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Bifurcation Anti–Control Tactics of a Fractional–Order Stable Chemical Reaction System

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Abstract

Establishing suitable differential dynamical models to describe the real natural phenomenon in chemistry and physics has become a very hot topic in nowadays society. In this present research, we deal with a fractional-order chemical reaction system. Taking advantage of the fixed point theorem, we prove the existence and uniqueness of the fractional-order chemical reaction system. Using the inequality skill, we prove the non-negativeness of the fractional-order chemical reaction system. By applying a suitable function, we prove the uniform boundedness of the solution to the fractional-order chemical reaction system. With the aid of a hybrid controller including state feedback and parameter perturbation, we discuss the Hopf bifurcation anti-control issue of the fractional-order stable chemical reaction system. A novel delay-independent condition ensuring the stability and the onset of Hopf bifurcation of the involved fractional-order stable chemical reaction system is set up. The study manifests that the delay in the hybrid controller plays a vital role in stabilizing the system and controlling the occurrence of Hopf bifurcation of the fractional-order stable chemical reaction system. In order to

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validate the derived key conclusions, MATLAB simulations are executed and bifurcation plots are given. The obtained results of this article have momentous theoretical guiding value in controlling the chemical compositions. The exploration idea can also be utilized to investigate the bifurcation control and bifurcation anti-control problems in lots of other fractional-order differential systems in numerous disciplines.

1 Introduction

Many nonlinear phenomena (for example, periodic oscillation, stability, boundedness, chaos, etc.) occurring in chemical reaction models have attracted much attention from numerous scholars [45]. Making use of some suitable mathematical tools, we can effectively explore the various dynamics of chemical reaction models and reveal the relation of different chemical variables. Then we can better grasp the inherent law among different chemical variables and serve humanity. In particular, a differential equation is a very vital tool to probe into the dynamical behavior of chemical reaction models. During the past decades, many interesting works on chemical reaction models have been carried out and many valuable achievements are constantly presented. For example, Wang and Li [27] dealt with the microscopic dynamical behavior of a chaotic chemical reaction model. Geysermans and Nicolis [12] explored the thermodynamic fluctuations and chaotic behavior of a chemical reaction system. Zhu and Li [45] revealed the influence of intrinsic fluctuations on bistable behavior in a chemical reaction system. Voorsluijs and Decker [26] discussed the emergence of chaos in a spatially confined reactive model. Huang and Yang [17] analyzed the chaoticity issue of some chemical attractors. In detail, one can see [7, 8, 14, 25].

In 1996, Geysermans and Baras [11] explored a homogeneous chaotic Willamowshi-Rossler system. The balance equations of this system own a well-defined microscopic counterpart and all the reaction obey the “elementary" steps as follows:

$$\begin{cases} \mathcal{A}_1 + \mathcal{X} \overset{\mu_1}{\to} 2\mathcal{X}, \mathcal{A}_1 + \mathcal{X} \overset{\mu_{-1}}{\leftarrow} 2\mathcal{X}, \\ \mathcal{X} + \mathcal{Y} \overset{\mu_2}{\to} 2\mathcal{Y}, \\ \mathcal{A}_5 + \mathcal{Y} \overset{\mu_3}{\to} \mathcal{A}_2, \\ \mathcal{X} + \mathcal{Z} \overset{\mu_4}{\to} \mathcal{A}_3, \\ \mathcal{A}_4 + \mathcal{Z} \overset{\mu_5}{\to} 2\mathcal{Z}, \mathcal{A}_4 + \mathcal{Z} \overset{\mu_{-5}}{\leftarrow} 2\mathcal{Z}, \end{cases} \tag{1}$$

which includes both autocatalytic steps with the constituents $\mathcal{X}$ and $\mathcal{Z}$, coupled via three other steps with the three constituents $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}$. The initial $(\mathcal{A}_1, \mathcal{A}_4, \mathcal{A}_5)$ and final $(\mathcal{A}_2, \mathcal{A}_3)$ product concentrations keep fixed. The distance from thermodynamic equilibrium is controlled by the values of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$. $\mu_{\pm j}(j = 1, 2, 3, 4, 5)$ denote the rate constant. In system (1.1), there are 15 free parameters. To reduce the number of free parameters, Geysermans and Baras [11] chose the rate coefficients $\mu_{-2} = 0, \mu_{-3} = 0$ and $\mu_{-4} = 0$. Note that the last two relations mean that $\mathcal{A}_2$ and $\mathcal{A}_3$ are continuously removed from the reactor [11, 12].

Suppose that there exists an ideal mixture and a well-stirred reactor, then the macroscopic rate equations of system (1) takes the following form:

$$\begin{cases} \frac{dv_1(t)}{dt} = b_1 v_1(t) - \mu_{-1} v_1^2(t) - v_1(t) v_2(t) - v_1(t) v_3(t), \\ \frac{dv_2(t)}{dt} = v_1(t) v_2(t) - b_5 v_2(t), \\ \frac{dv_3(t)}{dt} = b_4 v_3(t) - v_1(t) v_3(t) - \mu_{-5} v_3^2(t), \end{cases} \tag{2}$$

where $v_1(t), v_2(t)$ and $v_3(t)$ represent the mole fractions of $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ at the time t, The rate constants μ_1, μ_3 and μ_5 are incorporated in the parameters b_1, b_5 and b_4 (e.g., $b_1 = \mu_1[\mathcal{A}_1], \cdots$) and $b_1 > 0, b_4 > 0, b_5 > 0, \mu_{-1} > 0, \mu_{-5} > 0$ represent constants. For more relation information on system (2), one can see [11, 12]. In 2015, Xu and Wu [36] explored the control of chaos of system (2) by using delayed feedback control approach, i.e., they investigated the controlled chemical system as follows:

$$\begin{cases} \dfrac{dv_1(t)}{dt} = b_1 v_1(t) - \mu_{-1} v_1^2(t) - v_1(t)v_2(t) - v_1(t)v_3(t) \\ \qquad\qquad + \nu_1[v_1(t) - v_1(t-\rho)], \\ \dfrac{dv_2(t)}{dt} = v_1(t)v_2(t) - b_5 v_2(t) + \nu_2[v_2(t) - v_2(t-\rho)], \\ \dfrac{dv_3(t)}{dt} = b_4 v_3(t) - v_1(t)v_3(t) - k_{-5} v_3^2(t) + \nu_3[v_3(t) - v_3(t-\rho)], \end{cases} \tag{3}$$

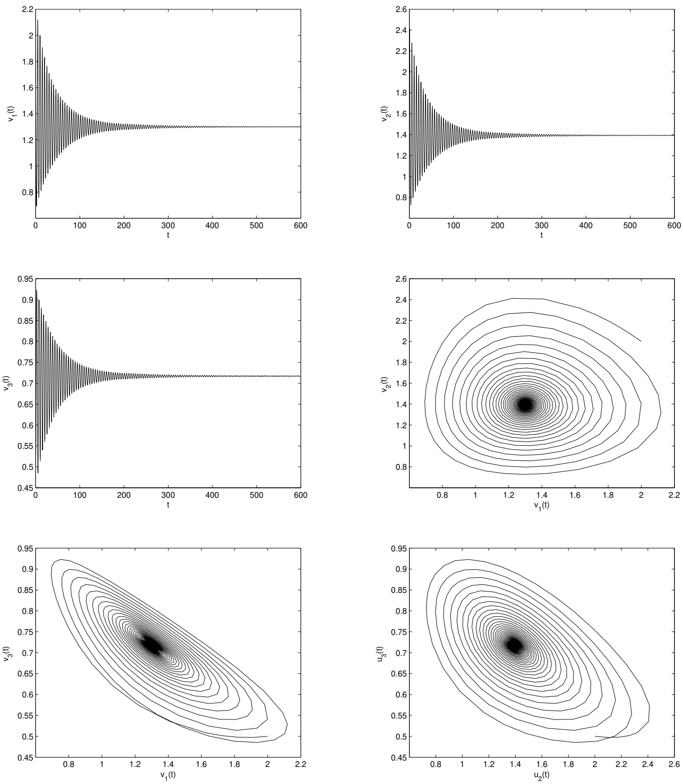
where $\nu_i (i = 1, 2, 3)$ represents a real constant and ρ represents a time delay.

Here we would like to point out that all the works mentioned above (see, [11, 12, 36]) focused on integer-order chemical systems. Studies in recent decades manifest that the fractional-order dynamical model has been regarded as a more efficient implementation to characterize practical phenomena than the conventional integer-order ones since the fractional-order dynamical model can display immense advantages in keeping memory and hereditary properties of a lot of materials and change process [15, 19, 22, 32, 34, 37, 39]. Nowadays, fractional-order dynamical models have been widely applied in many fields such as neural networks, control engineering, physical waves, biology, chemistry, economics and so forth [20,32,41]. A great deal of valuable results have been reported. For instance, Ghanbari and Djilali [13] carried out a Hopf bifurcation analysis for a fractional-order predator-prey system. Sekerci [23] reveled the climate change effects on the dynamics of a fractional-order predator-prey model. Wang et al. [28] probed into the stability and Hopf bifurcation for a generalized fractional-order delayed prey-predator system. Huang et al. [16] explored the Hopf bifurcation issue of fractional-order neural networks with leakage delays. For more related studies, one can see [1,2,15,24,33,35,38].

Inspired by the exploration above and on the basis of system (2), in order to describe the continuous change process of the mole fractions of $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ and characterize the memory trait and hereditary property of the variables $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$, we modify system (2) as the following fractional-order form:

$$
\begin{cases}
\dfrac{d^{\varsigma}v_1(t)}{dt^{\varsigma}} = b_1 v_1(t) - \mu_{-1} v_1^2(t) - v_1(t)v_2(t) - v_1(t)v_3(t), \\
\dfrac{d^{\varsigma}v_2(t)}{dt^{\varsigma}} = v_1(t)v_2(t) - b_5 v_2(t), \\
\dfrac{d^{\varsigma}v_3(t)}{dt^{\varsigma}} = b_4 v_3(t) - v_1(t)v_3(t) - \mu_{-5} v_3^2(t),
\end{cases}
\tag{4}
$$

where $\varsigma \in (0,1]$. The investigation shows that when $\varsigma = 0.94, b_1 = 2.5, \mu_{-1} = 0.3, b_5 = 1.3, b_4 = 3.2, \mu_{-5} = 2.65$, then system (4) displays a stable state which means that the three constituents $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}$ will tend to three different fixed real numbers with the increase of time t. The MATLAB simulation plots are given in Figure 1.



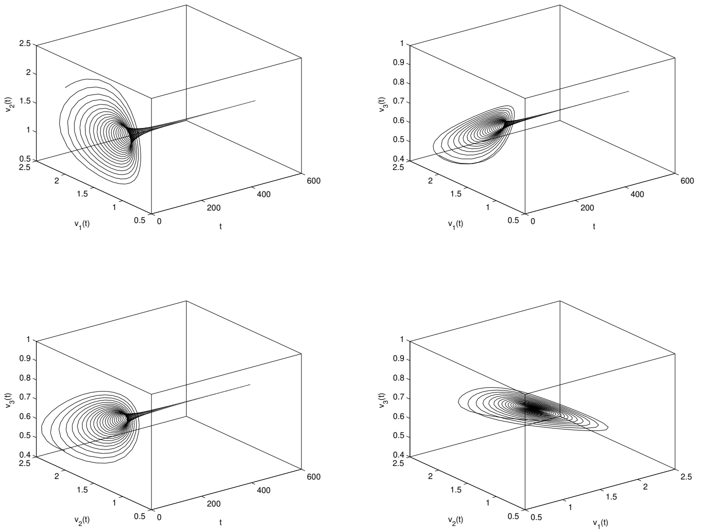


Figure 1. Software simulation figures of system (4) with $\varsigma = 0.94, b_1 = 2.5, \mu_{-1} = 0.3, b_5 = 1.3, b_4 = 3.2, \mu_{-5} = 2.65$.

In many cases, we expect to make the three constituents $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}$ of model (4) produce the periodic cycle state in chemical reaction. Mathematically, it involves the Hopf bifurcation anti-control. Hopf bifurcation anti-control is to design an appropriate controller to make the differential systems produce a family of periodic solutions around the equilibrium point. During the past decades, Hopf bifurcation anti-control has aroused much interest from numerous researchers due to the great application prospect in maintaining the balanced state of all variables of differential systems. There is some literature on the Hopf bifurcation anti-control issue of some differential models(e.g. [3–6, 29, 30, 40, 42]).

It is a pity that all the works above (see [3–6, 29, 30, 40, 42]) are only concerned with the Hopf bifurcation anti-control of integer-order differential models. Up to now, only very few works on the Hopf bifurcation anti-control of fractional-order differential models. This motivates us to explore this aspect.

In the present research, we shall focus on Hopf bifurcation anti-control of the fractional-order stable chemical reaction system (4) via a hybrid controller including state feedback and parameter perturbation. The chief highlights of this research lie in the following points:

• Based on the previous publications, a novel fractional-order stable chemical reaction system is proposed.

• The fractional-order stable chemical reaction system displays the Hopf bifurcation via a suitable hybrid controller including state feedback and parameter perturbation.

• The research method can be applied to control or anti-control Hopf bifurcation for many fractional-order differential systems in lots of areas.

This work is planned as follows. Some pre-requisite knowledge about fractional-order dynamical system is present in Section 2. In Section 3, we analyze the existence and uniqueness, non-negativeness and uniformly boundedness of the solution to model (4). In Section 4, a suitable hybrid controller including state feedback and parameter perturbation is effectively designed to make the fractional-order stable chemical reaction system (4) generate Hopf bifurcation. In Section 5, software simulation results are presented to sustain the established conclusions. Section 6 completes this article.

2 Basic theory

In this section, we present some related definitions and lemmas on fractional-order differential system.

Definition 2.1. [22] *The fractional integral of order ς of the function $h(\varrho)$ is defined as follows:*

$$\mathcal{I}^{\varsigma}h(\varrho) = \frac{1}{\Gamma(\varsigma)}\int_{\varrho_0}^{\varrho}(\varrho - s)^{\varsigma-1}h(s)ds,$$

where $\varrho \geq \varrho_0, \varsigma > 0$, and $\Gamma(s) = \int_0^{\infty}\varrho^{s-1}e^{-\varrho}d\varrho$ denotes the Gamma function.

Definition 2.2. [3] *The Caputo fractional order derivative of order ς of* *the function* $h(\varrho) \in ([\varrho_0, \infty), R)$ *is given by:*

$$\mathcal{D}^{\varsigma} h(\varrho) = \frac{1}{\Gamma(k-\varsigma)} \int_{\varrho_0}^{\varrho} \frac{h^{(k)}(s)}{(\varrho - s)^{\varsigma-k+1}} ds,$$

where $\varrho \geq \varrho_0$ *and* k *stands for a positive integer* ($\varsigma \in [k-1, k)$). *Especially, when* $\varsigma \in (0,1)$, *then*

$$\mathcal{D}^{\varsigma} h(\varrho) = \frac{1}{\Gamma(1-\varsigma)} \int_{\varrho_0}^{\varrho} \frac{h'(s)}{(\varrho - s)^{\varsigma}} ds.$$

Definition 2.3. **[15]** *Consider the following system:*

$$\frac{dv_l^{\varsigma}(t)}{dt^{\varsigma}} = h_l(v_l(t)), l = 1, 2, \cdots, k, \tag{5}$$

where $\varsigma \in (0,1], v_l(t) = (v_1(t), v_2(t), \cdots, v_k(t)), h_l(t) = (h_1(t), h_2(t), \cdots, h_k(t))$. *Then* $(v_1^*, v_2^*, \cdots, v_k^*)$ *is the equilibrium point of system (5) if* $h_l(v_l^*) = 0$.

Lemma 2.1. **[19]** *Consider the fractional-order system* $\mathcal{D}^{\varsigma} v = \mathcal{Q} v, v(0) = v_0$ *where* $0 < \varsigma < 1, v \in R^k, \mathcal{Q} \in R^{k \times k}$. *Denote* $\chi_l (l = 1, 2, \cdots, k)$ *the root of the characteristic equation of* $\mathcal{D}^{\varsigma} v = \mathcal{Q} v$. *Then system* $\mathcal{D}^{\varsigma} v = \mathcal{Q} v$ *is asymptotically stable if and only if* $|arg(\chi_l)| > \frac{\varsigma\pi}{2} (l = 1, 2, \cdots, k)$. *Besides, this system is stable if and only if* $|arg(\chi_j)| > \frac{\varsigma\pi}{2} (l = 1, 2, \cdots, k)$ *and all critical eigenvalues satisfying* $|arg(\chi_l)| = \frac{\varsigma\pi}{2} (l = 1, 2, \cdots, k)$ *have geometric multiplicity one.*

Lemma 2.2. **[10]** *Consider the fractional-order system* $\mathcal{D}^{\varsigma} v(t) = \mathcal{Q}_1 v(t) + \mathcal{Q}_2 v(t-\rho)$, *where* $v(t) = \omega(t), t \in [-\rho, 0], \varsigma \in (0,1], v \in R^m, \mathcal{Q}_1, \mathcal{Q}_2 \in R^{m \times m}, \varsigma \in R^{+(m \times m)}$. *Then the characteristic equation of the system is* $\det |s^{\varsigma} \mathcal{I} - \mathcal{Q}_1 - \mathcal{Q}_2 e^{-s\rho}| = 0$. *Then the zero solution of the system is asymptotically stable if all roots of the equation* $\det |s^{\varsigma} \mathcal{I} - \mathcal{Q}_1 - \mathcal{Q}_2 e^{-s\rho}| = 0$ *possess negative real part.*

Lemma 2.3. **[21]** *Let* $\varsigma \in (0,1], g(t) \in C[\xi_1, \xi_2]$ *and* $\mathcal{D}^{\varrho} g(t) \in C[\alpha_1, \alpha_2]$. *If* $\mathcal{D}^{\varsigma} h(t) \geq 0, t \in (\xi_1, \xi_2)$, *then* $g(t)$ *is a non-decreasing function for* $t \in [\xi_1, \xi_2]$. *If* $\mathcal{D}^{\varsigma} g(t) \leq 0, t \in (\xi_1, \xi_2)$, *then* $g(t)$ *is a non-increasing function for* $t \in [\xi_1, \xi_2]$.

Lemma 2.4. [18] *Suppose that* $\phi(t) \in C[t_0, \infty)$ *and satisfies*

$$\begin{cases} D^{\varsigma}\phi(t) \leq -\kappa_1\phi(t) + \kappa_2, \\ \phi(t_0) = \phi_{t_0}, \end{cases}$$

where $\varsigma \in (0,1), \kappa_1, \kappa_2 \in R, \kappa_1 \neq 0, t_0 \geq 0,$ *then*

$$\phi(t) \leq \left(\phi(t_0) - \frac{\kappa_2}{\kappa_1}\right) E_{\varsigma}[-\kappa_1(t-t_0)^{\varsigma}] + \frac{\kappa_2}{\kappa_1}.$$

3 Existence and uniqueness, non-negativeness and uniformly boundedness

In this section, we will prove the existence and uniqueness, non-negativeness, boundedness of the solution of system (4) by virtue of fixed point theorem, mathematical inequity skill and an appropriate function.

Theorem 3.1. *Denote* $\Theta = \{v_1, v_2, v_3) \in R^3 : \max\{|v_1|, |v_2|, |v_3|\} \leq \mathcal{H}\}$, *where* $\mathcal{H} > 0$ *represents a constant. For every* $(v_{10}, v_{20}, v_{30}) \in \Theta$, *model (4) with the initial value* (v_{10}, v_{20}, v_{30}) *has a unique solution* $V = (v_1, v_2, v_3) \in \Theta$.

Proof Define the mapping as follows:

$$g(V) = (g_1(V), g_2(V), g_3(V)), \tag{6}$$

where

$$\begin{cases} g_1(V) = b_1v_1(t) - \mu_{-1}v_1^2(t) - v_1(t)v_2(t) - v_1(t)v_3(t), \\ g_2(V) = v_1(t)v_2(t) - b_5v_2(t), \\ g_3(V) = b_4v_3(t) - v_1(t)v_3(t) - \mu_{-5}v_3^2(t). \end{cases} \tag{7}$$

$\forall\ V, \bar{V} \in \Theta$, one derives

$$\begin{aligned} &||g(V) - g(\bar{V})|| \\ &= |b_1v_1(t) - \mu_{-1}v_1^2(t) - v_1(t)v_2(t) - v_1(t)v_3(t) \\ &\quad - [b_1\bar{v}_1(t) - \mu_{-1}\bar{v}_1^2(t) - \bar{v}_1(t)\bar{v}_2(t) - \bar{v}_1(t)\bar{v}_3(t)]| \end{aligned}$$

$$
\begin{aligned}
&+ |v_1(t)v_2(t) - b_5v_2(t) - [\bar{v}_1(t)\bar{v}_2(t) - b_5\bar{v}_2(t)]| \\
&+ |b_4v_3(t) - v_1(t)v_3(t) - \mu_{-5}v_3^2(t) \\
&- [b_4\bar{v}_3(t) - \bar{v}_1(t)\bar{v}_3(t) - \mu_{-5}\bar{v}_3^2(t)]| \\
\leq\ & b_1|v_1(t) - \bar{v}_1(t)| + 2\mu_{-1}\mathcal{H}|v_1(t) - \bar{v}_1(t)| + \mathcal{H}|v_1(t) - \bar{v}_1(t)| \\
&+ \mathcal{H}|v_2(t) - \bar{v}_2(t)| + \mathcal{H}|v_1(t) - \bar{v}_1(t)| + \mathcal{H}|v_3(t) - \bar{v}_3(t)| \\
&+ \mathcal{H}|v_1(t) - \bar{v}_1(t)| + \mathcal{H}|v_2(t) - \bar{v}_2(t)| + b_5|v_2(t) - \bar{v}_2(t)| \\
&+ b_4|v_3(t) - \bar{v}_3(t)| + \mathcal{H}|v_1(t) - \bar{v}_1(t)| + \mathcal{H}|v_3(t) - \bar{v}_3(t)| \\
&+ 2\mu_{-5}\mathcal{H}|v_3(t) - \bar{v}_3(t)| \\
=\ & [b_1 + (2\mu_{-1} + 4)\mathcal{H}|v_1(t) - \bar{v}_1(t)| \\
&+ (b_5 + 2\mathcal{H})\mathcal{H}|v_2(t) - \bar{v}_2(t)| \\
&+ (b_4 + 2\mathcal{H})|v_3(t) - \bar{v}_3(t)|,
\end{aligned}
$$

then

$$||g(V) - g(\bar{V})|| \leq \mathcal{G}||V - \bar{V}||, \tag{8}$$

where

$$\mathcal{G} = \max\{b_1 + (2\mu_{-1} + 4)\mathcal{H}, b_5 + 2\mathcal{H}, b_4 + 2\mathcal{H}\}. \tag{9}$$

Then $g(V)$ obeys Lipschitz condition with respect to V (see [18]). In view of fixed point theorem, we can conclude that Theorem 3.1 is true. ■

Theorem 3.2. *(1) All solutions of model (4) beginning with R_+^3 are non-negative; (2) If $\mu_{-1} > 1, \mu_{-5} > 1$ hold, then all solutions of system (4) starting with R_+^3 are uniformly bounded.*

Proof Let the initial condition of system (4) be $V(t_0) = (v_1(t_0), v_2(t_0), v_3(t_0))$. Assume that there exists a constant $t_\star$ satisfying $t \in (t_0, t_\star)$ such that

$$
\begin{cases}
v_1(t) = 0, t \in (t_0, t_\star), \\
v_1(t_\star) = 0, \\
v_1(t_\star^+) < 0.
\end{cases} \tag{10}
$$

By system (4), we get

$$\mathcal{D}^\varsigma v_1(t)|_{v_1(t_\star)=0} = 0. \tag{11}$$

By Lemma 1 of [9], one has $v_1(t_\star^+) = 0$, which contradicts (10). Thus

$v_1(t) \geq 0, \ \forall \ t \geq t_0$. In a same way, we can also prove that $v_2(t) \geq 0, v_3(t) \geq 0, \ \forall \ t \geq t_0$. The proof of (1) finishes. ■

Let

$$W(t) = v_1(t) + v_2(t) + v_3(t). \tag{12}$$

Then

$$\begin{aligned}
\mathcal{D}^{\varsigma} W(t) + b_5 W(t) &= \mathcal{D}^{\varsigma} v_1(t) + \mathcal{D}^{\varsigma} v_2(t) + \mathcal{D}^{\varsigma} v_3(t) \\
&\quad + b_5 v_1(t) + b_5 v_2(t) + b_5 v_3(t) \\
&= b_1 v_1(t) - \mu_{-1} v_1^2(t) - v_1(t)v_2(t) - v_1(t)v_3(t) \\
&\quad + v_1(t)v_2(t) - b_5 v_2(t) + b_4 v_3(t) - v_1(t)v_3(t) \\
&\quad - \mu_{-5} v_3^2(t) + b_5 v_1(t) + b_5 v_2(t) + b_5 v_3(t) \\
&= (b_1 + b_5) v_1(t) - \mu_{-1} v_1^2(t) - 2 v_1(t) v_3(t) \\
&\quad + (b_4 + b_5) v_3(t) - \mu_{-5} v_3^2(t) \\
&\leq (b_1 + b_5) v_1(t) - \mu_{-1} v_1^2(t) + v_1^2(t) + v_3^2(t) \\
&\quad + (b_4 + b_5) v_3(t) - \mu_{-5} v_3^2(t) \\
&= (b_1 + b_5) v_1(t) - (\mu_{-1} - 1) v_1^2(t) \\
&\quad + (b_4 + b_5) v_3(t) - (\mu_{-5} - 1) v_3^2(t),
\end{aligned}$$

then

$$\mathcal{D}^{\varsigma} W(t) + b_5 W(t) \leq \frac{b_1 + b_5}{4(\mu_{-1} - 1)} + \frac{b_4 + b_5}{4(\mu_{-5} - 1)}. \tag{13}$$

By Lemma 2.4, we get

$$W(t) \to \frac{b_1 + b_5}{4b_5(\mu_{-1} - 1)} + \frac{b_4 + b_5}{4b_5(\mu_{-5} - 1)}, \text{ as } t \to \infty. \tag{14}$$

The proof Theorem 3.2 completes. ■

4 Bifurcation anti-control via hybrid controller

In this section, we will study the Hopf bifurcation anti-control issue of the fractional-order chemical reaction system (4). Let $(v_{1\star}, v_{2\star}, v_{3\star})$ be the equilibrium point of the fractional-order chemical reaction system (4), then

$$\begin{cases} b_1 v_{1\star} - \mu_{-1} v_{1\star}^2 - v_{1\star} v_{2\star} - v_{1\star} v_{3\star} = 0, \\ v_{1\star} v_{2\star} - b_5 v_{2\star} = 0, \\ b_4 v_{3\star} - v_{1\star} v_{3\star} - \mu_{-5} v_{3\star}^2 = 0. \end{cases} \tag{15}$$

If the following condition

$$(A_1) \quad b_4 > b_5, (b_1 - \mu_{-1})\mu_{-5} > b_4 - b_5$$

holds, then system (4) has the unique positive equilibrium point $E(v_{1\star}, v_{2\star}, v_{3\star})$, where

$$\begin{cases} v_{1\star} = b_5, \\ v_{2\star} = \dfrac{(b_1 - \mu_{-1} b_5)\mu_{-5} - b_4 + b_5}{\mu_{-5}}, \\ v_{3\star} = \dfrac{b_4 - b_5}{\mu_{-5}}. \end{cases} \tag{16}$$

Following the idea of [51,52], we get the following fractional-order controlled chemical reaction system:

$$\begin{cases} \dfrac{d^\varsigma v_1(t)}{dt^\varsigma} = \alpha[b_1 v_1(t) - \mu_{-1} v_1^2(t) - v_1(t) v_2(t) - v_1(t) v_3(t)] \\ \qquad\qquad + \beta[v_1(t) - v_1(t-\rho)], \\ \dfrac{d^\varsigma v_2(t)}{dt^\varsigma} = \alpha[v_1(t) v_2(t) - b_5 v_2(t)] + \beta[v_2(t) - v_2(t-\rho)], \\ \dfrac{d^\varsigma v_3(t)}{dt^\varsigma} = \alpha[b_4 v_3(t) - v_1(t) v_3(t) - \mu_{-5} v_3^2(t)] \\ \qquad\qquad + \beta[v_3(t) - v_3(t-\rho)], \end{cases} \tag{17}$$

where α, β are feedback gain parameters. System (17) has the same equilibrium points as those in system (4). Namely, the equilibrium point is $E(v_{1\star}, v_{2\star}, v_{3\star})$. Let

$$\begin{cases} \bar{v}_1(t) = v_1(t) - v_{1\star}, \\ \bar{v}_2(t) = v_2(t) - v_{2\star}, \\ \bar{v}_3(t) = v_3(t) - v_{3\star}, \end{cases} \tag{18}$$

then system (17) can be rewritten as

$$\begin{cases} \dfrac{d^{\varsigma}\bar{v}_1(t)}{dt^{\varsigma}} = \alpha[b_1(\bar{v}_1(t) + v_{1\star}) - \mu_{-1}(\bar{v}_1(t) \\ \qquad + v_{1\star})^2 - (\bar{v}_1(t) + v_{1\star})(\bar{v}_2(t) + v_{2\star}) \\ \qquad - (\bar{v}_1(t) + v_{1\star})(\bar{v}_3(t) + v_{3\star})] + \beta[\bar{v}_1(t) - \bar{v}_1(t-\rho)], \\ \dfrac{d^{\varsigma}\bar{v}_2(t)}{dt^{\varsigma}} = \alpha[(\bar{v}_1(t) + v_{1\star})(\bar{v}_2(t) + v_{2\star}) - b_5(\bar{v}_2(t) + v_{2\star})] \\ \qquad + \beta[\bar{v}_2(t) - \bar{v}_2(t-\rho)], \\ \dfrac{d^{\varsigma}\bar{v}_3(t)}{dt^{\varsigma}} = \alpha[b_4(\bar{v}_3(t) + v_{3\star}) - (\bar{v}_1(t) + v_{1\star})(\bar{v}_3(t) + v_{3\star}) \\ \qquad - \mu_{-5}(\bar{v}_3(t) + v_{3\star})^2] + \beta[\bar{v}_3(t) - \bar{v}_3(t-\rho)]. \end{cases} \tag{19}$$

The linear system of (19) around $(0, 0, 0)$ is given by

$$\begin{cases} \dfrac{d^{\varsigma}\bar{v}_1(t)}{dt^{\varsigma}} = c_1\bar{v}_1(t) + c_2\bar{v}_2(t) + c_2\bar{v}_3(t) + c_3\bar{v}_1(t-\rho), \\ \dfrac{d^{\varsigma}\bar{v}_2(t)}{dt^{\varsigma}} = c_4\bar{v}_1(t) + c_5\bar{v}_2(t) + c_3\bar{v}_2(t-\rho), \\ \dfrac{d^{\varsigma}\bar{v}_3(t)}{dt^{\varsigma}} = c_6\bar{v}_1(t) + c_7\bar{v}_3(t) + c_3\bar{v}_3(t-\rho). \end{cases} \tag{20}$$

where

$$\begin{cases} c_1 = 2b_1 - 2\alpha\mu_{-1}v_{1\star} - v_{2\star} - v_{3\star} + \beta, \\ c_2 = -v_{1\star}, \\ c_3 = -\beta, \\ c_4 = \alpha v_{2\star}, \\ c_5 = \alpha v_{1\star} - b_5 + \beta, \\ c_6 = -\alpha v_{3\star}, \\ c_7 = \alpha b_4 - \alpha v_{1\star} - 2v_{3\star}\mu_{-5} + \beta. \end{cases} \tag{21}$$

Denote $\bar{v}_i (i = 1, 2, 3)$ by v_i in (20), then system (20) takes the form:

$$\begin{cases} \dfrac{d^\varsigma v_1(t)}{dt^\varsigma} = c_1 v_1(t) + c_2 v_2(t) + c_2 v_3(t) + c_3 v_1(t-\rho), \\ \dfrac{d^\varsigma v_2(t)}{dt^\varsigma} = c_4 v_1(t) + c_5 v_2(t) + c_3 v_2(t-\rho), \\ \dfrac{d^\varsigma v_3(t)}{dt^\varsigma} = c_6 v_1(t) + c_7 v_3(t) + c_3 v_3(t-\rho). \end{cases} \tag{22}$$

The characteristic equation of system (22) is

$$\det \begin{bmatrix} s^\varsigma - c_1 - c_3 e^{-s\rho} & -c_2 & -c_2 \\ -c_4 & s^\varsigma - c_5 - c_3 e^{-s\rho} & 0 \\ -c_6 & 0 & s^\varsigma - c_7 - c_3 e^{-s\rho} \end{bmatrix} = 0, \tag{23}$$

which leads to

$$s^{3\varsigma} + a_1 s^{2\varsigma} + a_2 s^\varsigma + a_3 + (a_4 s^{2\varsigma} + a_5 s^\varsigma + a_6) e^{-s\rho}$$

$$+ (a_7 s^\varsigma + a_8) e^{-2s\rho} + a_9 e^{-3s\rho} = 0, \tag{24}$$

where

$$\begin{cases} a_1 = -(c_1 + c_5 + c_7)), \\ a_2 = c_1 c_5 + c_7(c_1 + c_5) - c_2 c_6 - c_2 c_4, \\ a_3 = c_2 c_5 c_6 + c_2 c_4 c_7 - c_1 c_5 c_7, \\ a_4 = -3c_3, \\ a_5 = 2c_3(c_1 + c_5) + 2c_3 c_7 + 2c_3^2, \\ a_6 = c_1 c_3 c_5 + c_2 c_3 c_6 + c_2 c_3 c_4 - c_1 c_3 c_7 \\ \qquad - c_3 c_5 c_7 - c_1 c_3^2 - c_3^2 c_5, \\ a_7 = c_3^2, \\ a_8 = -c_7 c_3^2, \\ a_9 = -c_3^3. \end{cases} \tag{25}$$

When $\sigma = 0$, then Eq.(24) is

$$\lambda^3 + (a_1 + a_4)\lambda^2 + (a_2 + a_5 + a_7)\lambda + a_3 + a_6 + a_8 + a_9 = 0, \tag{26}$$

Assume that

$$(A_2)\begin{cases} M_1 = a_1 + a_4 > 0, \\ M_2 = \det\begin{bmatrix} a_1 + a_4 & 1 \\ a_3 + a_6 + a_8 + a_9 & a_2 + a_5 + a_7 \end{bmatrix} > 0, \\ M_3 = (a_3 + a_6 + a_8 + a_9)M_2 > 0 \end{cases}$$

is true, then the three roots $\lambda_1, \lambda_2, \lambda_3$ of Eq. (26) owns negative real parts. Thus the equilibrium point $E(v_{1\star}, v_{2\star}, v_{3\star})$ of model (24) with $\sigma = 0$ is locally asymptotically stable.

It follows from Eq. (24) that

$$\begin{aligned} &(s^{3\varsigma} + a_1 s^{2\varsigma} + a_2 s^{\varsigma} + a_3)e^{s\rho} + (a_4 s^{2\varsigma} + a_5 s^{\varsigma} + a_6) \\ &+(a_7 s^{\varsigma} + a_8)e^{-s\rho} + a_9 e^{-2s\rho} = 0. \end{aligned} \tag{27}$$

Suppose that $s = i\vartheta = \vartheta\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ is the root of Eq. (27). Then it follows from (27) that

$$\begin{aligned} &\left[\vartheta^{3\varsigma}\left(\cos\frac{3\varsigma\pi}{2} + i\sin\frac{3\varsigma\pi}{2}\right) + a_1\vartheta^{2\varsigma}(\cos\varsigma\pi + i\sin\varsigma\pi)\right. \\ &\left.+a_2\vartheta^{\varsigma}\left(\cos\frac{\varsigma\pi}{2} + i\sin\frac{\varsigma\pi}{2}\right) + a_3\right](\cos\vartheta\rho + i\sin\vartheta\rho) \\ &+\left[a_4\vartheta^{2\varsigma}(\cos\varsigma\pi + i\sin\varsigma\pi) + a_5\vartheta^{\varsigma}\left(\cos\frac{\varsigma\pi}{2} + i\sin\frac{\varsigma\pi}{2}\right) + a_6\right] \\ &+\left[a_7\vartheta^{\varsigma}\left(\cos\frac{\varsigma\pi}{2} + i\sin\frac{\varsigma\pi}{2}\right) + a_8\right](\cos\vartheta\rho - i\sin\vartheta\rho) \\ &+a_9(\cos 2\vartheta\rho - i\sin 2\vartheta\rho) = 0, \end{aligned} \tag{28}$$

which leads to

$$\begin{cases} (\varepsilon_1\vartheta^{3\varsigma} + \varepsilon_2\vartheta^{2\varsigma} + \varepsilon_3\vartheta^{\varsigma} + \varepsilon_4)\cos\vartheta\rho + (\varepsilon_5\vartheta^{3\varsigma} + \varepsilon_6\vartheta^{2\varsigma} + \varepsilon_7\vartheta^{\varsigma}) \\ \qquad \times\sin\vartheta\rho + \varepsilon_8\vartheta^{2\varsigma} + \varepsilon_9\vartheta^{\varsigma} + \varepsilon_{10} = -a_9\cos 2\vartheta\rho, \\ (-\varepsilon_5\vartheta^{3\varsigma} - \varepsilon_6\vartheta^{2\varsigma} + \varepsilon_{11}\vartheta^{\varsigma})\cos\vartheta\rho + (\varepsilon_1\vartheta^{3\varsigma} + \varepsilon_2\vartheta^{2\varsigma} + \varepsilon_3\vartheta^{\varsigma} + \varepsilon_{12}) \\ \qquad \times\sin\vartheta\rho + \varepsilon_{13}\vartheta^{2\varsigma} + \varepsilon_{14}\vartheta^{\varsigma} = a_9\sin 2\vartheta\rho, \end{cases} \tag{29}$$

where

$$\begin{cases} \epsilon_1 = \cos\dfrac{3\varsigma\pi}{2}, \\ \epsilon_2 = a_1\cos\varsigma\pi, \\ \epsilon_3 = (a_2+a_7)\cos\dfrac{\varsigma\pi}{2}, \\ \epsilon_4 = a_3+a_8, \\ \epsilon_5 = -\sin\dfrac{3\varsigma\pi}{2}, \\ \epsilon_6 = -a_1\sin\varsigma\pi, \\ \epsilon_7 = (a_2-a_7)\sin\dfrac{\varsigma\pi}{2}, \\ \epsilon_8 = a_4\cos\varsigma\pi, \\ \epsilon_9 = a_5\cos\dfrac{\varsigma\pi}{2}, \\ \epsilon_{10} = a_6, \\ \epsilon_{11} = (a_2+a_7)\sin\dfrac{\varsigma\pi}{2}, \\ \epsilon_{12} = a_3-a_8, \\ \epsilon_{13} = a_4\sin\varsigma\pi, \\ \epsilon_{14} = a_5\sin\dfrac{\varsigma\pi}{2}. \end{cases} \tag{30}$$

It follows from (29) that

$$\begin{aligned} &[(\varepsilon_1\vartheta^{3\varsigma}+\varepsilon_2\vartheta^{2\varsigma}+\varepsilon_3\vartheta^{\varsigma}+\varepsilon_4)\cos\vartheta\rho+(\varepsilon_5\vartheta^{3\varsigma}+\varepsilon_6\vartheta^{2\varsigma}+\varepsilon_7\vartheta^{\varsigma})\sin\vartheta\rho \\ &+\varepsilon_8\vartheta^{2\varsigma}+\varepsilon_9\vartheta^{\varsigma}+\varepsilon_{10}]^2+[(-\varepsilon_5\vartheta^{3\varsigma}-\varepsilon_6\vartheta^{2\varsigma}+\varepsilon_{11}\vartheta^{\varsigma})\cos\vartheta\rho \\ &+(\varepsilon_1\vartheta^{3\varsigma}+\varepsilon_2\vartheta^{2\varsigma}+\varepsilon_3\vartheta^{\varsigma}+\varepsilon_{12})\sin\vartheta\rho+\varepsilon_{13}\vartheta^{2\varsigma}+\varepsilon_{14}\vartheta^{\varsigma}]^2=a_9^2, \end{aligned} \tag{31}$$

which generates

$$\begin{aligned} &(\varrho_1\vartheta^{4\varsigma}+\varrho_2\vartheta^{3\varsigma}+\varrho_3\vartheta^{2\varsigma}+\varrho_4\vartheta^{\varsigma}+\varrho_5)\cos^2\vartheta\rho \\ &+(\delta_1\vartheta^{4\varsigma}+\delta_2\vartheta^{3\varsigma}+\delta_3\vartheta^{2\varsigma}+\delta_4\vartheta^{\varsigma}+\delta_5)\cos\vartheta\rho\sin\vartheta\rho \\ &+(\xi_1\vartheta^{5\varsigma}+\xi_2\vartheta^{4\varsigma}+\xi_3\vartheta^{3\varsigma}+\xi_4\vartheta^{2\varsigma}+\xi_5\vartheta^{\varsigma}+\xi_6)\cos\vartheta\rho \\ &+(\eta_1\vartheta^{5\varsigma}+\eta_2\vartheta^{4\varsigma}+\eta_3\vartheta^{3\varsigma}+\eta_4\vartheta^{2\varsigma}+\eta_5\vartheta^{\varsigma})\sin\vartheta\rho \\ &=\upsilon_1\vartheta^{6\varsigma}+\upsilon_2\vartheta^{5\varsigma}+\upsilon_3\vartheta^{4\varsigma}+\upsilon_4\vartheta^{3\varsigma}+\upsilon_5\vartheta^{2\varsigma}+\upsilon_6\vartheta^{\varsigma}+\upsilon_7, \end{aligned} \tag{32}$$

where

$$
\begin{cases}
\varrho_1 = -2\varepsilon_5(\varepsilon_{11} - \varepsilon_7), \\
\varrho_2 = 2(\varepsilon_1\varepsilon_4 - \varepsilon_6\varepsilon_{11} - \varepsilon_2\varepsilon_3 - \varepsilon_6\varepsilon_7), \\
\varrho_3 = \varepsilon_{11}^2 - \varepsilon_7^2 + 2\varepsilon_2\varepsilon_4 - 2\varepsilon_2\varepsilon_{12}, \\
\varrho_4 = 2\varepsilon_3\varepsilon_4 - \varepsilon_3\varepsilon_{12}, \\
\varrho_5 = \varepsilon_4^2 - \varepsilon_{12}^2, \\
\delta_1 = \varepsilon_6^2 + \varepsilon_{13}^2 + \varepsilon_1\varepsilon_7 - \varepsilon_1\varepsilon_{11}, \\
\delta_2 = 2(\varepsilon_6\varepsilon_9 + \varepsilon_{13}\varepsilon_{14}) + \varepsilon_2\varepsilon_7 + \varepsilon_4\varepsilon_5 - \varepsilon_5\varepsilon_{12} - \varepsilon_2\varepsilon_{11}, \\
\delta_3 = \varepsilon_9^2 + \varepsilon_{14}^2 + 2\varepsilon_6\varepsilon_{10} + \varepsilon_3\varepsilon_7 + \varepsilon_4\varepsilon_6 - \varepsilon_6\varepsilon_{12} - \varepsilon_3\varepsilon_{11}, \\
\delta_4 = 2\varepsilon_9\varepsilon_{10} + \varepsilon_4\varepsilon_7 + \varepsilon_{11}\varepsilon_{12}, \\
\delta_5 = \varepsilon_{10}^2, \\
\xi_1 = 2(\varepsilon_1\varepsilon_8 - \varepsilon_5\varepsilon_{13}), \\
\xi_2 = 2(\varepsilon_1\varepsilon_9 + \varepsilon_2\varepsilon_8 - \varepsilon_3\varepsilon_{14} - \varepsilon_6\varepsilon_{13}), \\
\xi_3 = 2(\varepsilon_1\varepsilon_{10} + \varepsilon_2\varepsilon_9 + \varepsilon_3\varepsilon_8 - \varepsilon_6\varepsilon_{14} + \varepsilon_{11}\varepsilon_{13}), \\
\xi_4 = 2(\varepsilon_2\varepsilon_9 + \varepsilon_2\varepsilon_{10} + \varepsilon_4\varepsilon_8 + \varepsilon_{11}\varepsilon_{14}), \\
\xi_5 = 2(\varepsilon_3\varepsilon_{10} + \varepsilon_4\varepsilon_9), \\
\xi_6 = 2\varepsilon_4\varepsilon_{10}, \\
\eta_1 = 2(\varepsilon_3\varepsilon_8 + \varepsilon_1\varepsilon_{13}), \\
\eta_2 = 2(\varepsilon_5\varepsilon_9 + \varepsilon_6\varepsilon_8 + \varepsilon_1\varepsilon_{14} + \varepsilon_2\varepsilon_{13}), \\
\eta_3 = 2(\varepsilon_5\varepsilon_{10} + \varepsilon_6\varepsilon_9 + \varepsilon_7\varepsilon_8 + \varepsilon_2\varepsilon_{14} + \varepsilon_3\varepsilon_{13}), \\
\eta_4 = 2(\varepsilon_6\varepsilon_{10} + \varepsilon_7\varepsilon_9 + \varepsilon_{12}\varepsilon_{13} + \varepsilon_{13}\varepsilon_{14}), \\
\eta_5 = 2(\varepsilon_7\varepsilon_{10} + \varepsilon_{12}\varepsilon_{14}, \\
\upsilon_1 = -(\varepsilon_1^2 + \varepsilon_5^2), \\
\upsilon_2 = 2(\varepsilon_1\varepsilon_2 + \varepsilon_5\varepsilon_6), \\
\upsilon_3 = \varepsilon_2^2 + \varepsilon_6^2 + 2(\varepsilon_1\varepsilon_3 + \varepsilon_5\varepsilon_7), \\
\upsilon_4 = 2(\varepsilon_1\varepsilon_{12} + \varepsilon_2\varepsilon_3 + \varepsilon_6\varepsilon_7), \\
\upsilon_5 = \varepsilon_3^2 + \varepsilon_7^2 + 2\varepsilon_2\varepsilon_{12}, \\
\upsilon_6 = 2\varepsilon_3\varepsilon_{12}, \\
\upsilon_7 = a_9^2 - \varepsilon_{12}^2.
\end{cases}
\tag{33}
$$

By $\sin\vartheta\rho = \pm\sqrt{1 - \cos^2\vartheta\rho}$, then we can rewrite (32) as follows:

$$(\varrho_1\vartheta^{4\varsigma} + \varrho_2\vartheta^{3\varsigma} + \varrho_3\vartheta^{2\varsigma} + \varrho_4\vartheta^{\varsigma} + \varrho_5)\cos^2\vartheta\rho$$

$$
\begin{aligned}
&+(\delta_1\vartheta^{4\varsigma}+\delta_2\vartheta^{3\varsigma}+\delta_3\vartheta^{2\varsigma}+\delta_4\vartheta^{\varsigma}+\delta_5)\cos\vartheta\rho\left(\pm\sqrt{1-\cos^2\vartheta\rho}\right)\\
&+(\xi_1\vartheta^{5\varsigma}+\xi_2\vartheta^{4\varsigma}+\xi_3\vartheta^{3\varsigma}+\xi_4\vartheta^{2\varsigma}+\xi_5\vartheta^{\varsigma}+\xi_6)\cos\vartheta\rho\\
&+(\eta_1\vartheta^{5\varsigma}+\eta_2\vartheta^{4\varsigma}+\eta_3\vartheta^{3\varsigma}+\eta_4\vartheta^{2\varsigma}+\eta_5\vartheta^{\varsigma})\left(\pm\sqrt{1-\cos^2\vartheta\rho}\right)\\
&=\upsilon_1\vartheta^{6\varsigma}+\upsilon_2\vartheta^{5\varsigma}+\upsilon_3\vartheta^{4\varsigma}+\upsilon_4\vartheta^{3\varsigma}+\upsilon_5\vartheta^{2\varsigma}+\upsilon_6\vartheta^{\varsigma}+\upsilon_7. \qquad (34)
\end{aligned}
$$

It follows from (34) that

$$\theta_1\cos^4\vartheta\rho+\theta_2\cos^3\vartheta\rho+\theta_3\cos^2\vartheta\rho+\theta_4\cos\vartheta\rho+\theta_5=0, \qquad (35)$$

where

$$
\begin{cases}
\theta_1=(\varrho_1\vartheta^{4\varsigma}+\varrho_2\vartheta^{3\varsigma}+\varrho_3\vartheta^{2\varsigma}+\varrho_4\vartheta^{\varsigma}+\varrho_5)^2\\
\qquad+(\delta_1\vartheta^{4\varsigma}+\delta_2\vartheta^{3\varsigma}+\delta_3\vartheta^{2\varsigma}+\delta_4\vartheta^{\varsigma}+\delta_5)^2,\\
\theta_2=2(\varrho_1\vartheta^{4\varsigma}+\varrho_2\vartheta^{3\varsigma}+\varrho_3\vartheta^{2\varsigma}+\varrho_4\vartheta^{\varsigma}+\varrho_5)\\
\qquad\times(\xi_1\vartheta^{5\varsigma}+\xi_2\vartheta^{4\varsigma}+\xi_3\vartheta^{3\varsigma}+\xi_4\vartheta^{2\varsigma}+\xi_5\vartheta^{\varsigma}+\xi_6)\\
\qquad+2(\delta_1\vartheta^{4\varsigma}+\delta_2\vartheta^{3\varsigma}+\delta_3\vartheta^{2\varsigma}+\delta_4\vartheta^{\varsigma}+\delta_5)\\
\qquad\times(\eta_1\vartheta^{5\varsigma}+\eta_2\vartheta^{4\varsigma}+\eta_3\vartheta^{3\varsigma}+\eta_4\vartheta^{2\varsigma}+\eta_5\vartheta^{\varsigma}),\\
\theta_3=(\xi_1\vartheta^{5\varsigma}+\xi_2\vartheta^{4\varsigma}+\xi_3\vartheta^{3\varsigma}+\xi_4\vartheta^{2\varsigma}+\xi_5\vartheta^{\varsigma}+\xi_6)^2\\
\qquad+(\eta_1\vartheta^{5\varsigma}+\eta_2\vartheta^{4\varsigma}+\eta_3\vartheta^{3\varsigma}+\eta_4\vartheta^{2\varsigma}+\eta_5\vartheta^{\varsigma})^2\\
\qquad-(\delta_1\vartheta^{4\varsigma}+\delta_2\vartheta^{3\varsigma}+\delta_3\vartheta^{2\varsigma}+\delta_4\vartheta^{\varsigma}+\delta_5)^2,\\
\theta_4=-2(\xi_1\vartheta^{5\varsigma}+\xi_2\vartheta^{4\varsigma}+\xi_3\vartheta^{3\varsigma}+\xi_4\vartheta^{2\varsigma}+\xi_5\vartheta^{\varsigma}+\xi_6)\\
\qquad\times(\upsilon_1\vartheta^{6\varsigma}+\upsilon_2\vartheta^{5\varsigma}+\upsilon_3\vartheta^{4\varsigma}+\upsilon_4\vartheta^{3\varsigma}+\upsilon_5\vartheta^{2\varsigma}+\upsilon_6\vartheta^{\varsigma}+\upsilon_7)\\
\qquad-2(\delta_1\vartheta^{4\varsigma}+\delta_2\vartheta^{3\varsigma}+\delta_3\vartheta^{2\varsigma}+\delta_4\vartheta^{\varsigma}+\delta_5)\\
\qquad\times(\eta_1\vartheta^{5\varsigma}+\eta_2\vartheta^{4\varsigma}+\eta_3\vartheta^{3\varsigma}+\eta_4\vartheta^{2\varsigma}+\eta_5\vartheta^{\varsigma}),\\
\theta_5=(\upsilon_1\vartheta^{6\varsigma}+\upsilon_2\vartheta^{5\varsigma}+\upsilon_3\vartheta^{4\varsigma}+\upsilon_4\vartheta^{3\varsigma}+\upsilon_5\vartheta^{2\varsigma}+\upsilon_6\vartheta^{\varsigma}+\upsilon_7)^2\\
\qquad(\eta_1\vartheta^{5\varsigma}+\eta_2\vartheta^{4\varsigma}+\eta_3\vartheta^{3\varsigma}+\eta_4\vartheta^{2\varsigma}+\eta_5\vartheta^{\varsigma})^2.
\end{cases} \qquad (36)
$$

By Matlab software, we can easily solve the value of $\cos\vartheta\rho$. Suppose that

$$\cos\vartheta\rho=l_1(\vartheta). \qquad (37)$$

Then one can get the value of $\sin\vartheta\rho$. Suppose that

$$\sin\vartheta\rho=l_2(\vartheta). \qquad (38)$$

By (37) and (38), we have

$$l_1^2(\vartheta) + l_2^2(\vartheta) = 1. \tag{39}$$

Making use of (39), we can obtain the value of ϑ (say ϑ_0). By (37), we get

$$\vartheta_i = \frac{1}{\vartheta_0}[\arccos l_1(\vartheta_0) + 2i\pi], i = 0, 1, 2, \cdots . \tag{40}$$

Define

$$\rho_\star = \min_{\{i=0,1,2,\cdots\}} \{\vartheta_i\}. \tag{41}$$

Then we can conclude that when $\rho = \rho_\star$, (24) owns a pair of imaginary roots $\pm i\vartheta_0$.

Next we make the following hypothesis:

$$(A_3) \;\; \mathcal{W}_{1R}\mathcal{W}_{2R} + \mathcal{W}_{1I}\mathcal{W}_{2I} > 0,$$

where

$$\begin{aligned}
\mathcal{W}_{1R} = {} & 3\varsigma\vartheta_0^{3\varsigma-1}\cos\frac{(3\varsigma-1)\pi}{2} + 2\varsigma a_1\vartheta_0^{2\varsigma-1}\cos\frac{(2\varsigma-1)\pi}{2} \\
& + \varsigma a_2\vartheta_0^{\varsigma-1}\cos\frac{(\varsigma-1)\pi}{2} + \left[2\varsigma a_4\vartheta_0^{2\varsigma-1}\cos\frac{(2\varsigma-1)\pi}{2}\right. \\
& \left.+\varsigma a_5\vartheta_0^{\varsigma-1}\cos\frac{(\varsigma-1)\pi}{2}\right]\cos\vartheta_0\rho_\star + \left[2\varsigma a_4\vartheta_0^{2\varsigma-1}\sin\frac{(2\varsigma-1)\pi}{2}\right. \\
& \left.+\varsigma a_5\vartheta_0^{\varsigma-1}\sin\frac{(\varsigma-1)\pi}{2}\right]\sin\vartheta_0\rho_\star \\
& + \varsigma a_7\vartheta_0^{\varsigma-1}\cos\frac{(\varsigma-1)\pi}{2}\cos 2\vartheta_0\rho_\star \\
& + \varsigma a_7\vartheta_0^{\varsigma-1}\sin\frac{(\varsigma-1)\pi}{2}\sin 2\vartheta_0\rho_\star, \\
\mathcal{W}_{1I} = {} & 3\varsigma\vartheta_0^{3\varsigma-1}\sin\frac{(3\varsigma-1)\pi}{2} + 2\varsigma a_1\vartheta_0^{2\varsigma-1}\sin\frac{(2\varsigma-1)\pi}{2} \\
& + \varsigma a_2\vartheta_0^{\varsigma-1}\sin\frac{(\varsigma-1)\pi}{2} - (2\varsigma a_4\vartheta_0^{2\varsigma-1}\cos\frac{(2\varsigma-1)\pi}{2}
\end{aligned}$$

$$
\begin{aligned}
&+\varsigma a_5\vartheta_0^{\varsigma-1}\cos\frac{(\varsigma-1)\pi}{2})\sin\vartheta_0\rho_\star+(2\varsigma a_4\vartheta_0^{2\varsigma-1}\sin\frac{(2\varsigma-1)\pi}{2}\\
&+\varsigma a_5\vartheta_0^{\varsigma-1}\sin\frac{(\varsigma-1)\pi}{2})\cos\vartheta_0\rho_\star-\varsigma a_7\vartheta_0^{\varsigma-1}\cos\frac{(\varsigma-1)\pi}{2}\sin 2\vartheta_0\rho_\star\\
&+\varsigma a_7\vartheta_0^{\varsigma-1}\sin\frac{(\varsigma-1)\pi}{2}\cos 2\vartheta_0\rho_\star,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_{2R}=&\left(a_4\vartheta_0^{2\varsigma}\cos\varsigma\pi+a_5\vartheta_0^{\varsigma}\cos\frac{\varsigma\pi}{2}+a_6\right)\vartheta_0\sin\vartheta_0\rho_\star+3a_9\vartheta_0\sin 3\vartheta_0\rho_\star\\
&-\left(a_4\vartheta_0^{2\varsigma}\sin\varsigma\pi+a_5\vartheta_0^{\varsigma}\sin\frac{\varsigma\pi}{2}\right)\vartheta_0\cos\vartheta_0\rho_\star\\
&+2\left(a_7\vartheta_0^{\varsigma}\cos\frac{\varsigma\pi}{2}+a_8\right)\vartheta_0\sin\vartheta_0\rho_\star+2a_7\vartheta_0^{\varsigma}\sin\frac{\varsigma\pi}{2}\vartheta_0\cos\vartheta_0\rho_\star,\\
\mathcal{W}_{2I}=&\left(a_4\vartheta_0^{2\varsigma}\cos\varsigma\pi+a_5\vartheta_0^{\varsigma}\cos\frac{\varsigma\pi}{2}+a_6\right)\vartheta_0\cos\vartheta_0\rho_\star+3a_9\vartheta_0\cos 3\vartheta_0\rho_\star\\
&+\left(a_4\vartheta_0^{2\varsigma}\sin\varsigma\pi+a_5\vartheta_0^{\varsigma}\sin\frac{\varsigma\pi}{2}\right)\vartheta_0\sin\vartheta_0\rho_\star\\
&-2\left(a_7\vartheta_0^{\varsigma}\cos\frac{\varsigma\pi}{2}+a_8\right)\vartheta_0\cos\vartheta_0\rho_\star+2a_7\vartheta_0^{\varsigma}\sin\frac{\varsigma\pi}{2}\vartheta_0\sin\vartheta_0\rho_\star.
\end{aligned}
$$

Lemma 4.1. *Assume that $s(\rho)=\tau_1(\rho)+i\tau_2(\rho)$ is the root of Eq. (24) near $\sigma=\sigma_\star$ such that $\tau_1(\rho_\star)=0,\tau_2(\rho_\star)=\vartheta_0$, then Re $\left(\frac{ds}{d\rho}\right)\Big|_{\rho=\rho_\star,\vartheta=\vartheta_0}>0$.*

Proof In view of Eq.(24), one gets

$$
\begin{aligned}
&\left(3\varsigma s^{3\varsigma-1}+2\varsigma a_1 s^{2\varsigma-1}+\varsigma a_2 s^{\varsigma-1}\right)\frac{ds}{d\rho}+(2\varsigma a_4 s^{2\varsigma-1}+\varsigma a_5 s^{\varsigma-1})e^{-s\rho}\frac{ds}{d\rho}\\
&-e^{-s\rho}\left(\frac{ds}{d\rho}\rho+s\right)\left(a_4 s^{2\varsigma}+a_5 s^{\varsigma}+a_6\right)+\varsigma a_7 s^{\varsigma-1}e^{-2s\rho}\frac{ds}{d\rho}\\
&-2e^{-2s\rho}\left(\frac{ds}{d\rho}\rho+s\right)(a_7 s^{\varsigma}+a_8)-3a_9e^{-3s\rho}\left(\frac{ds}{d\rho}\rho+s\right)=0. \qquad (42)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\big[3\varsigma s^{3\varsigma-1}+2\varsigma a_1 s^{2\varsigma-1}+\varsigma a_2 s^{\varsigma-1}+(2\varsigma a_4 s^{2\varsigma-1}+\varsigma a_5 s^{\varsigma-1})e^{-s\rho}\\
&-e^{-s\rho}\rho\left(a_4 s^{2\varsigma}+a_5 s^{\varsigma}+a_6\right)+\varsigma a_7 s^{\varsigma-1}e^{-2s\rho}\\
&-2e^{-2s\rho}\rho(a_7 s^{\varsigma}+a_8)-3a_9e^{-3s\rho}\rho\big]\frac{ds}{d\rho}\\
=&\ e^{-s\rho}s\left(a_4 s^{2\varsigma}+a_5 s^{\varsigma}+a_6\right)+2e^{-2s\rho}s(a_7 s^{\varsigma}+a_8)+3a_9 s e^{-3s\rho}, \qquad (43)
\end{aligned}
$$

which implies

$$\left(\frac{d\lambda}{d\sigma}\right)^{-1} = \frac{\mathcal{W}_1(\lambda)}{\mathcal{W}_2(\lambda)} - \frac{\rho}{s}, \tag{44}$$

where

$$\begin{cases} \mathcal{W}_1(\lambda) = 3\varsigma s^{3\varsigma-1} + 2\varsigma a_1 s^{2\varsigma-1} + \varsigma a_2 s^{\varsigma-1} + (2\varsigma a_4 s^{2\varsigma-1} \\ \qquad\qquad + \varsigma a_5 s^{\varsigma-1})e^{-s\rho} + \varsigma a_7 s^{\varsigma-1} e^{-2s\rho}, \\ \mathcal{W}_2(\lambda) = e^{-s\rho} s \left(a_4 s^{2\varsigma} + a_5 s^{\varsigma} + a_6\right) \\ \qquad\qquad + 2e^{-2s\rho} s \left(a_7 s^{\varsigma} + a_8\right) + 3a_9 s e^{-3s\rho}. \end{cases} \tag{45}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\sigma}\right)^{-1}\right]_{\rho=\rho_\star,\vartheta=\vartheta_0} = \mathrm{Re}\left[\frac{\mathcal{W}_1(\lambda)}{\mathcal{W}_2(\lambda)}\right]_{\rho=\rho_\star,\vartheta=\vartheta_0} = \frac{\mathcal{W}_{1R}\mathcal{W}_{2R} + \mathcal{W}_{1I}\mathcal{W}_{2I}}{\mathcal{W}_{2R}^2 + \mathcal{W}_{2I}^2}. \tag{46}$$

By $(\mathcal{A}_3)$, one derives

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\rho}\right)^{-1}\right]_{\rho=\rho_\star,\vartheta=\vartheta_0} > 0, \tag{47}$$

which ends the proof. ■

According to the analysis above, the following assertion is easily derived.

Theorem 4.1. *If $(A_1), (A_2), (A_3)$ are fulfilled, then the equilibrium point $E(v_{1\star}, v_{2\star}, v_{3\star})$ of system (17) is locally asymptotically stable for $\rho \in [0, \rho_\star)$ and a Hopf bifurcation of system (17) happens near the equilibrium point $E(v_{1\star}, v_{2\star}, v_{3\star})$ when $\rho = \rho_\star$.*

Remark 4.1. *In [11, 12, 36], the authors only dealt with the dynamics of the integer-order chemical reaction system. They were not concerned with the fractional-order case. In this study, based on the earlier works, we set up a new fractional-order chemical reaction system. The existence and uniqueness, non-negativeness, boundedness of the solution of the fractional-order chemical reaction system are considered. In addition, we discussed the Hopf bifurcation anti-control issue of the fractional-order chemical reaction system. The research method is different from that in [11, 12, 36]. Thus we argue that the obtained results of this article are* *completely new and supplement the works of [11, 12, 36] to some extent.*

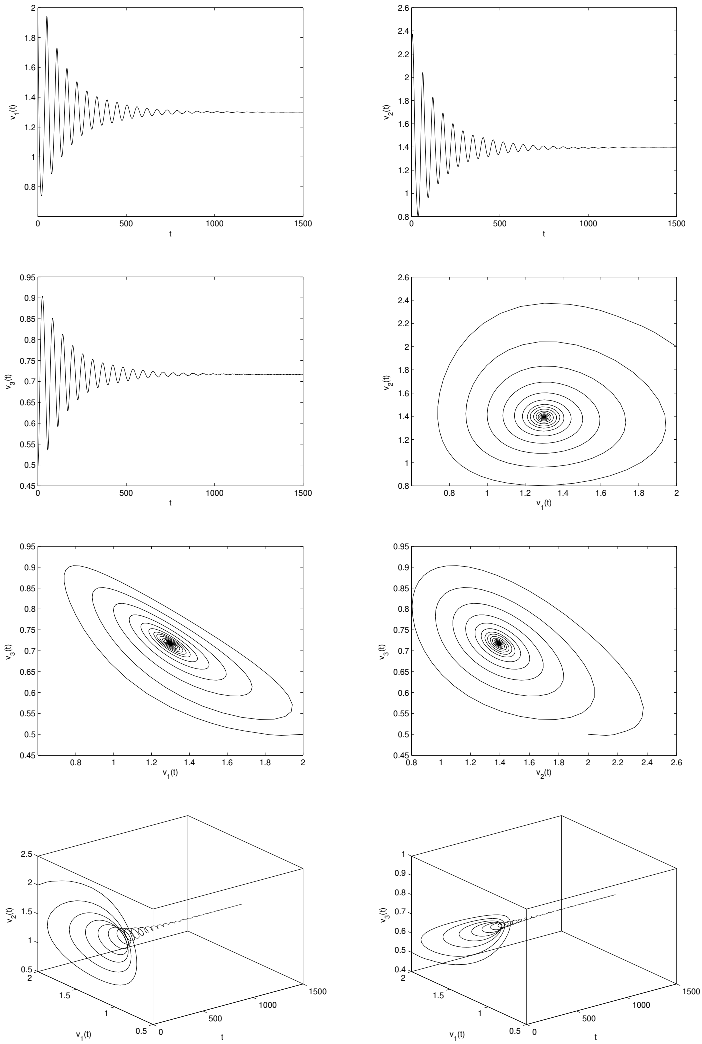
5 Numerical experiment

Consider the following fractional-order chemical reaction system:

$$\begin{cases} \frac{d^{0.94}v_1(t)}{dt^{0.94}} = \alpha[2.5v_1(t) - 0.3v_1^2(t) - v_1(t)v_2(t) - v_1(t)v_3(t)] \\ \qquad\qquad + \beta[v_1(t) - v_1(t-\rho)], \\ \frac{d^{0.94}v_2(t)}{dt^{0.94}} = \alpha[v_1(t)v_2(t) - 1.3v_2(t)] + \beta[v_2(t) - v_2(t-\rho)], \\ \frac{d^{0.94}v_3(t)}{dt^{0.94}} = \alpha[3.2v_3(t) - v_1(t)v_3(t) - 2.65v_3^2(t)] \\ \qquad\qquad + \beta[v_3(t) - v_3(t-\rho)]. \end{cases} \tag{48}$$

Clearly, system (48) has the unique positive equilibrium point $(1.300, 1.4685, 0.7170)$. Let $\alpha = 0.2, \beta = 0.8$. Making use of computer software, one derives $\vartheta_0 = 0.4997$ and $\rho_\star = 0.38$. The hypotheses (A_1)-(A_3) of Theorem 4.1 are fulfilled. In order to verify the stability of the positive equilibrium point $(1.300, 1.4685, 0.7170)$ and the generation of Hopf bifurcation of the fractional-order chemical reaction system (48), both delay values are selected. Let $\rho = 0.32 < \rho_\star = 0.38$. Then computer simulation plots are presented in Figure 2. According to Figure 2, one knows that the equilibrium point $(1.300, 1.4685, 0.7170)$ the fractional-order chemical reaction system (48) maintains locally asymptotically stable situation. In other words, the state variables v_1, v_2, v_3 will tardily tend to the values $1.300, 1.4685, 0.7170$, respectively. Chemically speaking, it manifests that the three constituents $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}$ will tardily tend to the values $1.300, 1.4685, 0.7170$, respectively. Let $\rho = 0.45 < \rho_\star = 0.38$. Then computer simulation plots are presented in Figure 3. According to Figure 3, one knows that the positive equilibrium point $(1.300, 1.4685, 0.7170)$ the fractional-order chemical reaction system (48) maintains a periodic vibratory level in the vicinity of the positive equilibrium point $(1.300, 1.4685, 0.7170)$. In other words, the state variables v_1, v_2, v_3 will move around the values $1.300, 1.4685, 0.7170$, respectively. Chemically speaking, it manifests that the three constituents $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}$ will change near the values $1.300, 1.4685, 0.7170$, respectively.

The relation among ς, ϑ_0 and $\rho_\star$ is listed in Table 1. Furthermore, the bifurcation plots are presented to indicate that the bifurcation value is $\rho_\star = 0.38$ (see Figures 4-6).



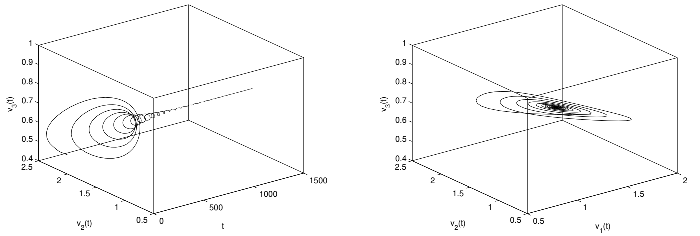
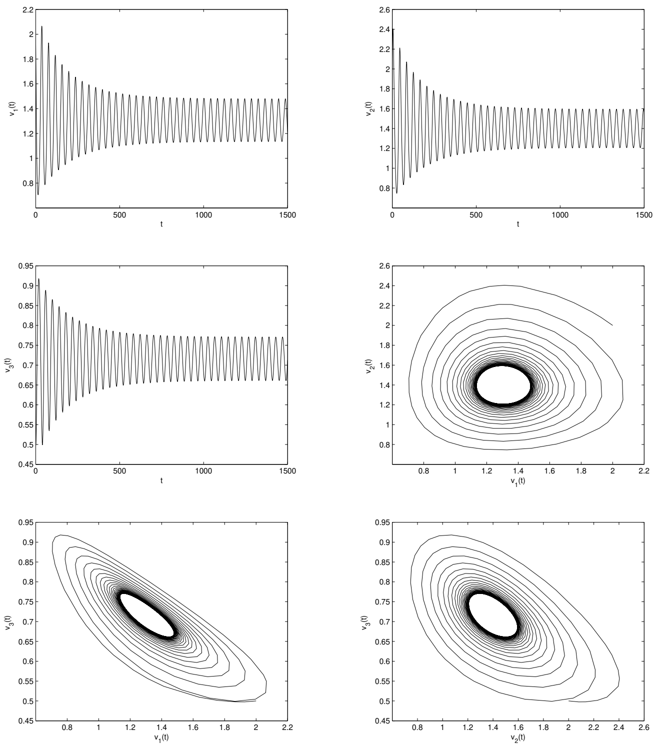


Figure 2. Numerical experiment results of the fractional-order chemical reaction system (48) with $\rho = 0.32 < \rho_\star = 0.38$. The positive equilibrium point $(1.300, 1.4685, 0.7170)$ of the fractional-order chemical reaction system (48) keeps locally asymptotically stable level. The three constituents $\mathcal{X} \to 1.300$, $\mathcal{Z} \to 1.4685$ and $\mathcal{Y} \to 0.7170$ with the increase of time t.



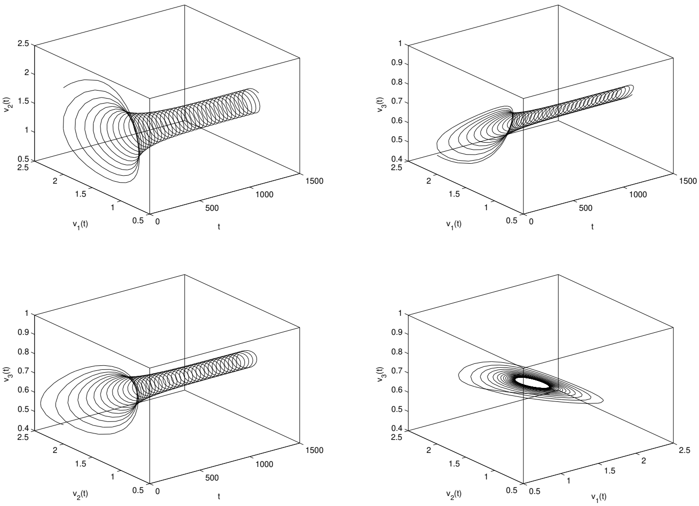


Figure 3. Numerical experiment results of the fractional-order chemical reaction system (48) with $\rho = 0.45 > \rho_\star = 0.38$. The positive equilibrium point $(1.300, 1.4685, 0.7170)$ of the fractional-order chemical reaction system (48) keeps periodic oscillatory level near $(1.300, 1.4685, 0.7170)$. the three constituents $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}$ will vibrate near the values $1.300, 1.4685, 0.7170$ with the increase of time t, respectively.

Table 1. The correlation of ς, ϑ_0 and $\rho_\star$ of the fractional-order chemical reaction system (48).

ς	ϑ_0	$\rho_\star$
0.18	0.1413	0.09
0.25	0.1984	0.13
0.33	0.2525	0.17
0.46	0.2914	0.20
0.57	0.3289	0.23
0.64	0.3888	0.28
0.77	0.4118	0.30
0.85	0.4344	0.32
0.94	0.4997	0.38

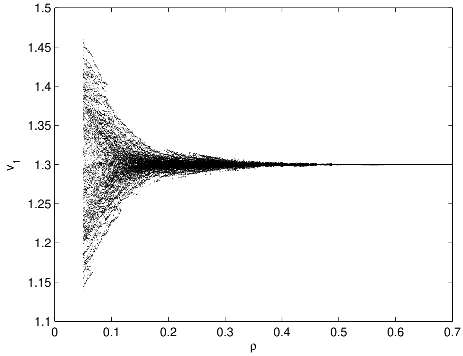


Figure 4. Bifurcation plot of the fractional-order chemical reaction system (48): the time delay ρ versus the state variable v_1. The bifurcation value is $\rho_\star = 0.38$.

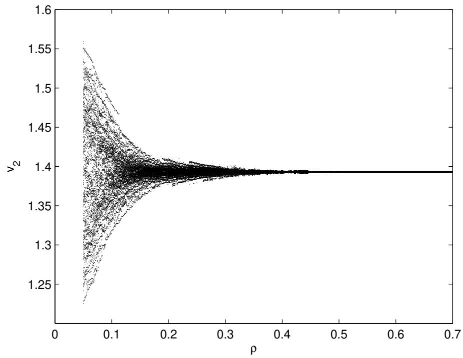


Figure 5. Bifurcation plot of the fractional-order chemical reaction system (48): the time delay ρ versus the state variable v_2. The bifurcation value is $\rho_\star = 0.38$.

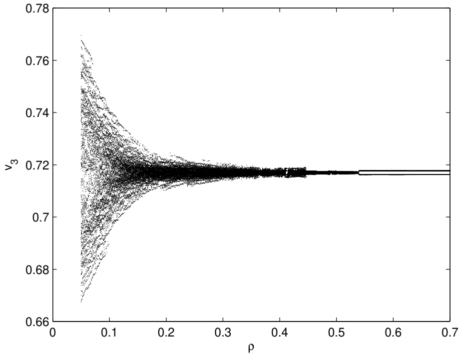


Figure 6. Bifurcation plot of the fractional-order chemical reaction system (48): the time delay ρ versus the state variable v_3. The bifurcation value is $\rho_\star = 0.38$.

6 Conclusions

Recent several decades, the fractional-order differential equation has displayed potential applications in many natural sciences. Based on the previous studies, we propose a new fractional-order chemical reaction system. We have investigated the existence and uniqueness, non-negativeness and uniformly boundedness of the solution of the fractional-order chemical reaction system. Under some parameter conditions, the fractional-order chemical reaction system always display a stable state. By virtue of a hybrid controller including state feedback and parameter perturbation, we have explored the Hopf bifurcation anti-control problem of the fractional-order stable chemical reaction system. Applying the stability and bifurcation theory of fractional-order differential equation, we obtain a new delay-independent condition that guarantees the stability and occurrence of Hopf bifurcation of the involved fractional-order stable chemical reaction model. Numerical experiments are carried out to verify the effectiveness of the designed hybrid controller. The role of the delay in the designed hybrid controller is fully revealed. The established conclusions have important theoretical value in controlling the constituents $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}$ in chemistry. Moreover, the bifurcation anti-control method can be applied to explore the bifurcation control and anti-control issue of fractional-order dynamical models in numerous areas.

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