MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **88** (2022) 767–797
ISSN: 0340–6253
doi: 10.46793/match.88-3.767Z

Matching Forcing Polynomials of Constructable Hexagonal Systems

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(Received March 1, 2022)

Abstract

Harary et al. put forward the concept on the minimum cardinality over all subsets of perfect matching M that are not included by any other ones, to be the forcing number for M. A counting polynomial for perfect matchings possessing the same forcing number was introduced by Zhang et al., using the name 'forcing polynomial'. This research deduces recurrence formulas of forcing polynomials for monotonic constructable hexagonal systems and constructable hexagonal systems with one turning. From them, a characterization of continuity of forcing spectrum for hexagonal systems with forcing edges can be derived.

1 Introduction

Given a graph G, denote $E(G)$ and $V(G)$ for its edge set and vertex set, respectively. A set M of disjoint edges of G is called *perfect matching* (or *PM* for convenience) if it covers all vertices of G. If all the edges of a cycle appear in M and $E(G) \setminus M$ alternately, then we call it M*-alternating*

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cycle. Harary et al. [6] introduced the forcing number for M, and Klein and Randić [7, 11] presented a similar chemical concept as innate degree of freedom a few years ago.

Some relevant definitions about forcing are as follows. A subset S of M is defined as its *forcing set* if it is not included by any other PMs of G. Namely $M \backslash S$ is the only PM in $G - V(S)$, where $G - V(S)$ represents for the subgraph of G deleting the ends of edges that belong to S. The minimum cardinality of all forcing sets of M is defined as the *forcing number* of M, and denoted by $f(G, M)$. If an edge forms a forcing set of G, then we call it a *forcing edge*. The *maximum* (resp. *minimum*) *forcing number* of G, denoted by $F(G)$ (resp. $f(G)$), refers to the maximum (resp. minimum) value of $f(G, M)$ over all PMs M of G. The forcing numbers of all PMs of G form its *forcing spectrum*.

Recently, a related concept called *anti-forcing number* for M in G was proposed [8]. In detail, it is the minimum cardinality over all subsets of $E(G) \setminus M$, whose removal from G makes M to be the only PM in the rest. Similar to forcing, anti-forcing edge and anti-forcing spectrum of a graph could be defined.

In this paper, we mainly talk about *hexagonal system*, which is a 2-connected plane graph such that every interior face is a regular hexagon. A hexagonal system is called *constructable*, or briefly *CHS*, if it can be dissected into $m+1$ paths $P_1, P_2, \ldots, P_{m+1}$ by parallel lines $L_1, L_2, \ldots, L_m$ that are perpendicular to some of its edges, such that P_1 and P_{m+1} are of even length, and all the other paths are of odd length. We can see that all the hexagons which intersect L_i form a linear hexagonal chain for $i = 1, 2, \ldots, m$, called the ith *row* of CHS. For convenience, we always place CHS satisfying that each L_i is horizontal, see Fig. 1.

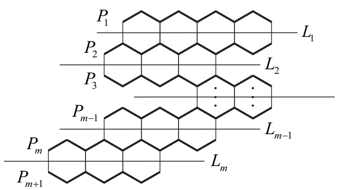

Figure 1. A CHS.

In fact, if two hexagonal systems H and H' are isomers and H has larger $F(H)$ than that of another one H', then H is more stable than H'. This is because $F(H)$ equals the *Clar number* [13], namely the maximum number of disjoint alternating hexagons. Furthermore, for any PM M with $f(H,M) = F(H)$, $f(H,M)$ equals the maximum number of disjoint M-alternating hexagons [23]. In the following, we will prove that for monotonic CHS and CHS with one turning, and an arbitrary PM M of them, $f(H,M)$ coincides with the maximum number of disjoint M-alternating hexagons. It is worth mentioning that by a similar proof as that in Ref. [16], one can see that the stated fact holds for general CHS, and general hexagonal system without coronene as nice subgraph.

Recently, some researchers focused on matching forcing and anti-forcing problems of CHS. Zhang and Deng showed that the forcing spectrum of a particular class of CHS presented in this paper, say hexagonal system with forcing edges, is an integer interval either from 1 to the Clar number or with only the gap 2 [16]; and the anti-forcing spectra of monotonic CHS and CHS with one turning are integer intervals [4]. The conclusion of forcing spectrum can be showed by its forcing polynomial in the following. If the reader want to know more conclusions about forcing number of a PM, then one can see from Refs. [2, 17, 19, 20].

For the sake of distribution for forcing number in PMs, Zhang et al. [18] introduced the *forcing polynomial* of a graph G as

$$F(G,x) = \sum_{M\in\mathcal{M}(G)} x^{f(G,M)} = \sum_{i=f(G)}^{F(G)} \omega(G,i)x^i, \tag{1}$$

where $\mathcal{M}(G)$ denotes the collection of all PMs of G, and $\omega(G,i)$ represents for the number of PMs of G with forcing number i. Furthermore, the forcing spectrum $\{f(G,M) : M \in \mathcal{M}(G)\}$ coincides with the collection of degrees of forcing polynomial $F(G,x)$. Here we make a convention that $F(G,x) = 1$ if G contains no vertices, called *empty graph*. This is because $\emptyset$ is the only PM of an empty graph, which has forcing number 0.

Afterwards, forcing polynomials for benzenoid parallelograms [22], catacondensed hexagonal systems [18], rectangle grids [21], and pyrene systems [3] were obtained.

In the following text, we will deduce forcing polynomials of two classes of CHSs. In Section 3, we focus on monotonic CHS using the preliminaries in Section 2. According to which vertical edge in the last row that belongs to PM, we classify all the PMs and give an edge subset S_M that is contained in some minimum forcing set for PM M in each class. By the variation of forcing numbers between M and $M \setminus S_M$, the forcing polynomial was obtained. Similarly, we derive the forcing polynomial of CHS with one turning in Section 4. As consequences, we get a characterization of continuity of forcing spectrum for hexagonal systems with forcing edges and forcing polynomials for some particular examples.

2 Some preliminaries

Given PM M in a graph G, we now give two equivalent definitions of forcing set, and a method of calculation on forcing number for M.

Theorem 2.1. [1, 12] *Let G be a graph possessing a PM M. A subset $S \subseteq M$ is a forcing set of M if and only if each M-alternating cycle of G intersects S.*

From the above theorem we have that the forcing number of M is bounded below by the maximum number of disjoint M-alternating cycles. Furthermore, planar bipartite graphs satisfy the lower bound with equality [10], especially for hexagonal systems.

For $S \subseteq M$, an edge e of $G - V(S)$ is said to be *forced* by $V(S)$ if it is contained in every PM of $G - V(S)$, thus belongs to M. Hence given $S \subseteq M$ and an edge e forced by $V(S)$, we know

$$f(G - V(S), M \setminus S) = f(G - V(S) - V(e), M \setminus (S \cup \{e\})).$$

Let $G \ominus V(S)$ denote the subgraph obtained from $G - V(S)$ by deleting the ends of all the edges forced by $V(S)$, and $M \ominus V(S) = M \cap E(G \ominus V(S))$ be a PM of $G \ominus V(S)$. Obviously we can deduce the following lemma.

Lemma 2.2. *Let G be a graph possessing a PM M. A subset $S \subseteq M$ is a forcing set of M if and only if $G \ominus V(S)$ is empty.*

Lemma 2.3. *Given a graph G possessing a PM M, and a collection $\mathcal{C}$ consisting of disjoint M-alternating cycles of G. Given a subset $S \subseteq M$, which consists of precisely one edge from each cycle in $\mathcal{C}$. If $V(S)$ forces all the other edges in $M \cap E(\mathcal{C})$, then*

$$f(G, M) = f(G \ominus V(S), M \ominus V(S)) + |S|.$$

Proof. Given a minimum forcing set S' for $M \ominus V(S)$ in $G \ominus V(S)$. Then $f(G \ominus V(S), M \ominus V(S)) = |S'|$. Since $G \ominus V(S \cup S') = (G \ominus V(S)) \ominus V(S')$ is empty, by Lemma 2.2 $S \cup S'$ is a forcing set for M in G. Suppose S_0 is another forcing set for M in G such that $|S_0| < |S \cup S'|$. Then either $|S_0 \cap E(G \ominus V(S))| < |S'|$, or $|S_0 \cap (E(G) \setminus E(G \ominus V(S)))| < |S| = |\mathcal{C}|$. By assumption, $E(\mathcal{C}) \subseteq E(G) \setminus E(G \ominus V(S))$ can be deduced. It follows that either $S_0 \cap E(G \ominus V(S))$ could not be a forcing set for $M \ominus V(S)$ in $G \ominus V(S)$, or $\mathcal{C}$ possessing an M-alternating cycle containing no edges of $S_0 \cap (E(G) \setminus E(G \ominus V(S)))$. This implies that there is an M-alternating cycle in G containing no edges of S_0, which contradicts Theorem 2.1. Hence $f(G, M) = |S| + |S'|$. ■

By the above lemma, from a special collection of disjoint M-alternating cycles we can find a subset that is contained in some minimum forcing set of a PM M. In particular, for hexagonal systems we can find a minimum forcing set directly.

3 Monotonic CHS

A CHS is called *left-monotonic* (resp. *right-monotonic*) if the leftmost hexagon in each row is located on the left (resp. right) to the leftmost hexagon of the row immediately above. Since inverting a right-monotonic CHS upside down derives a left-monotonic CHS, we only talk about left one in the following and call it *monotonic CHS* for convenience, see Fig. 2. In order to label each hexagon, we suppose the length between the center of two adjacent hexagon is 1. Denote the hexagon in the top right corner by $C_{1,1}$, and its center by O. From O draw two rays OA and OB perpendicular to the bottom left oblique edge and the left vertical edge of $C_{1,1}$, respectively. For a hexagon with center W, if two lines can be drawn through W such that one is parallel to OB and intersects OA at the point W_A and the other is parallel to OA and intersects OB at the point W_B, and the length of OW_A and OW_B are i and j respectively, then we denote it by $C_{i+1,j+1}$, see Fig. 2.

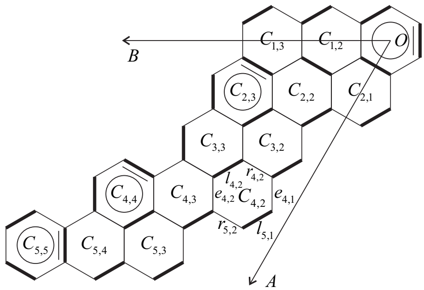

Figure 2. $CHS(3,3,3,4,5;1,1,2,2,3)$ with PM $(0,3,3,4,4)$.

If a monotonic CHS has $m(\geqslant 1)$ rows, and the leftmost and rightmost hexagons in ith row are C_{i,k_i} and C_{i,h_i} respectively ($k_{i+1} \geqslant k_i \geqslant h_{i+1} \geqslant h_i$ for $i = 1, 2, \ldots, m-1$), then we denote it by $CHS(k_1, k_2, \ldots, k_m; h_1, h_2, \ldots, h_m)$, or briefly $CHS(\{k_s; h_s\}_{s=1}^{m})$. Furthermore, for $i = 1, 2, \ldots, m$ and $j = h_i, h_i + 1, \ldots, k_i$, denote the left vertical edge of $C_{i,j}$ by $e_{i,j}$, and along the clockwise direction denote the rest edges by $l_{i,j}$, $r_{i,j}$, $e_{i,j-1}$, $l_{i+1,j-1}$, and $r_{i+1,j}$, see Fig. 2.

In Figs. 3(a-h), we illustrate some special examples of monotonic CHS, where $CHS(\{k_s; 1\}_{s=1}^{m})$ is a truncated parallelogram, $CHS(\{k; 1\}_{s=1}^{m})$ is a benzenoid parallelogram, $CHS(k_1; 1)$ is a linear hexagonal chain, $CHS(k_1, k_2; 1, h_2)$ is a double linear hexagonal chain, $CHS(k_1, k_2, k_3; 1, h_2, h_3)$ is a triple linear hexagonal chain, and $CHS(1, 2, \ldots, k, k; 1, 1, 2, \ldots, k)$ and $CHS(2, 3, \ldots, k, k; 1, 2, \ldots, k)$ are zigzag hexagonal chains with even and odd number of hexagons respectively. And the graph showed in Fig. 3(h) is a hexagonal chain with $n(\geqslant 1)$ maximal linear hexagonal chains, in which containing $r_1, r_2, \ldots, r_n$ ($r_i \geqslant 2$ for $i = 1, 2, \ldots, n$) hexagons in turn.

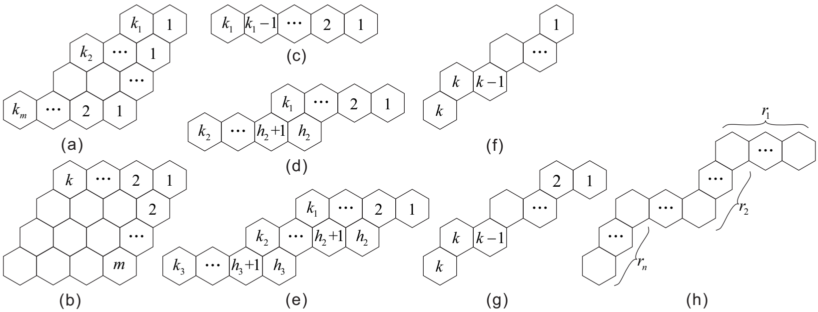

Figure 3. Examples of monotonic CHS.

Zhang and Li [14] proved that a CHS has a PM, and every PM of a CHS has only one vertical edge in every row. An immediate conclusion is as follows.

Lemma 3.1. [14] *We have a bijection g from the set of PMs M of $CHS(\{k_s;h_s\}_{s=1}^{m})$ to the set of non-decreasing sequences $(u_1,u_2,\ldots,u_m)$ with $u_i\in\{h_i-1,h_i,\ldots,k_i\}$ for $1\leqslant i\leqslant m$. In detail, corresponding relationship $g(M)=(u_1,u_2,\ldots,u_m)$ if $e_{i,u_i}\in M$ for $1\leqslant i\leqslant m$.*

By the above conclusion, a sequence $g(M)$ could denote a PM M of a monotonic CHS. Let $g(M)=(g_1(M),g_2(M),\ldots,g_m(M))$. As an example, (0,3,3,4,4) could denote the PM showed in Fig. 2, by a set of bold lines, of $CHS(3,3,3,4,5;1,1,2,2,3)$.

In the following, we will give forcing polynomial for $CHS(\{k_s;h_s\}_{s=1}^{m})$. For convenience, from now on we define $CHS(\{b_s;d_s\}_{s=1}^{i})=CHS(\{b_s;d_s\}_{s=1}^{i-1})$ if $b_i<d_i$ and $i\geqslant 1$, and $CHS(\{b_s;d_s\}_{s=1}^{j})$ as an empty graph if $j=0$. And we make a convention

$$\mathcal{M}(\{k_s;h_s\}_{s=1}^{m})=\mathcal{M}(CHS(\{k_s;h_s\}_{s=1}^{m})),$$
$$F(\{k_s;h_s\}_{s=1}^{m})=F(CHS(\{k_s;h_s\}_{s=1}^{m}),x).$$

According to different values of $g_m(M)$, we divide $\mathcal{M}(\{k_s;h_s\}_{s=1}^{m})$ into k_m-h_m+2 subsets:

$$\mathcal{M}_i(\{k_s;h_s\}_{s=1}^{m})=\{M\in\mathcal{M}(\{k_s;h_s\}_{s=1}^{m}):g_m(M)=i\}$$

for $i = h_m - 1, h_m, \ldots, k_m$. By Eq. (1), we have

$$F(\{k_s; h_s\}_{s=1}^m) = \sum_{i=h_m-1}^{k_m} \sum_{M \in \mathcal{M}_i(\{k_s;h_s\}_{s=1}^m)} x^{f(CHS(\{k_s;h_s\}_{s=1}^m),M)}$$
$$:= \sum_{i=h_m-1}^{k_m} F_i(\{k_s; h_s\}_{s=1}^m). \quad (2)$$

Theorem 3.2. *The forcing polynomial of* $CHS(\{k_s; h_s\}_{s=1}^m)$ *is*

$$F(\{k_s; h_s\}_{s=1}^m) = \sum_{i=h_m-1}^{k_m-1} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-1})x + \sum_{j=\min\{s:k_s=k_m\}}^{m} F(\{\min\{k_s, k_m - 1\}; h_s\}_{s=1}^{j-1})x, \quad (3)$$

where $F(\{b_s; d_s\}_{s=1}^i) = F(\{b_s; d_s\}_{s=1}^{i-1})$ *if* $b_i < d_i$ *and* $i \geqslant 1$, *and* $F(\{b_s; d_s\}_{s=1}^j) = 1$ *if* $j = 0$.

Proof. Given $i \in \{h_m - 1, h_m, \ldots, k_m - 1\}$ and $M \in \mathcal{M}_i(\{k_s; h_s\}_{s=1}^m)$. On the one hand $e_{m,i}$ belongs to M-alternating hexagon $C_{m,i+1}$, and on the other hand by Lemma 3.1 the edges $r_{m+1,i+1}$ and $l_{m,i+1}$ of $C_{m,i+1}$ are forced by $V(e_{m,i})$, see Fig. 4(a). By Lemma 2.3 we know

$$\begin{aligned} & f(CHS(\{k_s; h_s\}_{s=1}^m), M) \\ =& f(CHS(\{k_s; h_s\}_{s=1}^m) \ominus V(e_{m,i}), M \ominus V(e_{m,i})) + 1 \\ =& f(CHS(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-1}), M \cap E(CHS(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-1}))) + 1. \end{aligned}$$

It follows that

$$\begin{aligned} & F_i(\{k_s; h_s\}_{s=1}^m) \\ =& \sum_{M \in \mathcal{M}_i(\{k_s;h_s\}_{s=1}^m)} x^{f(CHS(\{\min\{k_s,i\};h_s\}_{s=1}^{m-1}),M \cap E(CHS(\{\min\{k_s,i\};h_s\}_{s=1}^{m-1})))+1} \\ =& \sum_{M \in \mathcal{M}(\{\min\{k_s,i\};h_s\}_{s=1}^{m-1})} x^{f(CHS(\{\min\{k_s,i\};h_s\}_{s=1}^{m-1}),M)} \cdot x \\ =& F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-1})x. \end{aligned} \quad (4)$$

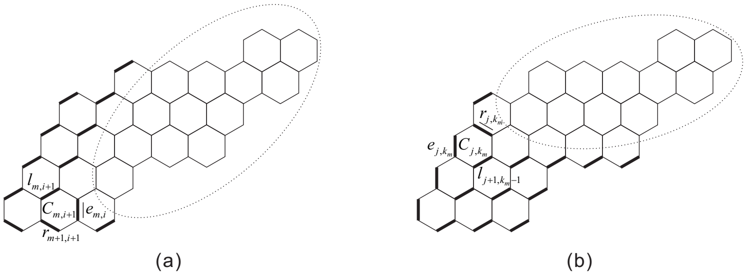

Figure 4. (a) $CHS(\{k_s;h_s\}_{s=1}^m)\ominus V(e_{m,i})$, (b) $CHS(\{k_s;h_s\}_{s=1}^m)\ominus V(r_{j,k_m})$.

Given $M\in\mathcal{M}_{k_m}(\{k_s;h_s\}_{s=1}^m)$. Let $p=\min\{s:k_s=k_m\}$ and $j=\min\{j:g_j(M)=k_m\}$. Then $p\leqslant j\leqslant m$ and $r_{j,k_m}\in M$. On the one hand r_{j,k_m} belongs to M-alternating hexagon C_{j,k_m}, and on the other hand by Lemma 3.1 the edges e_{j,k_m} and l_{j+1,k_m-1} of C_{j,k_m} are forced by $V(r_{j,k_m})$, see Fig. 4(b). By Lemma 2.3 we know

$$\begin{aligned}&f(CHS(\{k_s;h_s\}_{s=1}^m),M)\\=&f(CHS(\{k_s;h_s\}_{s=1}^m)\ominus V(r_{j,k_m}),M\ominus V(r_{j,k_m}))+1\\=&f(CHS(\{\min\{k_s,k_m-1\};h_s\}_{s=1}^{j-1}),\\&M\cap E(CHS(\{\min\{k_s,k_m-1\};h_s\}_{s=1}^{j-1})))+1.\end{aligned}$$

It follows that

$$\begin{aligned}&F_{k_m}(\{k_s;h_s\}_{s=1}^m)\\=&\sum_{j=p}^{m}\sum_{\substack{M\in\mathcal{M}_{k_m}(\{k_s;h_s\}_{s=1}^m)\\ j=\min\{j:g_j(M)=k_m\}}}\\&x^{f(CHS(\{\min\{k_s,k_m-1\};h_s\}_{s=1}^{j-1}),M\cap E(CHS(\{\min\{k_s,k_m-1\};h_s\}_{s=1}^{j-1})))+1}\\=&\sum_{j=p}^{m}\sum_{M\in\mathcal{M}(\{\min\{k_s,k_m-1\};h_s\}_{s=1}^{j-1})}x^{f(CHS(\{\min\{k_s,k_m-1\};h_s\}_{s=1}^{j-1}),M)}\cdot x\\=&\sum_{j=p}^{m}F(\{\min\{k_s,k_m-1\};h_s\}_{s=1}^{j-1})x.\end{aligned}\tag{5}$$

Substituting Eqs. (4,5) into Eq. (2), we could obtain Eq. (3). ■

From the proof above, we can derive the following conclusion and a fast algorithm to find a minimum forcing set of PM for $CHS(\{k_s; h_s\}_{s=1}^m)$.

Corollary 3.3. [16] *For PM M of $CHS(\{k_s; h_s\}_{s=1}^m)$, the forcing number of M equals the maximum number of disjoint M-alternating hexagons.*

Algorithm 3.4.
Input: $CHS(k_1, k_2, \ldots, k_m; h_1, h_2, \ldots, h_m)$ with PM $(a_1, a_2, \ldots, a_m)$.
Output: A minimum forcing set S of $(a_1, a_2, \ldots, a_m)$.
(1) Let $s \leftarrow m$; $t \leftarrow m$; $S \leftarrow \emptyset$;
  while $s \geqslant 1$ **do** $l_s \leftarrow k_s$, $s \leftarrow s-1$.
(2) **While** $t \geqslant 1$ **do**
  if $l_t \geqslant h_t$ **then**
   if $a_t = l_t$ **then**
    $i \leftarrow \min\{i : a_i = a_t\}$, $j \leftarrow i-1$, $t \leftarrow i-1$, $S \leftarrow S \cup \{r_{i,a_i}\}$,
    while $j \geqslant 1$ **do** $l_j \leftarrow \min\{l_j, l_i - 1\}$, $j \leftarrow j-1$;
   else $i \leftarrow t$, $j \leftarrow i-1$, $t \leftarrow i-1$, $S \leftarrow S \cup \{e_{i,a_i}\}$,
    while $j \geqslant 1$ **do** $l_j \leftarrow \min\{l_j, a_i\}$, $j \leftarrow j-1$;
  else $t \leftarrow t-1$.
(3) Output S.

For instance, by inputting $CHS(3,3,3,4,5;1,1,2,2,3)$ and PM $g(M) = (0,3,3,4,4)$, we could get output $\{e_{5,4}, r_{4,4}, r_{2,3}, e_{1,0}\}$ showed by a set of double lines in Fig. 2. On the other hand, we use a set of solid cycles to illustrate a set of disjoint M-alternating cycles $\{C_{5,5}, C_{4,4}, C_{2,3}, C_{1,1}\}$.

Hexagonal systems possessing anti-forcing edges are exactly $CHS(\{k_s; 1\}_{s=1}^m)$ [9]. From the above theorem, we can derive the forcing polynomial. For convenience, we make a convention $F(\{k_s\}_{s=1}^m) = F(\{k_s; 1\}_{s=1}^m)$.

Corollary 3.5. *The forcing polynomial of $CHS(\{k_s; 1\}_{s=1}^m)$ (see Fig.3(a)) is*

$$\begin{aligned} &F(\{k_s\}_{s=1}^m) \\ =& \sum_{i=0}^{k_m-1} F(\{\min\{k_s, i\}\}_{s=1}^{m-1})x + \sum_{j=\min\{s:k_s=k_m\}}^{m} F(\{\min\{k_s, k_m-1\}\}_{s=1}^{j-1})x, \end{aligned}$$

where $F(\{b_s\}_{s=1}^{i}) = F(\{b_s\}_{s=1}^{i-1})$ *if* $b_i = 0$ *and* $i \geqslant 1$, *and* $F(\{b_s\}_{s=1}^{j}) = 1$ *if* $j = 0$.

Zhang and Deng [16] obtained the continuity of forcing spectrum for $CHS(\{k_s; 1\}_{s=1}^{m})$ by Z-transform graph, and here we show the result by degrees of forcing polynomial. For convenience, SD is short for set of degrees.

Corollary 3.6. [16] *The forcing spectrum of* $CHS(\{k_s; 1\}_{s=1}^{m})$ *is an integer interval from 1.*

Proof. By the induction for the number m of rows. When $m = 1$, the conclusion can be obtained by the following Example 3.8. Let $m \geqslant 2$. By Corollary 3.5, we have

$$F(\{k_s\}_{s=1}^{m}) = \sum_{i=1}^{k_m-1} F(\{\min\{k_s, i\}\}_{s=1}^{m-1})x + x + \sum_{j=\min\{s:k_s=k_m\}}^{m} F(\{\min\{k_s, k_m-1\}\}_{s=1}^{j-1})x.$$

By inductive hypothesis, we can derive that SD of $F(\{\min\{k_s, i\}\}_{s=1}^{m-1})x$ is an integer interval from 2, SD of $F(\{\min\{k_s, k_m-1\}\}_{s=1}^{j-1})x$ is an integer interval from 1 or 2, and x has degree 1. Then the forcing spectrum is an integer interval from 1. ■

We now give some forcing polynomials of particular monotonic CHSs.

Example 3.7. [22] *The forcing polynomial of* $CHS(\{k; 1\}_{s=1}^{m}) = M(k, m)$ *(see Fig. 3(b)) is*

$$F(\{k\}_{s=1}^{m}) = \sum_{i=0}^{k-1} F(\{i\}_{s=1}^{m-1})x + \sum_{j=1}^{m} F(\{k-1\}_{s=1}^{j-1})x,$$

where $F(\{b\}_{s=1}^{n}) = 1$ *if* $b = 0$ *or* $n = 0$.

From the above recurrence formula, explicit form of forcing polynomial for benzenoid parallelograms can be deduced, which has already been obtained in Ref. [22].

Example 3.8. [18] *The forcing polynomial of* $CHS(k_1;1)$ *(see Fig. 3(c)) is*

$$F(k_1)=\sum_{i=0}^{k_1-1}x+x=(k_1+1)x.$$

Example 3.9. *The forcing polynomial of* $CHS(k_1,k_2;1,h_2)$ *(see Fig. 3(d)) is*

$$\begin{aligned}&F(k_1,k_2;1,h_2)\\&=\sum_{i=h_2-1}^{k_2-1}F(\min\{k_1,i\})x+\sum_{j=\min\{s:k_s=k_2\}}^{2}F(\{\min\{k_s,k_2-1\}\}_{s=1}^{j-1})x\\&=\sum_{i=k_1}^{k_2-1}F(k_1)x+\sum_{i=h_2}^{k_1-1}F(i)x+F(h_2-1)x+\beta\\&=\left[(k_2-k_1)(k_1+1)+\frac{(k_1+h_2+1)(k_1-h_2)}{2}\right]x^2+\alpha+\beta,\end{aligned}$$

where

$$\alpha=\begin{cases}x & \text{if } h_2=1,\\ h_2x^2 & \text{if } h_2\neq 1,\end{cases}\qquad \beta=\begin{cases}2x & \text{if } k_2=k_1=1,\\ k_2x^2+x & \text{if } k_2=k_1\neq 1,\\ (k_1+1)x^2 & \text{if } k_2\neq k_1.\end{cases}$$

In particular, the forcing polynomials of $CHS(k,k;1,h_2)$ and $CHS(k_1,k_2;1,h_2)$ $(k\neq 1,\ k_2\neq k_1,\ h_2\neq 1)$ are

$$\begin{aligned}&F(k,k;1,h_2)\\=&\frac{(k+h_2+1)(k-h_2)}{2}x^2+h_2x^2+kx^2+x=\frac{k^2-h_2^2+3k+h_2}{2}x^2+x,\\&F(k_1,k_2;1,h_2)\\=&\left[(k_2-k_1)(k_1+1)+\frac{(k_1+h_2+1)(k_1-h_2)}{2}\right]x^2+h_2x^2+(k_1+1)x^2\\=&\frac{2k_1k_2-k_1^2-h_2^2+k_1+2k_2+h_2+2}{2}x^2.\end{aligned}$$

Example 3.10. *The forcing polynomial of* $CHS(k_1,k_2,k_3;1,h_2,h_3)$ *(see*

Fig. 3(e)) is

$$F(k_1,k_2,k_3;1,h_2,h_3)=\sum_{i=h_3-1}^{k_3-1}F(\{\min\{k_s,i\};h_s\}_{s=1}^{2})x + \sum_{j=\min\{s:k_s=k_3\}}^{3}F(\{\min\{k_s,k_3-1\};h_s\}_{s=1}^{j-1})x = \gamma+\chi,$$

where

$$\gamma=\begin{cases}\sum_{i=k_2}^{k_3-1}F(k_1,k_2;1,h_2)x+\sum_{i=k_1}^{k_2-1}F(k_1,i;1,h_2)x+\sum_{i=h_3-1}^{k_1-1}F(i,i;1,h_2)x & \text{if } h_3\leqslant k_1,\\ \sum_{i=k_2}^{k_3-1}F(k_1,k_2;1,h_2)x+\sum_{i=h_3-1}^{k_2-1}F(k_1,i;1,h_2)x & \text{if } h_3>k_1,\end{cases}$$

$$\chi=\begin{cases}F(k_1,k_2;1,h_2)x & \text{if } k_3\neq k_2,\\ F(k_1,k_3-1;1,h_2)x+F(k_1)x & \text{if } k_3=k_2\neq k_1,\\ F(k_3-1,k_3-1;1,h_2)x+F(k_3-1)x+x & \text{if } k_3=k_2=k_1.\end{cases}$$

In particular, we can derive the following forcing polynomials.

$$\begin{aligned}&F(3,4,5;1,2,3)=2F(3,4;1,2)x+F(3,3;1,2)x+F(2,2;1,2)x\\ =&2\times\frac{2\times3\times4-3^2-2^2+3+2\times4+2+2}{2}x^3\\ &+\left[\frac{3^2-2^2+3\times3+2}{2}x^2+x\right]x+\left[\frac{2^2-2^2+3\times2+2}{2}x^2+x\right]x\\ =&38x^3+2x^2,\\ &F(3,4,4;1,2,4)=F(3,3;1,2)x+F(3,3;1,2)x+F(3)x\\ =&2\left[\frac{3^2-2^2+3\times3+2}{2}x^2+x\right]x+4x^2=16x^3+6x^2.\end{aligned}$$

Example 3.11. [18] *The forcing polynomial of zigzag hexagonal chain Z_n*

with $n(\geqslant 3)$ hexagons (see Figs. 3(f,g)) is

$$\begin{aligned} F(Z_{2k},x) =& F(1,2,\dots,k,k;1,1,2,\dots,k) \\ =& 2F(1,2,\dots,k-1,k-1;1,1,2,\dots,k-1)x \\ & + F(1,2,\dots,k-1;1,1,2,\dots,k-2)x \\ =& 2F(Z_{2k-2},x)x + F(Z_{2k-3},x)x, \\ F(Z_{2k-1},x) =& F(2,3,\dots,k,k;1,2,\dots,k) \\ =& 2F(2,3,\dots,k-1,k-1;1,2,\dots,k-1)x \\ & + F(2,3,\dots,k-1;1,2,\dots,k-2)x \\ =& 2F(Z_{2k-3},x)x + F(Z_{2k-4},x)x. \end{aligned}$$

From Example 3.11, explicit form of forcing polynomial for zigzag hexagonal chains can be deduced, which has already been obtained in Ref. [18]. In fact, recurrence formula of forcing polynomial for an arbitrary hexagonal chain (see Fig. 3(h)) can be deduced from Eq. (3), which coincides with that in Ref. [18].

4 CHS with one turning

We now turn to *CHS with one turning*. It can be obtained from two monotonic CHSs, say $CHS(\{k_s;h_s\}_{s=1}^{m})$ and $CHS(\{k'_t;h'_t\}_{t=1}^{m'})$ with $m,m'\geqslant 2$ and $k_m-h_m=k'_{m'}-h'_{m'}$. First place the first one in left-monotonic way and the second one in right-monotonic way, and then paste the mth row of the first one and the m'th row of the second one, see Fig. 5. The pasted row is called *turning row*. We denote the CHS by $CHS(k_1,k_2,\dots,k_m;h_1,h_2,\dots,h_m|k'_1,k'_2,\dots,k'_{m'};h'_1,h'_2,\dots,h'_{m'})$, or briefly $CHS(\{k_s;h_s\}_{s=1}^{m}|\{k'_t;h'_t\}_{t=1}^{m'})$. What's more, its labels of hexagons and edges follow the corresponding two monotonic ones with only an apostrophe in the second one. Note that the hexagons in the turning row have two labels, such as $C_{m,i}=C'_{m',i-h_m+h'_{m'}}$. Denote $p(y)=\min\{p:k_p\geqslant y\}$ and $q=\min\{q:k'_q=k'_{m'}\}$. Similar to Lemma 3.1, we have the following lemma.

Lemma 4.1. [14] *We have a bijection g' from the set of PMs M of $CHS(\{k_s;h_s\}_{s=1}^{m}|\{k'_t;h'_t\}_{t=1}^{m'})$ to the set of binary non-decreasing sequences*

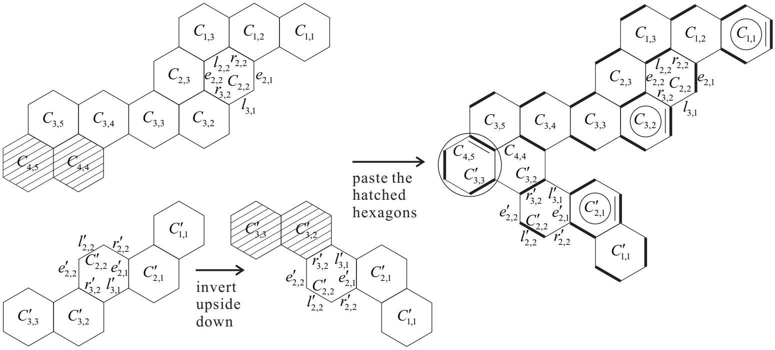

Figure 5. The way of obtaining $CHS(\{k_s;h_s\}_{s=1}^{m}|\{k'_t;h'_t\}_{t=1}^{m'})$.

$((u_1,u_2,\dots,u_m),(u'_1,u'_2,\dots,u'_{m'}))$ *with* $u_m-h_m=u'_{m'}-h'_{m'}$, $u_i\in\{h_i-1,h_i,\dots,k_i\}$, $u'_j\in\{h'_j-1,h'_j,\dots,k'_j\}$ *for* $1\leqslant i\leqslant m$, $1\leqslant j\leqslant m'$. *In detail, corresponding relationship* $g'(M)=((u_1,u_2,\dots,u_m),(u'_1,u'_2,\dots,u'_{m'}))$ *if* $e_{i,u_i},e'_{j,u'_j}\in M$ *for* $1\leqslant i\leqslant m$, $1\leqslant j\leqslant m'$.

By the above conclusion, a binary sequence $g'(M)$ could denote a PM M of a CHS with one turning. Let $g'(M)=((g_1^1(M),g_2^1(M),\dots,g_m^1(M)),(g_1^2(M),g_2^2(M),\dots,g_{m'}^2(M)))$. As an example, ((0,1,1,5),(0,0,3)) may denote the PM M in $CHS(3,3,5,5;1,2,2,4|1,2,3;1,1,2)$ illustrated in Fig. 5. In the figure, we also show a minimum forcing set $\{e'_{2,0},r_{4,5},e_{3,1},e_{1,0}\}$ of M, and a set $\{C'_{2,1},C_{4,5},C_{3,2},C_{1,1}\}$ of disjoint M-alternating cycles.

For convenience, in the following part we make a convention

$$\mathcal{H}=CHS(\{k_s;h_s\}_{s=1}^{m}|\{k'_t;h'_t\}_{t=1}^{m'}),\quad \mathcal{M}=\mathcal{M}(\mathcal{H}),\quad \mathcal{F}=F(\mathcal{H},x).$$

Now we can give a recurrence formula of $\mathcal{F}$. According to different values of $g_m^1(M)$, we divide $\mathcal{M}$ into k_m-h_m+2 subsets:

$$\mathcal{M}_i=\{M\in\mathcal{M}:g_m^1(M)=i\}$$

for $i=h_m-1,h_m,\dots,k_m$. By Eq. (1), we obtain

$$\mathcal{F}=\sum_{i=h_m-1}^{k_m}\sum_{M\in\mathcal{M}_i}x^{f(\mathcal{H},M)}:=\sum_{i=h_m-1}^{k_m}\mathcal{F}_i. \tag{6}$$

Theorem 4.2. *The forcing polynomial $\mathcal{F}$ of $\mathcal{H}$ has the following form:*
(1) *if either $p(k_m) = m$ or $q = m'$, then*

$$\mathcal{F} = \sum_{i=h_m-1}^{k_m-1} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-1}) F(\{\min\{k'_t, i - h_m + h'_{m'}\}; h'_t\}_{t=1}^{m'-1}) x + \sum_{j=p(k_m)}^{m} \sum_{i=q}^{m'} F(\{\min\{k_s, k_m - 1\}; h_s\}_{s=1}^{j-1}) F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{i-1}) x; \tag{7}$$

(2) *if $p(k_m) < m$ and $q < m'$, and the maximal zigzag hexagonal chain $\mathcal{Z}$ starting from C_{m,k_m} (see Figs. 3 (f,g)) contains n hexagons ($2 \leqslant n \leqslant \min\{2m-1, 2k_m\}$), namely $\mathcal{Z} = C_{m,k_m} C_{m-1,k_m} C_{m-1,k_m-1} C_{m-2,k_m-1} C_{m-2,k_m-2} \cdots C_{m-\lfloor \frac{n}{2} \rfloor, k_m - \lfloor \frac{n-1}{2} \rfloor}$, then*

$$\begin{aligned}
\mathcal{F} = & \sum_{i=h_m-1}^{k_m-1} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-1}) F(\{\min\{k'_t, i - h_m + h'_{m'}\}; h'_t\}_{t=1}^{m'-1}) x \\
& + \sum_{j=p(k_m)}^{m-1} F(\{\min\{k_s, k_m - 1\}; h_s\}_{s=1}^{j-1}) F(\{k'_t; h'_t\}_{t=1}^{m'-1}) x \\
& + \sum_{i=h_{m-1}-1}^{k_{m-1}-1} \sum_{j=q}^{m'-1} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-2}) F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{j-1}) x^2 \\
& + \sum_{i=h_{m-1}-1}^{k_m-2} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-2}) F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{m'-1}) x^2 \\
& + \sum_{w=1}^{\lfloor \frac{n}{2} \rfloor - 1} \sum_{i=p(k_m-w)}^{m-w-1} \sum_{j=h'_{m'-1}-1}^{k'_{m'}-1} F(\{\min\{k_s, k_m - w - 1\}; h_s\}_{s=1}^{i-1}) \\
& \cdot F(\{\min\{k'_t, j\}; h'_t\}_{t=1}^{m'-2}) x^{w+2} \\
& + \sum_{w=1}^{\lfloor \frac{n}{2} \rfloor - 1} \sum_{i=h_{m-w-1}-1}^{k_m-w-2} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-w-2}) \\
& \cdot F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{m'-1}) x^{w+2} + \delta,
\end{aligned} \tag{8}$$

where

$$\delta = \begin{cases} \sum_{j=h'_{m'-1}-1}^{k'_{m'}-1} F(\{\min\{k_s, k_m - \frac{n}{2}\}; h_s\}_{s=1}^{m-\frac{n}{2}-1}) \\ \cdot F(\{\min\{k'_t, j\}; h'_t\}_{t=1}^{m'-2}) x^{\frac{n}{2}+1} & \text{if } n \text{ is even}, \\ F(\{k_s; h_s\}_{s=1}^{m-\frac{n+1}{2}}) F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{m'-1}) x^{\frac{n+1}{2}} & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Given $i \in \{h_m - 1, h_m, \dots, k_m - 1\}$ and $M \in \mathcal{M}_i$. On the one hand $e_{m,i}$ belongs to M-alternating hexagon $C_{m,i+1}$, and on the other hand by Lemma 4.1 the edges $r_{m+1,i+1}$ and $l_{m,i+1}$ of $C_{m,i+1}$ are forced by $V(e_{m,i})$, see Fig. 6(a). By Lemma 2.3 we know

$$f(\mathcal{H}, M) = f(\mathcal{H} \ominus V(e_{m,i}), M \ominus V(e_{m,i})) + 1.$$

Furthermore, from Fig. 6(a) we observe that

$$\begin{aligned} \mathcal{H} \ominus V(e_{m,i}) =& CHS(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-1}) \\ & \cup CHS(\{\min\{k'_t, i - h_m + h'_{m'}\}; h'_t\}_{t=1}^{m'-1}). \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{F}_i =& \sum_{M \in \mathcal{M}_i} x \cdot x^{f(CHS(\{\min\{k_s,i\};h_s\}_{s=1}^{m-1}), M \cap E(CHS(\{\min\{k_s,i\};h_s\}_{s=1}^{m-1})))} \\ & \cdot x^{f(CHS(\{\min\{k'_t,i-h_m+h'_{m'}\};h'_t\}_{t=1}^{m'-1}), M \cap E(CHS(\{\min\{k'_t,i-h_m+h'_{m'}\};h'_t\}_{t=1}^{m'-1})))} \\ =& \sum_{M \in \mathcal{M}(\{\min\{k_s,i\};h_s\}_{s=1}^{m-1})} x \cdot x^{f(CHS(\{\min\{k_s,i\};h_s\}_{s=1}^{m-1}), M)} \\ & \cdot \sum_{M' \in \mathcal{M}(\{\min\{k'_t,i-h_m+h'_{m'}\};h'_t\}_{t=1}^{m'-1})} x^{f(CHS(\{\min\{k'_t,i-h_m+h'_{m'}\};h'_t\}_{t=1}^{m'-1}), M')} \\ =& F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-1}) F(\{\min\{k'_t, i - h_m + h'_{m'}\}; h'_t\}_{t=1}^{m'-1}) x. \end{aligned} \quad (9)$$

In the remaining part, we calculate $\mathcal{F}_{k_m}$ according to different values of $p(k_m)$ and q.

Case 1. $p(k_m) \leqslant m$ and $q = m'$. Given $M \in \mathcal{M}_{k_m}$. Let $j = \min\{j : g_j^1(M) = k_m\}$. Then $p(k_m) \leqslant j \leqslant m$ and $r_{j,k_m} \in M$. On the one hand

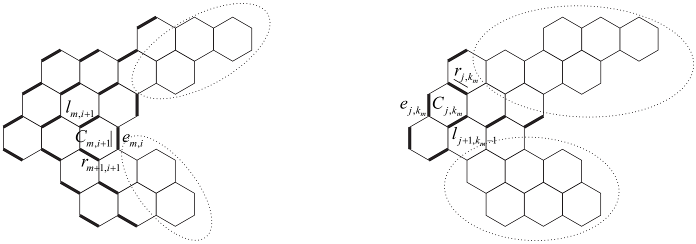

Figure 6. (a) $\mathcal{H} \ominus V(e_{m,i})$, and (b) $\mathcal{H} \ominus V(r_{j,k_m})$.

r_{j,k_m} belongs to M-alternating hexagon C_{j,k_m}, and on the other hand by Lemma 4.1 the edges e_{j,k_m} and l_{j+1,k_m-1} of C_{j,k_m} are forced by $V(r_{j,k_m})$, see Fig. 6(b). By Lemma 2.3 we know

$$f(\mathcal{H}, M) = f(\mathcal{H} \ominus V(r_{j,k_m}), M \ominus V(r_{j,k_m})) + 1.$$

Furthermore, from Fig. 6(b) we observe that

$$\mathcal{H} \ominus V(r_{j,k_m}) = CHS(\{\min\{k_s, k_m - 1\}; h_s\}_{s=1}^{j-1}) \cup CHS(\{k'_t; h'_t\}_{t=1}^{m'-1}).$$

Similar to the calculation of Eq. (9), we have

$$\mathcal{F}_{k_m} = \sum_{j=p(k_m)}^{m} F(\{\min\{k_s, k_m - 1\}; h_s\}_{s=1}^{j-1}) F(\{k'_t; h'_t\}_{t=1}^{m'-1}) x. \qquad (10)$$

Substituting Eqs. (9,10) into Eq. (6), we immediately obtain Eq. (7) in this case.

Case 2. $p(k_m) = m$ and $q < m'$. Since inverting the CHS upside down derives another one satisfying Case 1, we can derive

$$\mathcal{F}_{k_m} = \sum_{i=q}^{m'} F(\{k_s; h_s\}_{s=1}^{m-1}) F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{i-1}) x. \qquad (11)$$

Substituting Eqs. (9,11) into Eq. (6), we immediately obtain Eq. (7) in this case.

Case 3. $p(k_m) < m$ and $q < m'$. For $1 \leqslant w \leqslant \lfloor \frac{n}{2} \rfloor$, denote

$$P(m-w) = \sum_{M \in \mathcal{M}_{k_m},\ g^2_{m'-1}(M) \leqslant k'_{m'}-1,\ g^1_{m-i}(M) = k_m - i \text{ for } 1 \leqslant i \leqslant w} x^{f(\mathcal{H},M)}.$$

Let $\mathcal{M}^1 = \{M \in \mathcal{M}_{k_m} : g^1_{m-1}(M) = k_m\}$, $\mathcal{M}^2 = \{M \in \mathcal{M}_{k_m} : g^2_{m'-1}(M) = k'_{m'}\}$, $\mathcal{M}^3 = \{M \in \mathcal{M}_{k_m} : g^1_{m-1}(M) = k_m, g^2_{m'-1}(M) = k'_{m'}\}$, $\mathcal{M}^4 = \{M \in \mathcal{M}_{k_m} : g^1_{m-1}(M) \leqslant k_m - 2, g^2_{m'-1}(M) \leqslant k'_{m'} - 1\}$, and $\mathcal{M}^5 = \{M \in \mathcal{M}_{k_m} : g^1_{m-1}(M) = k_m - 1, g^2_{m'-1}(M) \leqslant k'_{m'} - 1\}$. Then we have

$$\begin{aligned} \mathcal{F}_{k_m} = & \sum_{M \in \mathcal{M}^1} x^{f(\mathcal{H},M)} + \sum_{M \in \mathcal{M}^2} x^{f(\mathcal{H},M)} - \sum_{M \in \mathcal{M}^3} x^{f(\mathcal{H},M)} \\ & + \sum_{M \in \mathcal{M}^4} x^{f(\mathcal{H},M)} + \sum_{M \in \mathcal{M}^5} x^{f(\mathcal{H},M)} \\ := & P^1 + P^2 - P^3 + P^4 + P(m-1). \end{aligned} \tag{12}$$

Given $M \in \mathcal{M}^1$. Let $j = \min\{j : g^1_j(M) = k_m\}$. Then $p(k_m) \leqslant j \leqslant m-1$ and $r_{j,k_m} \in M$. In fact, C_{j,k_m} is an M-alternating hexagon containing r_{j,k_m}. Similar to Case 1, by Lemmas 2.3 and 4.1 we can derive

$$f(\mathcal{H}, M) = f(\mathcal{H} \ominus V(r_{j,k_m}), M \ominus V(r_{j,k_m})) + 1.$$

Furthermore, we can observe from Fig. 7(a) that

$$\mathcal{H} \ominus V(r_{j,k_m}) = CHS(\{\min\{k_s, k_m - 1\}; h_s\}_{s=1}^{j-1}) \cup CHS(\{k'_t; h'_t\}_{t=1}^{m'-1}).$$

Similar to the calculation of Eq. (9), we have

$$P^1 = \sum_{j=p(k_m)}^{m-1} F(\{\min\{k_s, k_m - 1\}; h_s\}_{s=1}^{j-1}) F(\{k'_t; h'_t\}_{t=1}^{m'-1}) x. \tag{13}$$

Given $M \in \mathcal{M}^2$. Similar to the calculation of P^1, we can derive

$$P^2 = \sum_{j=q}^{m'-1} F(\{k_s; h_s\}_{s=1}^{m-1}) F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{j-1}) x.$$

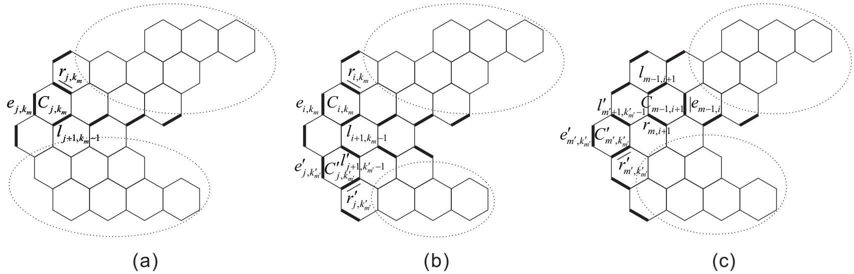

Figure 7. Illustration of calculations for P^1, P^3 and P^4.

Given $M \in \mathcal{M}^3$. Let $i = \min\{i : g_i^1(M) = k_m\}$ and $j = \min\{j : g_j^2(M) = k'_{m'}\}$. Then $p(k_m) \leqslant i \leqslant m-1$, $q \leqslant j \leqslant m'-1$, and $r_{i,k_m}, r'_{j,k'_{m'}} \in M$. On the one hand r_{i,k_m} belongs to M-alternating hexagon C_{i,k_m}, $r'_{j,k'_{m'}}$ belongs to M-alternating hexagon $C'_{j,k'_{m'}}$, and the two hexagons are disjoint. On the other hand by Lemma 4.1 the edges e_{i,k_m} and l_{i+1,k_m-1} of C_{i,k_m}, and the edges $e'_{j,k'_{m'}}$ and $l'_{j+1,k'_{m'}-1}$ of $C'_{j,k'_{m'}}$ are forced by $V(\{r_{i,k_m}, r'_{j,k'_{m'}}\})$, see Fig. 7(b). By Lemma 2.3 we know

$$f(\mathcal{H}, M) = f(\mathcal{H} \ominus V(\{r_{i,k_m}, r'_{j,k'_{m'}}\}), M \ominus V(\{r_{i,k_m}, r'_{j,k'_{m'}}\})) + 2.$$

Furthermore, it is observed from Fig. 7(b) that

$$\begin{aligned}&\mathcal{H} \ominus V(\{r_{i,k_m}, r'_{j,k'_{m'}}\})\\ =&CHS(\{\min\{k_s, k_m-1\}; h_s\}_{s=1}^{i-1}) \cup CHS(\{\min\{k'_t, k'_{m'}-1\}; h'_t\}_{t=1}^{j-1}).\end{aligned}$$

Similar to the calculation of Eq. (9), we have

$$\begin{aligned}P^3 =& \sum_{i=p(k_m)}^{m-1} \sum_{j=q}^{m'-1} F(\{\min\{k_s, k_m-1\}; h_s\}_{s=1}^{i-1})\\ &\cdot F(\{\min\{k'_t, k'_{m'}-1\}; h'_t\}_{t=1}^{j-1})x^2,\\ P^2 - P^3 =& \sum_{i=h_{m-1}-1}^{k_{m-1}-1} \sum_{j=q}^{m'-1} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-2})\\ &\cdot F(\{\min\{k'_t, k'_{m'}-1\}; h'_t\}_{t=1}^{j-1})x^2. \qquad (14)\end{aligned}$$

Given $M \in \mathcal{M}^4$. Let $i = g^1_{m-1}(M)$. Then $h_{m-1} - 1 \leqslant i \leqslant k_m - 2$. In fact, $C_{m-1,i+1}$ and $C'_{m',k'_{m'}}$ are disjoint M-alternating hexagons containing $e_{m-1,i}$ and $r'_{m',k'_{m'}}$, respectively. Similar to the calculation of P^3, by Lemmas 2.3 and 4.1 and Fig. 7(c) we can derive

$$P^4 = \sum_{i=h_{m-1}-1}^{k_m-2} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-2}) F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{m'-1}) x^2. \tag{15}$$

It remains to consider $P(m-1)$. Given $M \in \mathcal{M}^5$. The value of $P(m-1)$ varies with different values of n, and we distinguish according to the following subcases.

Subcase 3.1. $n = 2$. Then the last hexagon of $\mathcal{Z}$ is C_{m-1,k_m}, and C_{m,k_m-1} is nonexistent, which implies that r_{m,k_m-1} is nonexistent. Let $j = g^2_{m'-1}(M)$. Then $h'_{m'-1} - 1 \leqslant j \leqslant k'_{m'} - 1$. In fact, C_{m-1,k_m} and $C'_{m'-1,j+1}$ are disjoint M-alternating hexagons containing r_{m,k_m} and $e'_{m'-1,j}$, respectively. Similar to the calculation of P^3, by Lemmas 2.3 and 4.1 and Fig. 8(a) we can derive

$$P(m-1) = \sum_{j=h'_{m'-1}-1}^{k'_{m'}-1} F(\{\min\{k_s, k_m - 1\}; h_s\}_{s=1}^{m-2}) F(\{\min\{k'_t, j\}; h'_t\}_{t=1}^{m'-2}) x^2.$$

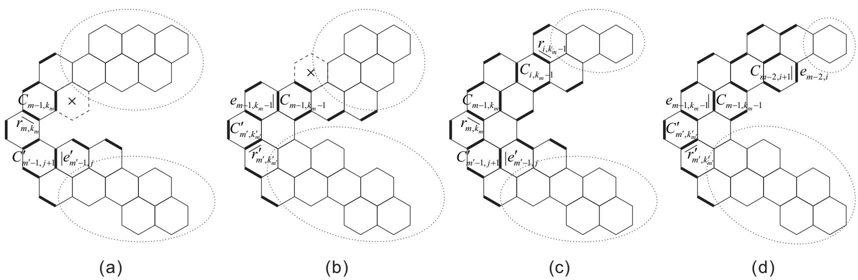

Figure 8. Illustration of calculation for $P(m-1)$.

Subcase 3.2. $n = 3$. Then the last hexagon of $\mathcal{Z}$ is C_{m-1,k_m-1}, and

C_{m-2,k_m} is nonexistent, which implies that e_{m-2,k_m-1} is nonexistent. In fact, C_{m-1,k_m-1} and $C'_{m',k'_{m'}}$ are disjoint M-alternating hexagons containing e_{m-1,k_m-1} and $r'_{m',k'_{m'}}$, respectively. Similar to the calculation of P^3, by Lemmas 2.3 and 4.1 and Fig. 8(b) we can derive

$$P(m-1) = F(\{k_s; h_s\}_{s=1}^{m-2}) F(\{\min\{k'_t, k'_{m'}-1\}; h'_t\}_{t=1}^{m'-1}) x^2.$$

Subcase 3.3. $n \geqslant 4$. Then $m \geqslant 3$, $k_m \geqslant 2$, and $p(k_m - w) \leqslant m - w - 1$ for $1 \leqslant w \leqslant \lfloor \frac{n}{2} \rfloor - 1$. Let $\mathcal{M}^6 = \{M \in \mathcal{M}^5 : g^1_{m-2}(M) = k_m - 1\}$, $\mathcal{M}^7 = \{M \in \mathcal{M}^5 : g^1_{m-2}(M) \leqslant k_m - 3\}$, and $\mathcal{M}^8 = \{M \in \mathcal{M}^5 : g^1_{m-2}(M) = k_m - 2\}$. Then we have

$$\begin{aligned} P(m-1) = & \sum_{M \in \mathcal{M}^6} x^{f(\mathcal{H},M)} + \sum_{M \in \mathcal{M}^7} x^{f(\mathcal{H},M)} + \sum_{M \in \mathcal{M}^8} x^{f(\mathcal{H},M)} \\ := & P^6 + P^7 + P(m-2). \end{aligned} \tag{16}$$

Given $M \in \mathcal{M}^6$. Let $i = \min\{i : g^1_i(M) = k_m - 1\}$ and $j = g^2_{m'-1}(M)$. Then $p(k_m - 1) \leqslant i \leqslant m - 2$ and $h'_{m'-1} - 1 \leqslant j \leqslant k'_{m'} - 1$. In fact, C_{i,k_m-1}, C_{m-1,k_m} and $C'_{m'-1,j+1}$ are disjoint M-alternating hexagons containing r_{i,k_m-1}, r_{m,k_m} and $e'_{m'-1,j}$, respectively. Similar to the calculation of P^3, by Lemmas 2.3 and 4.1 and Fig. 8(c) we can derive

$$\begin{aligned} P^6 = & \sum_{i=p(k_m-1)}^{m-2} \sum_{j=h'_{m'-1}-1}^{k'_{m'}-1} F(\{\min\{k_s, k_m - 2\}; h_s\}_{s=1}^{i-1}) \\ & \cdot F(\{\min\{k'_t, j\}; h'_t\}_{t=1}^{m'-2}) x^3. \end{aligned} \tag{17}$$

Given $M \in \mathcal{M}^7$. Let $i = g^1_{m-2}(M)$. Then $h_{m-2} - 1 \leqslant i \leqslant k_m - 3$. In fact, $C_{m-2,i+1}$, C_{m-1,k_m-1} and $C'_{m',k'_{m'}}$ are disjoint M-alternating hexagons containing $e_{m-2,i}$, e_{m-1,k_m-1}, and $r'_{m',k'_{m'}}$, respectively. Similar to the calculation of P^3, by Lemmas 2.3 and 4.1 and Fig. 8(d) we derive

$$P^7 = \sum_{i=h_{m-2}-1}^{k_m-3} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-3}) F(\{\min\{k'_t, k'_{m'}-1\}; h'_t\}_{t=1}^{m'-1}) x^3. \tag{18}$$

Substituting Eqs. (17,18) into Eq. (16), we obtain $P(m-1)$ in this case.

Similar to the above three subcases, we deduce $P(m-w) =$

$$\begin{cases} \displaystyle\sum_{j=h'_{m'-1}-1}^{k'_{m'}-1} F(\{\min\{k_s, k_m - w\}; h_s\}_{s=1}^{m-w-1}) F(\{\min\{k'_t, j\}; h'_t\}_{t=1}^{m'-2}) x^{w+1} & \text{if } n = 2w, \\ F(\{k_s; h_s\}_{s=1}^{m-w-1}) F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{m'-1}) x^{w+1} & \text{if } n = 2w+1, \\ \displaystyle\sum_{i=p(k_m-w)}^{m-w-1} \sum_{j=h'_{m'-1}-1}^{k'_{m'}-1} F(\{\min\{k_s, k_m - w - 1\}; h_s\}_{s=1}^{i-1}) \\ \cdot F(\{\min\{k'_t, j\}; h'_t\}_{t=1}^{m'-2}) x^{w+2} \\ + \displaystyle\sum_{i=h_{m-w-1}-1}^{k_m-w-2} F(\{\min\{k_s, i\}; h_s\}_{s=1}^{m-w-2}) F(\{\min\{k'_t, k'_{m'} - 1\}; h'_t\}_{t=1}^{m'-1}) x^{w+2} \\ + P(m-w-1) & \text{if } n \geqslant 2w+2. \end{cases}$$

for $1 \leqslant w \leqslant \lfloor \frac{n}{2} \rfloor$. Substituting each $P(m-w)$ into $P(m-1)$ and substituting Eqs. (13-15) into Eq. (12), we immediately obtain $\mathcal{F}_{k_m}$ in this case. Furthermore, substituting the result and Eq. (9) into Eq. (6), we could get Eq. (8). ∎

From the proof above, we can deduce the following conclusion.

Corollary 4.3. [16] *For every PM M in $CHS(\{k_s; h_s\}_{s=1}^{m} | \{k'_t; h'_t\}_{t=1}^{m'})$, the forcing number of M coincides with the maximum number of disjoint M-alternating hexagons.*

In Fig. 9, we illustrate some particular examples of CHS with one turning. Hexagonal systems possessing forcing edges are $CHS(\{k_s; 1\}_{s=1}^{m})$ and $CHS(\{k_s; 1\}_{s=1}^{m} | \{k'_t; 1\}_{t=1}^{m'})$ (see Fig. 9(a)) [5, 15]. For the second class, from the above theorem we deduce the forcing polynomial and a characterization of the continuity of forcing spectrum as follows, while by Z-transform graph Zhang and Deng [16] have already talked about the forcing spectrum.

Corollary 4.4. [16] *The forcing spectrum of $CHS(\{k_s; 1\}_{s=1}^{m} | \{k'_t; 1\}_{t=1}^{m'})$ is an integer interval from 1 if either $p(k_m) = 1$ or $q = 1$, and an integer interval from 1 with the only gap 2 if $p(k_m) > 1$ and $q > 1$.*

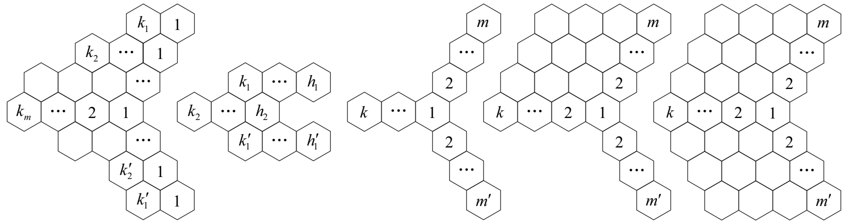

Figure 9. Examples of CHS with one turning.

Proof. From Theorem 4.2, we know that the forcing polynomial has the following same part no matter what the values of $p(k_m)$ and q are:

$$\sum_{i=1}^{k_m-1} F(\{\min\{k_s,i\}\}_{s=1}^{m-1})F(\{\min\{k'_t,i\}\}_{t=1}^{m'-1})x + x.$$

By Corollary 3.6 we can derive that SD of $F(\{\min\{k_s,i\}\}_{s=1}^{m-1})F(\{\min\{k'_t, i\}\}_{t=1}^{m'-1})x$ is an integer interval from 3, and the second term has degree 1. For the rest part of the forcing polynomial, denoted by I, we distinguish according to different values of $p(k_m)$ and q.

Case 1. $p(k_m) = m$ and $q = 1$, or $p(k_m) = 1$ and $q = m'$. By symmetry, we only need to talk about the first case. By Eq. (7), we know

$$I = \sum_{i=2}^{m'} F(\{k_s\}_{s=1}^{m-1})F(\{k'_{m'} - 1\}_{t=1}^{i-1})x + F(\{k_s\}_{s=1}^{m-1})x.$$

By Corollary 3.6 we have that SD of $F(\{k_s\}_{s=1}^{m-1})F(\{k'_{m'} - 1\}_{t=1}^{i-1})x$ is an integer interval from 3, and SD of $F(\{k_s\}_{s=1}^{m-1})x$ is an integer interval from 2, which implies that the forcing spectrum is an integer interval from 1.

Case 2. $p(k_m) = m$ and $1 < q \leqslant m'$, or $1 < p(k_m) < m$ and $q = m'$. By symmetry, we only need to check the first case. By Eq. (7), we know

$$I = \sum_{i=q}^{m'} F(\{k_s\}_{s=1}^{m-1})F(\{\min\{k'_t, k'_{m'} - 1\}\}_{t=1}^{i-1})x.$$

By Corollary 3.6 we have that SD of each term is an integer interval from

3, which implies that the forcing spectrum is an integer interval from 1 with the only gap 2.

Case 3. $1 < p(k_m) < m$ and $1 < q < m'$. Then $m, m' > 2$. By Eq. (8), we know

$$
\begin{aligned}
I = &\sum_{j=p(k_m)}^{m-1} F(\{\min\{k_s, k_m - 1\}\}_{s=1}^{j-1}) F(\{k'_t\}_{t=1}^{m'-1}) x \\
&+ \sum_{i=1}^{k_{m-1}-1} \sum_{j=q}^{m'-1} F(\{\min\{k_s, i\}\}_{s=1}^{m-2}) F(\{\min\{k'_t, k'_{m'} - 1\}\}_{t=1}^{j-1}) x^2 \\
&+ \sum_{j=q}^{m'-1} F(\{\min\{k'_t, k'_{m'} - 1\}\}_{t=1}^{j-1}) x^2 + F(\{\min\{k'_t, k'_{m'} - 1\}\}_{t=1}^{m'-1}) x^2 \\
&+ \sum_{i=1}^{k_m-2} F(\{\min\{k_s, i\}\}_{s=1}^{m-2}) F(\{\min\{k'_t, k'_{m'} - 1\}\}_{t=1}^{m'-1}) x^2 \\
&+ \sum_{w=1}^{\lfloor \frac{n}{2} \rfloor - 1} \sum_{i=0}^{k_m-w-2} F(\{\min\{k_s, i\}\}_{s=1}^{m-w-2}) F(\{\min\{k'_t, k'_{m'} - 1\}\}_{t=1}^{m'-1}) x^{w+2} \\
&+ \sum_{w=1}^{\lfloor \frac{n}{2} \rfloor - 1} \sum_{i=p(k_m-w)}^{m-w-1} \sum_{j=0}^{k'_{m'}-1} F(\{\min\{k_s, k_m - w - 1\}\}_{s=1}^{i-1}) \\
&\cdot F(\{\min\{k'_t, j\}\}_{t=1}^{m'-2}) x^{w+2} + \varepsilon, \qquad (19)
\end{aligned}
$$

where

$$
\varepsilon = \begin{cases} \displaystyle\sum_{j=0}^{k'_{m'}-1} F(\{\min\{k_s, k_m - \frac{n}{2}\}\}_{s=1}^{m-\frac{n}{2}-1}) F(\{\min\{k'_t, j\}\}_{t=1}^{m'-2}) x^{\frac{n}{2}+1} & \text{if } n \text{ is even,} \\ F(\{k_s\}_{s=1}^{m-\frac{n+1}{2}}) F(\{\min\{k'_t, k'_{m'} - 1\}\}_{t=1}^{m'-1}) x^{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}
$$

By Corollary 3.6, we have that SD of terms in the 1st, 3th and 4th summations are integer intervals from 3, SD of terms in the 2nd and 5th summations are integer intervals from 4. If $\lfloor \frac{n}{2} \rfloor \geqslant 2$, then the 7th summation is nonzero. Furthermore for $w = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, SD of $F(\{\min\{k'_t, j\}\}_{t=1}^{m'-2}) x^{w+2}$ is an integer interval from $w+2$ or $w+3$, and both appear since the existent of $j = 0$ and $j = k'_{m'} - 1$, which implies that SD of $F(\{\min\{k_s, k_m - w - 1\}\}_{s=1}^{i-1})F(\{\min\{k'_t, j\}\}_{t=1}^{m'-2})x^{w+2}$ is an integer interval from $w+2$, $w+3$ or $w+4$, and the $w+3$ one must appear. Hence SD of terms in the 7th summation is an integer interval from 3 or 4 including $\lfloor \frac{n}{2} \rfloor + 2$. Similarly, if $\lfloor \frac{n}{2} \rfloor \geqslant 2$, then SD of terms in the 6th summation is an integer interval from 4. And SD of ε is an integer interval from $\lfloor \frac{n}{2} \rfloor + 1$, $\lfloor \frac{n}{2} \rfloor + 2$, or $\lfloor \frac{n}{2} \rfloor + 3$. Then the forcing spectrum is an integer interval from 1 with the only gap 2.

Case 4. $p(k_m) = 1$ and $1 \leqslant q < m'$, or $1 < p(k_m) < m$ and $q = 1$. By symmetry, we only need to talk about the first case. The case of $k_m = 1$ can be obtained in the following Example 4.6, and here we suppose $k_m > 1$. Note that if $m > k_m$, then $n = 2k_m$; if $m \leqslant k_m$, then $n = 2m - 1$. By Eq. (8), we can derive the forcing polynomial same as Eq. (19). However, we can rewrite the 1st summation as follows

$$\sum_{j=2}^{m-1} F(\{\min\{k_s, k_m - 1\}\}_{s=1}^{j-1})F(\{k'_t\}_{t=1}^{m'-1})x + F(\{k'_t\}_{t=1}^{m'-1})x,$$

which implies that SD of term is an integer interval from 2. Furthermore, SD of term in each of the following summation is an integer interval, but from different values. In detail, from 2, 3 or 4 in the 2nd summation; from 2 or 3 in the 3th summation; from 3 in the 4th summation; from 3 or 4 in the 5th summation if it is nonzero; from 4 in the 6th summation if it is nonzero and $\lfloor \frac{n}{2} \rfloor \geqslant 2$; from 3 including $\lfloor \frac{n}{2} \rfloor + 1$ in the 7th summation if $\lfloor \frac{n}{2} \rfloor \geqslant 2$; from $\lfloor \frac{n}{2} \rfloor + 1$ or $\lfloor \frac{n}{2} \rfloor + 2$ in ε. Then the forcing spectrum is an integer interval from 1. ■

We now give some forcing polynomials of particular CHSs with one turning.

Example 4.5. *If $k_2 - k_1 \leqslant k'_2 - k'_1$, then the forcing polynomial F_b of $CHS(k_1, k_2; 1, h_2|k'_1, k'_2; 1, h'_2)$ (see Fig. 9(b)) has the following form:*

(1) *if* $k_2' > k_1'$, *then*

$$F_b = \sum_{i=h_2-1}^{k_2-1} F(\min\{k_1, i\})F(\min\{k_1', i-h_2+h_2'\})x + \sum_{j=p(k_2)}^{2} F(\{\min\{k_s, k_2-1\}\}_{s=1}^{j-1})F(k_1')x$$

$$= \sum_{i=k_1}^{k_2-1} F(k_1)F(k_1')x + \sum_{i=k_1'-h_2'+h_2}^{k_1-1} F(i)F(k_1')x + \sum_{i=h_2}^{k_1'-h_2'+h_2-1} F(i)F(i-h_2+h_2')x + \lambda + \eta$$

$$= [(k_2-k_1)(k_1+1)(k_1'+1) + \frac{(k_1'+1)(k_1+k_1'-h_2'+h_2+1)(k_1-k_1'+h_2'-h_2)}{2} + \frac{(k_1'-h_2'+h_2+1)(k_1'-h_2'+h_2)(2k_1'-2h_2'+2h_2+1)}{6} - \frac{h_2(h_2+1)(2h_2+1)}{6} + \frac{(h_2'-h_2)(k_1'-h_2'+2h_2+1)(k_1'-h_2')}{2}\Bigg] x^3 + \lambda + \eta;$$

(2) *if* $k_2' = k_1'$, *then*

$$F_b = \sum_{i=h_2}^{k_2-1} F(i)F(i-h_2+h_2')x + \lambda + F(k_1')x + \sum_{i=0}^{k_1-1} x^2 + \sum_{i=0}^{k_2-2} F(k_2'-1)x^2 + \kappa,$$

$$= [\frac{k_2(k_2+1)(2k_2+1) - h_2(h_2+1)(2h_2+1)}{6} + \frac{(h_2'-h_2)(k_2+h_2+1)(k_2-h_2)}{2}\Bigg] x^3 + \lambda + (k_1'+k_1+1)x^2 + \mu + \kappa,$$

where

$$\lambda = \begin{cases} x & \text{if } h_2 = h_2' = 1, \\ h_2 x^2 & \text{if } h_2 \neq 1 \text{ and } h_2' = 1, \\ h_2' x^2 & \text{if } h_2 = 1 \text{ and } h_2' \neq 1, \\ h_2 h_2' x^3 & \text{if } h_2 \neq 1 \text{ and } h_2' \neq 1, \end{cases} \qquad \eta = \begin{cases} k_2(k_1'+1)x^3 + (k_1'+1)x^2 \\ \qquad\qquad \text{if } k_2 = k_1, \\ (k_1+1)(k_1'+1)x^3 \\ \qquad\qquad \text{if } k_2 \neq k_1, \end{cases}$$

$$\mu = \begin{cases} (k_2-1)x^2 & \text{if } k_2' = 1, \\ (k_2-1)k_2' x^3 & \text{if } k_2' \neq 1, \end{cases} \qquad \kappa = \begin{cases} k_2' x^2 & \text{if } k_2 = 1 \text{ or } k_2' = 1, \\ k_2' x^3 & \text{if } k_2 \neq 1 \text{ and } k_2' \neq 1. \end{cases}$$

In particular, we can derive the following forcing polynomials.

$$\begin{aligned}
&F(3,4;1,2|2,3;1,1) \\
=&\Bigg[(4-3)(3+1)(2+1) + \frac{(2+1)(3+2-1+2+1)(3-2+1-2)}{2} \\
+&\frac{(2-1+2+1)(2-1+2)(2\times 2-2+2\times 2+1) - 2(2+1)(2\times 2+1)}{6} \\
+&\frac{(1-2)(2-1+2\times 2+1)(2-1)}{2}\Bigg]x^3 + 2x^2 + (3+1)(2+1)x^3 \\
=&30x^3 + 2x^2, \\
&F(4,4;1,2|3,4;1,2) \\
=&\Bigg[\frac{(3+1)(4+3-2+2+1)(4-3+2-2)}{2} \\
&+\frac{(3-2+2+1)(3-2+2)(2\times 3+1) - 2(2+1)(2\times 2+1)}{6}\Bigg]x^3 \\
&+ 2\times 2x^3 + 4(3+1)x^3 + (3+1)x^2 = 45x^3 + 4x^2, \\
&F(1,1;1,1|2,2;1,2) \\
=&\Bigg[\frac{(1+1)(2+1) - (1+1)(2+1)}{6} + \frac{(2-1)(1+1+1)(1-1)}{2}\Bigg]x^3 \\
&+ 2x^2 + (2+1+1)x^2 + 2(1-1)x^3 + 2x^2 = 8x^2, \\
&F(4,4;1,2|4,4;1,2) \\
=&\Bigg[\frac{4(4+1)(2\times 4+1) - 2(2+1)(2\times 2+1)}{6}\Bigg]x^3 \\
&+ 2\times 2x^3 + (4+4+1)x^2 + 4(4-1)x^3 + 4x^3 = 45x^3 + 9x^2.
\end{aligned}$$

Example 4.6. [18] *The forcing polynomial F_c of $CHS(1,\ldots,1,k;1,1,\ldots,1|1,\ldots,1,k;1,1,\ldots,1)$ (see Fig. 9(c)) has the following form:*

$$F_c = \begin{cases} \sum_{i=1}^{k-1} F(\{1\}_{s=1}^{m-1})F(\{1\}_{t=1}^{m'-1})x + x + F(\{1\}_{s=1}^{m-1})F(\{1\}_{t=1}^{m'-1})x \\ = kmm'x^3 + x & \text{if } k \geqslant 2, \\ x + \sum_{j=1}^{m-1} F(\{1\}_{t=1}^{m'-1})x + \sum_{j=1}^{m'-1} x^2 + x^2 = mm'x^2 + x & \text{if } k = 1. \end{cases}$$

Example 4.7. *The forcing polynomial F_d of $CHS(k,\ldots,k,k;1,1,\ldots,1|1,\ldots,1,k;1,1,\ldots,1)$ $(k \geqslant 2)$ (see Fig. 9(d)) is*

$$\begin{aligned} F_d =& \sum_{i=1}^{k-1} F(\{i\}_{s=1}^{m-1})F(\{1\}_{t=1}^{m'-1})x + x + \sum_{j=1}^{m} F(\{k-1\}_{s=1}^{j-1})F(\{1\}_{t=1}^{m'-1})x \\ =& m'F(M(k,m),x)x - m'x^2 + x. \end{aligned}$$

Example 4.8. *The forcing polynomial F_e of $CHS(\{k;1\}_{s=1}^{m}|\{k;1\}_{t=1}^{m'})$ $(k \geqslant 2)$ (see Fig. 9(e)) is*

$$\begin{aligned} F_e =& \sum_{i=0}^{k-1} F(\{i\}_{s=1}^{m-1})F(\{i\}_{t=1}^{m'-1})x + \sum_{j=1}^{m-1} F(\{k-1\}_{s=1}^{j-1})F(\{k\}_{t=1}^{m'-1})x \\ &+ \sum_{i=0}^{k-1}\sum_{j=1}^{m'-1} F(\{i\}_{s=1}^{m-2})F(\{k-1\}_{t=1}^{j-1})x^2 + \sum_{i=0}^{k-2} F(\{i\}_{s=1}^{m-2})F(\{k-1\}_{t=1}^{m'-1})x^2 \\ &+ \sum_{w=1}^{\min\{m-2,k-1\}} \sum_{i=1}^{m-w-1} \sum_{j=0}^{k-1} F(\{k-w-1\}_{s=1}^{i-1})F(\{j\}_{t=1}^{m'-2})x^{w+2} \\ &+ \sum_{w=1}^{\min\{m-2,k-1\}} \sum_{i=0}^{k-w-2} F(\{i\}_{s=1}^{m-w-2})F(\{k-1\}_{t=1}^{m'-1})x^{w+2} + \epsilon, \end{aligned}$$

where

$$\epsilon = \begin{cases} \sum_{j=0}^{k-1} F(\{j\}_{t=1}^{m'-2})x^{k+1} & \text{if } m > k, \\ F(\{k-1\}_{t=1}^{m'-1})x^{m} & \text{if } m \leqslant k. \end{cases}$$

***Acknowledgment*:** The author would like to sincerely thank the anonymous referees for the time they spent in checking the manuscript, as well as their many valuable suggestions. Additionally, this work is supported by Science and Technology Plan Foundation of Gansu Province of China (21JR7RA550), Gansu Provincial Department of Education: Youth Doctoral Fund Project (2021QB-090), and NSFC (11871256).

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