ number of v-branches are also included in T^*. Therefore, T^* contains exactly $(b_u - 1)$'s u-branches in B_u and $(b_v - 1)$'s v-branches in B_v. On the other hand, from Figure 2 we see that $T'_u(P_{uv}) = T^*_u(P_{uv}) \cup T_{S(v)}$ and $T'_v(P_{uv}) = T^*_v(P_{uv}) \cup T_{S(u)}$ are only two branches of T' not included in T^*. It implies that they must be one u-branch and one v-branch since otherwise T' will contain at most $b_u - 1$ numbers of u-branches or $b_v - 1$ numbers of v-branches. Hence $\{T_u(P_{uv}), T_v(P_{uv})\} = \{T'_u(P_{uv}), T'_v(P_{uv})\}$. If B_u and B_v are distinct orbits then $T^*_u(P_{uv}) \cup T_{S(u)} = T_u(P_{uv}) \simeq T'_u(P_{uv}) = T^*_u(P_{uv}) \cup T_{S(v)}$, and so $T_{S(u)} \simeq T_{S(v)}$. If B_u and B_v are identified then $T^*_u(P_{uv}) \cup T_{S(u)} = T_u(P_{uv}) \simeq T'_v(P_{uv}) = T^*_v(P_{uv}) \cup T_{S(u)}$, and so $T^*_u(P_{uv}) \simeq T^*_v(P_{uv})$. Thus there exists $\alpha \in Aut(T)$ such that $\alpha(u) = v$. It implies that T^* has an automorphism α^* containing transposition $(u\ v)$, i.e., u and v are similar in T^*.

We complete the proof. ■

It immediately follows the following three results from Lemma 3.1.

Theorem 3.1. *Under the assumption of Lemma 3.1. If* $T_{S(u)} \not\simeq T_{S(v)}$ *and* u, v *are not similar in* T^**, then* $T \approx T'$ *but* $T \ncong T'$*.*

If $T_u^*(P_{uv}) \not\simeq T_v^*(P_{uv})$ then u and v are not similar in T^*. From Theorem 3.1 we have the following results.

Corollary 3.1. *Under the assumption of Lemma 3.1. If* $T_{S(u)} \not\simeq T_{S(v)}$ *and* $T_u^*(P_{uv}) \not\simeq T_v^*(P_{uv})$*, then* $T \approx T'$ *but* $T \ncong T'$*.*

Corollary 3.2. *Under the assumption of Lemma 3.1. If* $T_{S(u)} \not\simeq T_{S(v)}$ *and* $Aut(T^*)$ *consists of the identity alone, then* $T \approx T'$ *but* $T \ncong T'$*.*

Example 3.1. Let T and T' be the trees as shown in Figure 3. Note that $T_x \not\simeq T_y$ and $T_u^*(P_{uv}) = P_3 \not\simeq P_1 = T_v^*(P_{uv})$. By Corollary 3.1, we have $T \approx T'$ but $T \ncong T'$, where $\mathbf{r}(T) = \mathbf{r}(T') = (7, 1, 3, 0, 1, 0, 1)$.

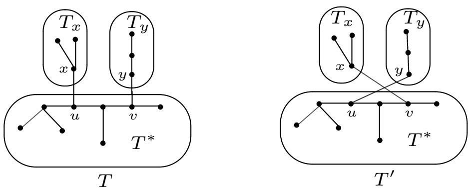


Figure 3. T and T'

Remark 3.1. As Example 3.1, by using Corollary 3.1 or Corollary 3.2, one can simply construct infinite pairs of EDV-equivalent trees.

4 The DEDV trees

In this section we completely determine the DEDV-starlike trees, and also characterize some other DEDV-trees.

Recall that a tree $T \in \mathcal{T}_n$ is a DEDV-tree if $T' \cong T$ whenever $T' \approx T$. Denote by $\mathcal{B}(T)$ the set of the trees that are constructed from T by branch-exchange transformation. More precisely, $T' \in \mathcal{B}(T)$ if and only if there exist $T = T_1$, T_2,...,$T_t = T'$ such that T_{i+1} is a branch-exchange of T_i for $i = 1, \ldots, t-1$. By Lemma 3.1, we have $T' \approx T$. According to definition we have the following result.

Lemma 4.1. *If $T \in \mathcal{T}_n$ is a DEDV-tree, then $T' \cong T$ whenever $T' \in \mathcal{B}(T)$, i.e., $\mathcal{B}(T) = \{T\}$.*

For a tree $T \in \mathcal{T}_n$, let $e = uv \in E(T)$. Recall that $T_u(e)$ and $T_v(e)$ are respectively the two components of $T - e$ containing root vertices u and v, and $\mu_T(e) = \min\{|T_u(e)|, |T_v(e)|\}$. For a vertex $u \in V(T)$, there exists at most one edge e incident to it such that $|T_u(e)| \leq \lfloor \frac{n}{2} \rfloor$. We call $T_u(e)$ a pendent subtree of T (with respect to root vertex u) if $\mu_T(e) = |T_u(e)| \leq \lfloor \frac{n}{2} \rfloor$. We will write only T_u when it does not lead to confusion. In particular, if T_u is really a path and $d_T(u) = 2$, it is called a pendent path; if it is a star with u as its center vertex, it is called a pendent star. A pendent subtree T_u is maximal if u is suspended from a branching vertex (or there is no any pendent subtree of order $|T_u| + 1$ including T_u). Let $\mathbf{r}(T) = (r_1, r_2, \ldots, r_{\lfloor \frac{n}{2} \rfloor})$. Note that $r_i = |\{e \in E(T) \mid \mu_T(e) = i\}|$. Thus r_i is just the number of pendent subtrees of T with order $|T_u| = i$. Particularly, if T_u is a pendent subtree with $|T_u| = 1$ then u is a leaf of T, in this case T_u contributes one to r_1; if $|T_u| = 2$ then subtree T_u is pendent path P_2, which contributes one to r_2. Naturally we have

Claim 4.1. *For $T, T' \in \mathcal{T}_n$, let $\mathbf{r}(T) = (r_1, r_2, \ldots, r_{\lfloor \frac{n}{2} \rfloor}) = \mathbf{r}(T')$. We have*
(a) T and T' have exactly r_1 pendent vertices;
(b) T and T' have exactly r_2 pendent P_2;
(c) T and T' have the same number of maximal pendent paths, the length of which is equal for the path with the smallest length;
(d) $r_1 + r_2 + \cdots + r_{\lfloor \frac{n}{2} \rfloor} = n - 1 = |E(T)| = |E(T')|$.

Since the pendent subtree of order three is either P_3 or S_3, r_3 is the total number of the pendent subtrees P_3 and S_3. However, if $r_1 = r_2 = r_3 = k$ then T must contain exactly k pendent paths P_3, since otherwise if T has a pendent star S_3 then T has at least $k + 1$ pendent vertices which contradicts $r_1 = k$.

Let $T_{l_1,l_2,\ldots,l_k}$ be a starlike tree with branching vertex u such that

$$T_{l_1,l_2,\ldots,l_k} - u = \cup_{i=1}^{k} P_{l_i}, \tag{2}$$

where P_{l_i} is the path with l_i vertices. By $T_{n;k}$ we denote the starlike tree with $T_{n;k} - u = k * P_s$(i.e., k copies of P_s), where $sk + 1 = n$.

Lemma 4.2. *Let $\mathbf{r}(T) = (r_1, r_2, \ldots, r_{\lfloor \frac{n}{2} \rfloor})$ be the edge division vector of $T \in \mathcal{T}_n$. Suppose that $r_1 = \cdots = r_s = k$. Then T has exactly k pendent paths P_s. Moreover*
(a) If $r_i = 0$ for $i > s$, then $T = T_{n;k}$.
(b) If $r_{s+1} = t < k$ and $r_i = 0$ for $i > s + 1$, then T is a starlike tree with center vertex u such that $T - u = (k - t) * P_s \cup t * P_{s+1}$, where $n - 1 = sk + t$.

Proof. By the arguments as $s = 3$ as above, we first claim that T has exactly k pendent P_s since otherwise T has leaves more than k.

(a) Since $r_i = 0$ for $i > s$, we have $r_1 + \cdots + r_s = n - 1$ by Claim 4.1(d). It implies that the endpoints of these k pendent P_s join at a center vertex u, that is $T = T_{n;k}$.

(b) Now T contains exactly k pendent P_s each of them contributes one leaf to T. Also note that $r_{s+1} = t$, we see that T has t pendent subtrees of order $s+1$. Let T_v be such a pendent subtree with respect to root v, where $|T_v| = s+1$. If $d_{T_v}(v) \geq 2$, let x_1 and x_2 be two adjacent vertices of v in T_v, then $F = T_v$ has two subtree F_{x_1} and F_{x_2}, where F_{x_i} is the component in $F - x_i v$ containing x_i for $i = 1, 2$. Clearly $|F_{x_i}| < s$. On the other aspect, since F_{x_i} is also a subtree of T, we have $F_{x_i} = P_{l_i}$ for $l_i \leq s$. Note that $s + 1 = |F| \geq l_1 + l_2 + 1$, we have $l_i < s$. Thus F_{x_i} must be a subtree P_s, however F does not contain any P_s, a contradiction. Therefore, $d_{T_v}(v) = 1$. Thus v has a unique adjacent vertex y in T_v such that $T_v - vy$ has a pendent subtree of order s that will be P_s with root vertex y, i.e., $T_v - vy = P_s \cup \{v\}$. It follows that $T_v = P_{s+1}$. Since $n - 1 = sk + t = (k - t)s + t(s+1)$, $r_{s+1} = t$ and $r_i = 0$ for $i > s + 1$, we have $(r_1 + r_2 + \cdots + r_{s+1}) = (k - t)s + t(s + 1)$. It means that $T - u = (k - t) * P_s \cup t * P_{s+1}$. ■

A starlike tree $T_{l_1,l_2,\ldots,l_k}$ is called balanced if $|l_i - l_j| \leq 1$ for $1 \leq i, j \leq k$. Clearly, the starlike trees described in (a) and (b) of Lemma 4.2

are balanced. Thus Lemma 4.2 implies that edge division vector uniquely determine balanced starlike trees.

Corollary 4.1. *The balanced starlike trees are DEDV-trees. Particularly, star S_n and path P_n are DEDV-trees.*

However, the following result indicates that starlike tree is not necessary to be balanced for a DEDV-tree. Let $(m * n) = (\overbrace{n, n, \ldots, n}^{m})$ denote a sequence of n of size m.

Proposition 4.1. *Starlike tree T_{l_1,l_2,l_3} is a DEDV-tree.*

Proof. Without loss of generality we assume that $1 \leq l_1 \leq l_2 \leq l_3$. According to definition, we have

$$\mathbf{r} = \mathbf{r}(T_{l_1,l_2,l_3}) = (l_1 * 3, (l_2 - l_1) * 2, (l_3 - l_2) * 1, 0, \ldots, 0).$$

Thus a tree T with $\mathbf{r}$ as above must have three leaves by Claim 4.1(a), and so $T = T_{l'_1,l'_2,l'_3}$ where $1 \leq l'_1 \leq l'_2 \leq l'_3$. Hence $\mathbf{r}(T) = (l'_1 * 3, (l'_2 - l'_1) * 2, (l'_3 - l'_2) * 1, 0, \cdots, 0) = \mathbf{r}$, which leads to $l_i = l'_i$ and so $T \cong T_{l_1,l_2,l_3}$. Therefore, T_{l_1,l_2,l_3} is a DEDV-tree. ■

Let $\mathcal{T}_{n,k}$ be the set of all starlike trees of the form $T_{l_1,l_2,\ldots,l_k}$ with order n. A starlike tree $T_{l_1,l_2,\ldots,l_k}$ described in (2) is called *weak balanced* if $l_i + l_j \geq \max\{l_q \mid 1 \leq q \leq k\}$ for $1 \leq i, j \leq k$. By a similar consideration of Proposition 4.1, we have the following result.

Lemma 4.3. *Let $T_{l_1,l_2,\ldots,l_k}, T_{l'_1,l'_2,\ldots,l'_k} \in \mathcal{T}_{n,k}$. Then $T_{l_1,l_2,\ldots,l_k} \approx T_{l'_1,l'_2,\ldots,l'_k}$ if and only if $T_{l_1,l_2,\ldots,l_k} \cong T_{l'_1,l'_2,\ldots,l'_k}$.*

Theorem 4.2. *Let $k \geq 4$. The starlike tree $T_{l_1,l_2,\ldots,l_k}$ is DEDV-tree if and only if it is weak balanced.*

Proof. Let $T = T_{l_1,l_2,\ldots,l_k}$ be a DEDV-tree as described in (2), and without loss of generality we assume that $1 \leq l_1 \leq l_2 \leq \cdots \leq l_k$. If T is not weak balanced, then $l_1 + l_2 < l_k$. There exists one edge $e = wz \in E(P_{l_k})$ such that $\mu(e) = |T_z| = l_1 + l_2 = |P_{l_1}| + |P_{l_2}|$. Consider the tree T' obtained from T by branch-exchange $P_{l_1} \cup P_{l_2}$ and T_z. By Theorem 3.1, we have $T' \approx T$ but $T' \ncong T$ since T' has two branching vertices, a contradiction.

Conversely, suppose that starlike tree $T = T_{l_1,l_2,\ldots,l_k}$ is weak balanced, i.e., $l_1 + l_2 \geq l_k$ and $l_k < \lfloor \frac{n}{2} \rfloor$. According to definition, we have

$$\mathbf{r}(T) = (l_1 * k, (l_2 - l_1) * (k-1), (l_3 - l_2) * (k-2), \ldots, (l_k - l_{k-1}) * 1, 0, \ldots, 0)$$

where $r_{l_k+1}(T) = \cdots = r_{\lfloor \frac{n}{2} \rfloor}(T) = 0$. Suppose that T' is the tree with $\mathbf{r}(T') = \mathbf{r}(T)$. Since $r_1 = r_2 = \cdots = r_{l_1} = k$, T' contains exactly k pendent paths P_{l_1} by Lemma 4.2, particularly T' has k leaves. First we will show that T' is also a starlike tree. Since otherwise T' has two branching vertices u_1 and u_2. Let $P' = u_1 u_1' \cdots u_2' u_2$ be the path in T' connecting u_1 and u_2. Notice that $d_{T'}(u_1), d_{T'}(u_2) \geq 3$, we see that $T'_{u_1}(u_1u_1')$ contains at least two pendent paths P_{l_1} and P_{l_2}, thus $|T'_{u_1}(u_1u_1')| \geq |P_{l_1}| + |P_{l_2}| + 1 = l_1 + l_2 + 1 > l_k$. Similarly, $|T'_{u_2}(u_2u_2')| \geq |P_{l_1}| + |P_{l_2}| + 1 = l_1 + l_2 + 1 > l_k$. Therefore, $\mu_{T'}(e) \geq \min\{|T'_{u_1}(u_1u_1')|, |T'_{u_2}(u_2u_2')|\} > l_k$ for any edge $e \in E(P')$. It implies that $r_i \neq 0$ for some $i > l_k$, which contradicts the assumption of $\mathbf{r}(T')$. Thus T' is a starlike tree such that $T' \in \mathcal{T}_{n,k}$ and $T' \approx T$. It immediately follows $T' \cong T = T_{l_1,l_2,\ldots,l_k}$ by Lemma 4.3. ■

A *double star* $S_{p,q}$ is the tree obtained from K_2 by attaching $p-1$ pendent vertices to one vertex and $q-1$ pendent vertices to the other vertex, where $p+q=n$. If T has exactly t pendent S_p, each of their centers is connected to the unique central vertex of T, such a tree T is called *power star* and denote by S_p^t (see Figure 4). From the definition of pendent star, we have the following result.

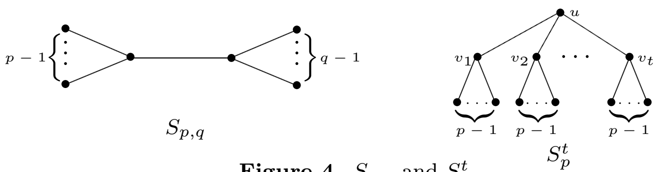


Figure 4. $S_{p,q}$ and S_p^t

Proposition 4.3. *The double star $S_{p,q}$ and power star S_p^t are DEDV-trees.*

Let $T = P_{s_1} \bullet_u P_{s_2} + P_{uv} + P_{t_1} \bullet_v P_{t_2}$ be a tree on n vertices with exactly two branching vertices u and v connecting by a path $P_{uv} = uu_1 \cdots u_{k-1}v$

such that $T_u(uu_1) = T_{s_1,s_2}$ and $T_v(u_{k-1}v) = T_{t_1,t_2}$, where $s_1+s_2 \leq t_1+t_2$, $s_1 \leq s_2$ and $t_1 \leq t_2$.

Proposition 4.4. *Let $T = P_{s_1} \bullet_u P_{s_2} + P_{uv} + P_{t_1} \bullet_v P_{t_2}$, where P_{uv} is a path of length k, $s_1 + s_2 \leq t_1 + t_2$, $s_1 \leq s_2$ and $t_1 \leq t_2$. Then T is a DEDV-tree if and only if the one of the following four conditions holds*
(i) $s_1 + s_2 = t_1 + t_2$,
(ii) $s_1 + s_2 > t_2$,
(iii) $s_1 + s_2 + k = t_2$ and $s_1 + s_2 > t_1$,
(iv) $s_1 + s_2 + k = t_2 = t_1$.

Proof. We prove necessity by proving its inverse proposition. Suppose that $s_1 + s_2 \neq t_1 + t_2$, we will show (ii), (iii) or (iv) holds. By the way of contradiction, we may assume that $s_1 + s_2 \leq t_2$ and consider the following two situations.

Case 1. $s_1 + s_2 = t_2$;

If $s_1 + s_2 = t_2$, we can construct a tree T' from T by branch-exchange the two balanced components $T_u(uu_1) - u = P_{s_1} \cup P_{s_2}$ and P_{t_2}. It is clear that $T' \approx T$ by Lemma 3.1. Note that $T' = T_{t_1,s_1,s_2,t_2+k}$ is a starlike tree and so $T' \ncong T$. This means that counterexamples can be found no matter how we choose the conditions under which (iii) and (iv) do not hold.

Case 2. $s_1 + s_2 < t_2$;

If $s_1 + s_2 + k \neq t_2$, we can get a tree T' from T by branch-exchange the two balanced components $T_u(uu_1) - u = P_{s_1} \cup P_{s_2}$ and $P_{s_1+s_2} \subseteq P_{t_2}$ of T. It is clear that $T' \approx T$ by Lemma 3.1. Note that T' is a tree with two branching vertices u' and v' connecting with a path $P_{u'v'}$ of $t_2-(s_1+s_2)+1$ vertices, i.e., $T' = P_{s_1} \bullet_{u'} P_{s_2} + P_{u'v'} + P_{t_1} \bullet_{v'} P_{s_1+s_2+k}$. Obviously, $T' \ncong T$ because of $s_1 + s_2 + k \neq t_2$, a contradiction.

If $s_1 + s_2 + k = t_2 = t_1 + k$, it also means $t_2 \neq t_1$, we can get a tree T' from T by branch-exchange the two balanced components $T_u(uu_1) - u = P_{s_1} \cup P_{s_2}$ and P_{t_1} of T. It is clear that $T' \approx T$ by Lemma 3.1. Note that $T' = T_{t_2,s_1,s_2,t_1+k}$ is a starlike tree and so $T' \ncong T$, a contradiction.

If $s_1 + s_2 + k = t_2 < t_1 + k$ and $t_2 \neq t_1$, we can get a tree T' from T by branch-exchange the two balanced components $T_u(uu_1) - u = P_{s_1} \cup P_{s_2}$

and $P_{s_1+s_2} \subseteq P_{t_1}$ of T. It is clear that $T' \approx T$ by Lemma 3.1. Note that T' is a tree with two branching vertices u' and v' connecting with a path $P_{u'v'}$ of $t_1-(s_1+s_2)+1$ vertices, i.e., $T' = P_{s_1} \bullet_{u'} P_{s_2} + P_{u'v'} + P_{s_1+s_2+k} \bullet_{v'} P_{t_2}$. Obviously, $T' \not\cong T$ because of $s_1+s_2+k \neq t_1$, a contradiction.

For the sufficiency, we may assume that $\alpha = s_1+s_2 \leq t_1+t_2 = \beta$, there exists a tree T' with $(r'_1, \ldots, r'_{\lfloor \frac{n}{2} \rfloor}) = \mathbf{r}(T') = \mathbf{r}(T) = (r_1, \ldots, r_{\lfloor \frac{n}{2} \rfloor})$. Since T has four leaves, we may assume P_{a_i} is the maximal pendent path of T' with a_i vertices, where $a_1 \leq a_2 \leq a_3 \leq a_4$.

Suppose (i) holds, i.e., $\alpha = \beta$. Without loss of generality, we assume that $s_1 \leq t_1 \leq t_2 \leq s_2$. It is easy to see that $r'_{s_1+s_2+1} = r_{s_1+s_2+1} = 1$ if $k = 1$ and 2 otherwise, but in any case T' has two pendent subtrees $T'_{u'}$ and $T'_{v'}$ such that $a_4 < |T'_{u'}| = |T'_{v'}| = s_1+s_2+1 \leq \frac{n}{2}$. It implies that $|T'_{u'}| = a_1+a_4+1$ and $|T'_{v'}| = a_2+a_3+1$ due to $\alpha = \beta$. Let $P'_{u'v'}$ be the path connecting u' and v', we have $T' = T'_{u'} + P'_{u'v'} + T'_{v'}$ and so $|P'_{u'v'}| = k+1$ by Claim 4.1(d). Moreover, we have $a_1 = s_1$ by Claim 4.1(c), and thus $a_4 = s_2$. Since $t_1+1 < s_1+s_2$, we have $r_{s_1+1} = \cdots = r_{t_1} = 3$ and $r_{t_1+1} = 2$. Similarly, $r'_{a_1+1} = \cdots = r'_{a_2} = 3$ and $r'_{a_2+1} = 2$. Thus one can verify that $t_1 \geq a_2$. By symmetry, we get $t_1 = a_2$ and so $t_2 = a_3$. Therefore, $T' \cong T$.

In what follows we always assume that $\alpha < \beta$ because of (i).

Suppose (ii) holds, i.e., $t_2 < s_1+s_2 < t_1+t_2$. First of all, we claim that T' has exactly two branching vertices. Otherwise $T' = T_{a_1,a_2,a_3,a_4}$. Since $r_{\alpha+1} \neq 0$, we have $a_4 \geq \alpha+1$ and so $r_i = r'_i \neq 0$ for $i = 1, \ldots, a_4$. On the other hand, we have $r_i = 0$ for $i = \max\{t_2, s_2\}+1, \ldots, \alpha$, and $\alpha < a_4$, a contradiction. Thus, we may assume that T' has two branching vertices u' and v' connecting with a path $P'_{u'v'}$. Then $T' = T'_{u'} + P'_{u'v'} + T'_{v'}$ with $a_{i_1}+a_{i_2}+1 = |T'_{u'}| < |T'_{v'}| = a_{i_3}+a_{i_4}+1$, where $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. Clearly, $t = \max\{s_2, t_2\} \geq a_4$ since $r_t \neq 0$ and $r_{t+1} = 0$. If $t > a_4$ then $r'_{a_4+1} = r_{a_4+1} \neq 0$. Therefore, $|T'_{u'}| = a_{i_1}+a_{i_2}+1 = a_4+1 \leq t < s_1+s_2 < \frac{n}{2}$. This is impossible since $r'_i \neq 0$ for $i \in [a_4+1, s_1+s_2]$ but $r_{t+1} = 0$. Thus $t = a_4$. Now by deleting P_t from T and T', respectively, we get $\tilde{r_i} = r_i - 1 = r'_i - 1 = \tilde{r'_i}$ for $i = 1, 2, \ldots, c = \min\{s_2, t_2\}$. It is easy to see that $\mathbf{r}(T_3) = (\tilde{r_1}, \ldots, \tilde{r_c}, 0, \ldots, 0)$ and $\mathbf{r}(T'_3) = (\tilde{r'_1}, \ldots, \tilde{r'_c}, 0, \ldots, 0)$ are the edge division vectors of $T_3 = T_{c_1,c_2,c_3}$ and $T'_3 = T_{a_1,a_2,a_3}$, respectively, where $\{c_1, c_2, c_3\} = \{s_1, s_2, t_1, t_2\} \setminus t$. By Proposition 4.1, $\{c_1, c_2, c_3\} = \{a_1, a_2, a_3\}$ and so $\{s_1, s_2, t_1, t_2\} = \{a_1, a_2, a_3, a_4\}$. It implies that $T'_{u'} = P_{s_1} \bullet_{u'} P_{s_2}$ and $T'_{v'} = P_{t_1} \bullet_{v'} P_{t_2}$. It follows that $T \cong T'$.

Suppose (iii) holds, i.e., $s_1 + s_2 + k = t_2$ and $s_1 + s_2 > t_1$. Since $s_1 + s_2 + k = t_2 < \lfloor \frac{n}{2} \rfloor$, we have $r'_{t_2} = r_{t_2} = 2$ and $r'_{t_2+1} = r_{t_2+1} = 0$. Then T' has two pendent subtrees $T'_{x'_1}$ and $T'_{x'_2}$ of order t_2 and no any of order $t_2 + 1$. Thus the root x'_i of $T'_{x'_i}$ appends with branching vertex x_i, i.e., $T'_{x_i} = T'_{x'_i} + x'_i x_i$ where $i = 1, 2$. First of all, both of $T'_{x'_1}$ and $T'_{x'_2}$ can not be path since otherwise $r'_{s_1+s_2} \geq 2$ but $r_{s_1+s_2} = 1$. Moreover, $x_1 = x_2 = v'$ and one of $T'_{x'_1}$ and $T'_{x'_2}$ must be a path since otherwise T' has at least five leaves. Thus we may assume that $T'_{x'_2} = P_{t_2}$. By deleting the P_{t_2} from T and T', similar as the arguments in the proof of (ii) we get $\{a_1, a_2, a_3, a_4\} = \{s_1, s_2, t_1, t_2\}$. Let P_{a_3} be the path, its root appends at v'. Then $a_3 = n - 2t_2 - 1 = t_1$. It follows that $T' = T'_{u'} + P'_{u'v'} + T'_{v'}$, where $T'_{u'} = P_{s_1} \bullet_{u'} P_{s_2}$ and $T'_{v'} = P_{t_1} \bullet_{v'} P_{t_2}$, and thus $T \cong T'$.

Suppose (iv) holds. Let $s = s_1 + s_2 + k = t_2 = t_1$, we have $r'_s = r_s = 3$ and $r'_i = r_i = 0$ for $i > s$. Then T' has three pendent subtrees $T'_{x'_i}$ of order s, its root x'_i appends with branching vertex x_i for $i = 1, 2, 3$, and no any of order $s + 1$. Since T' has four leaves, x_i can not distinct from each other. Thus we may assume that $x_1 = x_2 = x_3$ or $x_1 = x_2 \neq x_3$. Whichever happens, T' contains the maximal pendent path P_s. By deleting the P_s from T and T', similar as the arguments in the proof of (ii) we get $\{a_1, a_2, a_3, a_4\} = \{s_1, s_2, t_1, t_2\}$. It is clear that $T' \neq T_{a_1,a_2,a_3,a_4}$ because $r'_i = r_i = 3$ for $i \in [s_1 + s_2 + 1, t_2]$. Thus $T' = P_{s_1} \bullet_{u'} P_{s_2} + P'_{u'v'} + P_{t_1} \bullet_{v'} P_{t_2} \cong T$.

We complete this proof. ■

At the last of this section, we will give a method to construct DEDV-trees from some known DEDV-trees. We begin with some notions and symbols. Given a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and a graph H with root vertex u, the rooted product graph $G \diamond_u H$ is defined as the graph obtained from G and n copies of H by identifying vertex v_i in G with u in the i-th copy of H. Let P_s be the path of order s with one end point u as its root vertex, the rooted product graph $\bar{G}_s = G \diamond_u P_s$ is shown in Figure 5. The corona product graph $G \circ H$ is defined as the graph obtained from G and n copies of H by joining the vertex v_i of G to every vertex in the i-th copy of H. If we take $H = sK_1$ $(s \geq 1)$, then the corona product graph $\tilde{G}_s = G \circ sK_1$ is shown in Figure 5. Particularly, if G is taken as a tree $T \in \mathcal{T}_n$, we denote $\bar{T}_s = T \diamond_u P_s$ and $\tilde{T}_s = T \circ sK_1$. Let $\mathbf{r}(T) = (r_1, r_2, \ldots, r_{\lfloor \frac{n}{2} \rfloor})$ and $\mathbf{r}(\bar{T}_s) = (\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_{\lfloor \frac{ns}{2} \rfloor})$, $\mathbf{r}(\tilde{T}_s) = (\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_{\lfloor \frac{n(s+1)}{2} \rfloor})$. According to definition, one can simply verify that

$$\bar{r}_i = \begin{cases} n & \text{if } i \leq s-1 \\ r_k & \text{if } i = ks, \text{ where } k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \\ 0 & \text{if } s < i \neq ks, \end{cases} \tag{3}$$

$$\tilde{r}_i = \begin{cases} ns & \text{if } i = 1 \\ r_k & \text{if } i = k(s+1), \text{ where } k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \\ 0 & \text{if } 1 < i \neq k(s+1). \end{cases} \tag{4}$$

It is clear that $\mathbf{r}(\bar{T}_s)$ and $\mathbf{r}(\tilde{T}_s)$ are determined by $\mathbf{r}(T)$. Using the above symbols, we can state the following result.

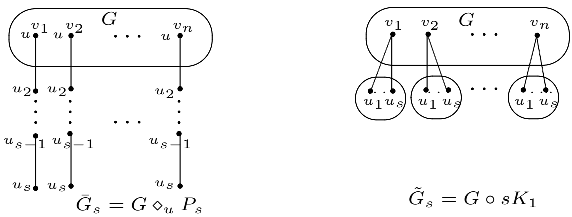


Figure 5. The rooted product graph $G \diamond_u P_s$ and corona product graph $G \circ sK_1$

Theorem 4.5. *Let T be a DEDV-tree of order n. Then we have*
(a) $\bar{T}_s = T \diamond_u P_s$ $(s \geq 1)$ is DEDV-tree.
(b) $\tilde{T}_s = T \circ sK_1$ $(s \geq 1)$ is DEDV-tree.

Proof. First we prove (a). Suppose that there is a tree H with $H \approx \bar{T}_s$ and H_x is any pendent subtree of H with root x. By Lemma 4.2, we see that H has exactly n pendent paths P_{s-1} due to $\bar{r}_i = n$ for $i = 1, 2, \ldots, s-1$, thus $H_x \cong P_i$ for $i = |H_x| < s$ and is included in P_{s-1}. Now by deleting these n pendent paths P_{s-1} from H, we get a subtree T' from H. Suppose that

$\mathbf{r}(T') = (r'_1, r'_2, \ldots, r'_{\lfloor \frac{n}{2} \rfloor})$. In what follows we only need to show that each vertex of T' joins one end point of path P_{s-1} in H and thus $\mathbf{r}(T') = \mathbf{r}(T)$ according to (3). Consequently, $T' \cong T$ due to T is a DEDV-tree.

From (3) we know that any pendent subtree of H with order less than s is a path by Lemma 4.2. Note that $\bar{r}_s = r_1 \neq 0$, H has pendent subtree H_x with $|H_x| = s$. Again by Lemma 4.2, the root x is pendent vertex of T' and so $H_x = P_s$. Let H_y be any pendent subtree of H with order $|H_y| = ks \geq s$, where $k \geq 1$, and assume that each vertex of $V(T') \cap V(H_y)$ joins one end point of P_{s-1}. Now if $i = |H_z| > ks$ then $i = k's$ for some $k' > k$ because $\bar{r}_i = 0$ if $s \nmid i$. Let $z_1, \ldots, z_t$ be adjacent vertices of z in H_z and H_{z_i} be the component of $H_z - z_i z$ containing z_i where $1 \leq i \leq t$. By induction hypothesis, each vertex of $V(T') \cap V(H_{z_i})$ joins one end point of P_{s-1} and hence $|H_{z_i}| = s|V(T') \cap V(H_{z_i})|$. Note that $|V(T') \cap V(H_z)| = |V(T') \cap V(H_{z_1})| + \cdots + |V(T') \cap V(H_{z_t})| + 1$, where $z \in V(T')$ contributes 1, and we have $s|V(T') \cap V(H_z)| = s|V(T') \cap V(H_{z_1})| + \cdots + s|V(T') \cap V(H_{z_t})| + s = |H_{z_1}| + \cdots + |H_{z_t}| + s$. Since $|H_z| = k's$ has the form of $s|V(T') \cap V(H_z)|$, we claim that z joins exactly one end point of P_{s-1}. Therefore, each vertex of T' joins one end point of P_{s-1} by induction. By considering the edge division vector of H we have

$$\bar{r}_i = \begin{cases} n & \text{if } i \leq s-1 \\ r'_k & \text{if } i = ks \\ 0 & \text{if } s < i \neq ks. \end{cases} \tag{5}$$

From (3) and (5) we see that $r_k = r'_k$ for $k \geq 1$, and so $\mathbf{r}(T) = \mathbf{r}(T')$.

As similar as (a), one can verify (b). ■

Example 4.1. According to Theorem 4.5, $(\bar{P_n})_s = P_n \diamond_u P_s$ and $(\tilde{P_n})_s = P_n \circ sK_1$ are DEDV-trees since P_n is DEDV-tree.

5 Construction for EDV-equivalent trees and DEDV-trees

In this section, we will use the branch-exchange transformation to construct EDV-equivalent trees, especially we give all the EDV-equivalent trees of vertices no more than 10 and consequently the corresponding DEDV-trees are also determined.

First of all, note that a tree of order less than 7 has at most two branching vertices, we get the following result by Theorem 4.2 and Proposition 4.4.

Proposition 5.1. *There is no any non-isomorphic EDV-equivalent trees of order less than* 7*, and equivalently any tree* T *of order* $n < 7$ *is a DEDV-tree.*

Except for the individual tree, all the trees of order $7 \leq n \leq 8$ have at most two branching vertices, select all trees that do not satisfy Theorem 4.2 and Proposition 4.4, which are listed in the Figure 6. Fortunately, we can verify that they are all families of non-isomorphic EDV-equivalent trees of order $7 \leq n \leq 8$.

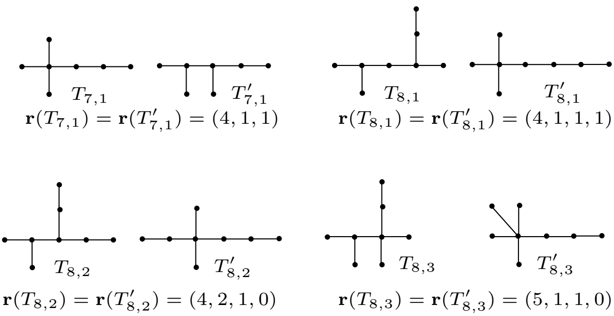


Figure 6. All the EDV-equivalent trees on $7, 8$ vertices

Proposition 5.2. *There is exactly one pair of non-isomorphic EDV-equivalent trees on* 7 *vertices, which are labelled as* $T_{7,1}$ *and* $T'_{7,1}$ *shown in Figure 6, where* $\mathbf{r}(T_{7,1}) = \mathbf{r}(T'_{7,1}) = (4, 1, 1)$.

Proposition 5.2 can also restate as that except of $T_{7,1}$ and $T'_{7,1}$ all the trees on 7 vertices are DEDV-trees.

Proposition 5.3. *There are exactly three pairs of non-isomorphic EDV-equivalent trees on* 8 *vertices, which are labelled as* $(T_{8,1}, T'_{8,1})$, $(T_{8,2}, T'_{8,2})$ *and* $(T_{8,3}, T'_{8,3})$ *shown in Figure 6, where* $\mathbf{r}(T_{8,1}) = \mathbf{r}(T'_{8,1}) = (4, 1, 1, 1)$, $\mathbf{r}(T_{8,2}) = \mathbf{r}(T'_{8,2}) = (4, 2, 1, 0)$ *and* $\mathbf{r}(T_{8,3}) = \mathbf{r}(T'_{8,3}) = (5, 1, 1, 0)$.

Proposition 5.3 can also restate as that except of $(T_{8,1}, T'_{8,1})$, $(T_{8,2}, T'_{8,2})$ and $(T_{8,3}, T'_{8,3})$ all the trees on 8 vertices are DEDV-trees.

The pairs of non-isomorphic EDV-equivalent trees on 7 and 8 vertices are described in Proposition 5.2 and Proposition 5.3. Similarly, we can exhaust all the non-isomorphic EDV-equivalent trees on 9 vertices, which correspond to 11 distinct edge division vectors: $(4, 1, 1, 2)$, $(4, 1, 2, 1)$, $(4, 2, 1, 1)$, $(4, 2, 2, 0)$, $(5, 1, 1, 1)$, $(5, 1, 2, 0)$, $(5, 2, 0, 1)$, $(5, 2, 1, 0)$, $(6, 1, 1, 0)$, $(6, 1, 0, 1)$, $(6, 0, 1, 1)$.

Proposition 5.4. *There are exactly* 11 *classes of non-isomorphic EDV-equivalent trees on* 9 *vertices, which are listed in Figure 7.*

From Figure 7 we see that there are four non-isomorphic trees corresponding to one edge division vectors $(5, 1, 1, 1)$. Additionally, we mention that except of the 11 classes trees list in Figure 7 all the trees on 9 vertices are DEDV-trees.

Remark 5.1. There are 106 connected trees on 10 vertices, 59 of which can be divided into 25 classes based on edge division vector that are EDV-equivalent trees, and the remaining 47 trees are DEDV-trees (see Appendix A).

Let u and v be two vertices of T. Recall that u and v are said to be similar if there is an automorphism α of T that contains the transposition $(u\ v)$ (i.e., $\alpha(u) = v$ and $\alpha(v) = u$), and not similar otherwise. In the following result, we give a simple method of constructing non-isomorphic EDV-equivalent trees that is a special case of the branch-exchange.

Theorem 5.5. *Let* T *and* $T' = T - ux + vx$ *be assumed as in Lemma 2.2 and* $n_u(uv) - n_v(uv) = |T_x(ux)|$. *Let* $\hat{T} = T - ux - T_x(ux)$. *If* u *and* v *are not similar vertices of* $\hat{T}$, *then* $T \approx T'$ *but* $T \ncong T'$.

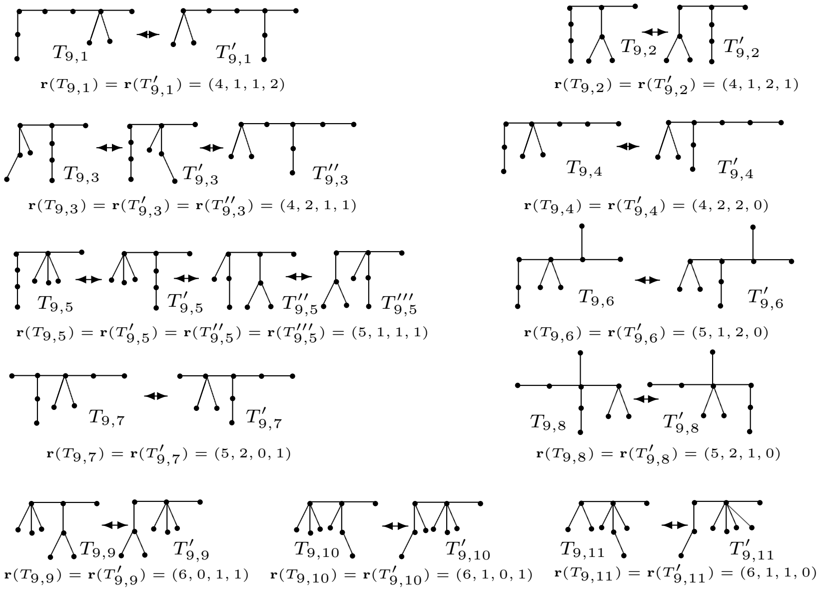


Figure 7. All the EDV-equivalent trees on 9 vertices

Proof. Since $n_u(uv) - n_v(uv) = |T_x(ux)|$, we have $T \approx T'$ from Lemma 2.2(2). Also note that $T = \hat{T} + xu + T_x(ux)$ and $T' = \hat{T} + xv + T_x(ux)$, we see that the two components $\hat{T}_u(uv)$ and $\hat{T}_v(uv)$ have the same number of vertices, i.e., $|\hat{T}_u(uv)| = n_u(uv) - |T_x(ux)| = n_v(uv) = |\hat{T}_v(uv)|$. If $T \cong T'$ then there exists an isomorphism α such that $\alpha(T) = T'$. It implies that α must fix the edge uv (i.e., α maps the edge uv of T to the edge uv of T'). Therefore, $\alpha(T_u(uv)) = T'_v(uv)$ and $\alpha(T_v(uv)) = T'_u(uv)$ because of $|T_u(uv)| = |T_v(uv)| + |T_x(ux)| = |T'_v(uv)| > |T_v(uv)| = |T_u(uv)| - |T_x(ux)| = |T'_u(uv)|$. Since $T_v(uv) = \hat{T}_v(uv)$ and $T'_u(uv) = \hat{T}_u(uv)$, we have $\alpha(\hat{T}_v(uv)) = \hat{T}_u(uv)$. This proves that u and v are similar vertices of $\hat{T}$, a contradiction. Therefore, $T \ncong T'$. ■

According to Theorem 5.5, we can give a method of constructing non-isomorphic EDV-equivalent trees. Given any tree $\hat{T}$ with one specified edge uv, we construct the trees T and T' shown in Figure 1. If $\hat{T}$ satisfies the following conditions:

(a) u is not similar with v in $\hat{T}$,

(b) $|\hat{T}_u(uv)| = |\hat{T}_v(uv)|$,
we have $T \approx T'$ but $T \ncong T'$. It leads the following conclusion.

Proposition 5.6. *There exist infinite pairs of EDV-equivalent trees that are not isomorphic such that they contain any given tree $\hat{T}$ as their subtree.*

Remark 5.2. It is an interesting problem to find the necessary and sufficient condition of the transformation for a tree T such that it can produce all the trees from T that are EDV-equivalent with T but not isomorphic to it.

We examine trees having the same edge division vector in Figure 6 and Figure 7 and find that each one can be obtained from another by using the branch-exchange transformation. We now propose a problem.

Problem 5.1. *Let T be a tree and $\mathcal{B}(T)$, the set of trees that can be obtained from T by using branch-exchange transformation. Then T is DEDV-tree if and only if $\mathcal{B}(T)$ does not contain any tree except of T itself.*

6 Topological indices on trees

The first topological index, named Wiener index, was introduced in [22] and is defined as $W(G) = \sum_{\{u,v\} \in V(G)} d(u, v)$. It was already known to Wiener that on the class of trees Wiener index can be represented by a function of $\mu(e)$ as follows:

$$\begin{aligned} W(T) &= \textstyle\sum_{e=xy \in E(T)} n_x(e) n_y(e) = \sum_{e=xy \in E(T)} n_x(e)(n - n_x(e)) \\ &= \textstyle\sum_{e \in E(T)} \mu(e)(n - \mu(e)). \end{aligned}$$

Let $\mathbf{r} = (r_1, r_2, \ldots, r_{\lfloor \frac{n}{2} \rfloor})$ be the edge division vector of T and $f(x) = x(n - x)$. As in the proof of Theorem 8 in [20], $W(T)$ can be further simplified as

$$W(T) = \sum_{e \in E(T)} \mu(e)(n - \mu(e)) = \sum_{1 \le i \le \lfloor \frac{n}{2} \rfloor} r_i f(i). \tag{6}$$

The authors in [20] introduced the notions bellow.

Definition 6.1. Let $F : \mathcal{T}_n \to \mathbb{R}$ be a topological index and let $f : \mathbb{N} \to \mathbb{R}$ be a real function defined for positive integers. The topological index F is an edge additive eccentric topological index if it holds that $F(T) = \sum_{e \in E(T)} f(\mu(e)) = \sum_{1 \le i \le \lfloor \frac{n}{2} \rfloor} r_i f(i)$. Function f is called the edge contribution function of index F.

In [20], the authors showed that several well known topological indices on the class of trees can be also represented by some functions of $\mu(e)$ described as Eq. (6), and they are edge additive eccentric topological indices. They are Wiener index [22], modified Wiener indices [10], variable Wiener indices [21] and Steiner k-Wiener index [14]. Recently, the authors of [18] extended the above conclusions to the more topological indices: hyper-Wiener index [15], Wiener-Hosoya index [16], degree distance [11], Gutman index [9] and second atom-bond connectivity index [7]. All these topological indices and their edge contribution functions are summarized in the Table 1.

Table 1. Some edge additive eccentric topological indices

Indices	Definition	Edge contribution function
Wiener index	$W(T) = \sum_{\{u,v\} \in V(T)} d(u,v)$	$f(x) = x(n-x)$
Modified Wiener indices	${}^{\lambda}W(T) = \sum_{\{u,v\} \in V(T)} d^{\lambda}(u,v)$	$f(x) = x^{\lambda}(n-x)^{\lambda}$
Variable Wiener indices	${}_{\lambda}W(T) = \frac{1}{2} \sum_{e=uv \in E(T)} (n^{\lambda} - n_u(e)^{\lambda} - n_v(e)^{\lambda})$	$f(x) = n^{\lambda} - x^{\lambda} - (n-x)^{\lambda}$
Steiner k-Wiener index	$SW_k(T) = \sum_{e=uv \in E(T)} \sum_{i=1}^{k-1} \binom{n_u(e)}{i}\binom{n_v(e)}{k-i}$	$f(x) = \binom{n}{k} - \binom{x}{k} - \binom{n-x}{k}$
hyper-Wiener index	$WW(T) = \sum_{e=uv \in E(T)} (\frac{1}{2} n_u(e) n_v(e) + \frac{1}{2} n_u(e)^2 n_v(e)^2)$	$f(x) = \frac{1}{2}x(n-x) + \frac{1}{2}x^2(n-x)^2$
Wiener-Hosoya index	$h(T) = \sum_{e=uv \in E(T)} [n_u(e) n_v(e) + (n_u(e)-1)(n_v(e)-1)]$	$f(x) = x(n-x) + (x-1)(n-x-1)$
degree distance	$D'(T) = \sum_{e=uv \in E(T)} (4 n_u(e) n_v(e) - n)$	$f(x) = 4x(n-x) - n$
Gutman index	$Gut(T) = \sum_{e=uv \in E(T)} [4 n_u(e) n_v(e) - (2n-1)]$	$f(x) = 4x(n-x) - (2n-1)$
second atom-bond connectivity index	$ABC_2(T) = \sum_{e=uv \in E(T)} \sqrt{\frac{n-2}{n_u(e)(n-n_u(e))}}$	$f(x) = \sqrt{n-2}x^{-\frac{1}{2}}(n-x)^{-\frac{1}{2}}$

Instead of dealing with the extremal problem for individual topological index one by one, we can use the concept of edge additive eccentric topological index to unify the problem of determining the extreme topological index in certain classes of trees. In the literature [18] the authors gave and summarized some results for the extremal topological indices involved in Table 1. Here, at last of the paper, we turn to consider whether trees with different structures can have the same topological indices? Based on the relation of $\langle \mathcal{T}_n, \preceq \rangle$ and the notion of edge additive eccentric topological index, the following results give us a way to solve this problem.

Theorem 6.1. *Let $F : \mathcal{T}_n \longrightarrow \mathbb{R}$ be an edge additive eccentric topological index and let $T, T' \in \mathcal{T}_n$. If $T \approx T'$, then $F(T) = F(T')$.*

Proof. Since F is an edge additive eccentric topological index, we know that it is defined by

$$F(T) = \sum_{e \in E(T)} f(\mu(e)),$$

where f is its edge contribution function. Let $\mathbf{r} = (r_1, r_2, \ldots, r_{\lfloor \frac{n}{2} \rfloor})$ and $\mathbf{r}' = (r'_1, r'_2, \ldots, r'_{\lfloor \frac{n}{2} \rfloor})$ be the edge division vectors of T and T', respectively. Hence we have

$$\begin{cases} F(T) = \sum_{1 \le i \le \lfloor \frac{n}{2} \rfloor} r_i f(i), \\ F(T') = \sum_{1 \le i \le \lfloor \frac{n}{2} \rfloor} r'_i f(i). \end{cases}$$

Note that $\mathbf{r} = \mathbf{r}'$ since $T \approx T'$. Therefore, we have $F(T) = F(T')$. ■

Theorem 6.1 shows that non-isomorphic EDV-equivalent trees have the same edge additive eccentric topological index value, which does not depend on individual form of topological index.

Notice that the pairs of non-isomorphic EDV-equivalent trees on 7, 8 and 9 vertices along with their edge division vectors are described in Figure 6 and Figure 7 (other trees that are not depicted are DEDV-trees). We give Table 2 as an example of application that list the values of various topological indices of these non-isomorphic EDV-equivalent trees. From Table 2 we see that pairs of these non-isomorphic EDV-equivalent trees have the same value of topological index although the value depending on individual form of topological index would be varied.

To determine the all EDV-equivalent trees of a given order we first of all had to generate the all connected trees by computer and then deter-

Table 2. The values of topological indices of all EDV-equivalent trees of order $7, 8$, and 9

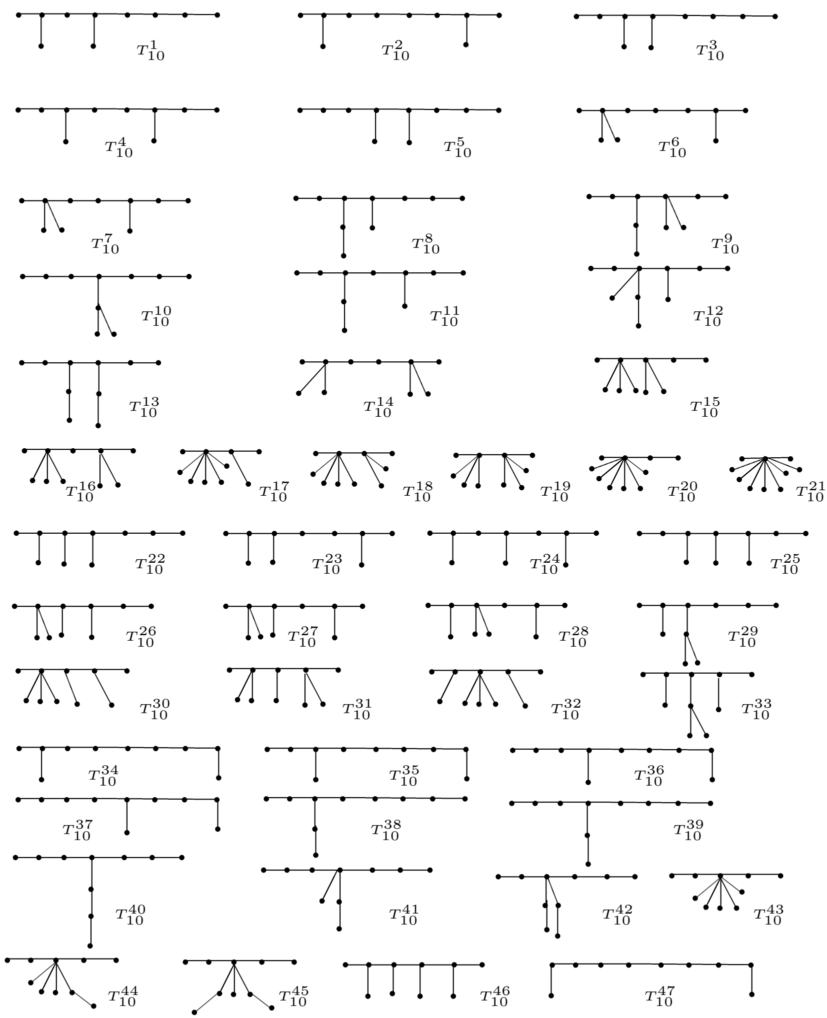
n	trees	edge division vector	$W(\cdot)$	$h(\cdot)$	$Gut(\cdot)$
7	$T_{7,1}, T'_{7,1}$	$(4, 1, 1)$	46	56	106
8	$T_{8,1}, T'_{8,1}$	$(4, 1, 1, 1)$	71	93	179
8	$T_{8,2}, T'_{8,2}$	$(4, 2, 1, 0)$	67	85	163
8	$T_{8,3}, T'_{8,3}$	$(5, 1, 1, 0)$	62	75	143
9	$T_{9,1}, T'_{9,1}$	$(4, 1, 1, 2)$	104	144	280
9	$T_{9,2}, T'_{9,2}$	$(4, 1, 2, 1)$	102	140	272
9	$T_{9,3}, T'_{9,3}, T''_{9,3}$	$(4, 2, 1, 1)$	98	132	256
9	$T_{9,4}, T'_{9,4}$	$(4, 2, 2, 0)$	96	128	248
9	$T_{9,5}, T'_{9,5}, T''_{9,5}, T'''_{9,5}$	$(5, 1, 1, 1)$	92	120	232
9	$T_{9,6}, T'_{9,6}$	$(5, 1, 2, 0)$	90	116	224
9	$T_{9,7}, T'_{9,7}$	$(5, 2, 0, 1)$	88	112	216
9	$T_{9,8}, T'_{9,8}$	$(5, 2, 1, 0)$	86	108	208
9	$T_{9,9}, T'_{9,9}$	$(6, 0, 1, 1)$	86	108	208
9	$T_{9,10}, T'_{9,10}$	$(6, 1, 0, 1)$	82	100	192
9	$T_{9,11}, T'_{9,11}$	$(6, 1, 1, 0)$	80	96	184

Table 3. Fractions of DEDV trees and EDV-equivalent trees

n	# trees	# DEDV trees	# EDV-equivalent trees	DEDV trees	EDV-equivalent trees
2	1	1	0	1	0
3	1	1	0	1	0
4	2	2	0	1	0
5	3	3	0	1	0
6	6	6	0	1	0
7	11	9	2	0.8181	0.1818
8	23	17	6	0.7391	0.2609
9	47	22	25	0.4681	0.5319
10	106	47	59	0.4434	0.5566

mine their edge division vectors. These would have to be stored and then compared. For example, the trees of order no more than 10 are described in Section 5. The results are in Table 3, where we give the fractions of EDV-equivalent trees and DEDV-trees. Notice that for $n \leq 6$ there are no EDV-equivalent trees, all of them are DEDV-trees. An interesting result from the table is that the fraction of EDV-equivalent trees is non-decreasing for small n. If this tendency continues, almost all trees will be the EDV-equivalent pairs in the table. Indeed, the fraction of trees that are DEDV-trees tends to zero as n tends to infinity. The conclusion may be that the present data give some indication that, the fraction of non-isomorphic EDV-equivalent pairs tends to one as n tends to infinity.

A All DEDV-trees on 10 vertices



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