MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **88** (2022) 605–624
ISSN: 0340–6253
doi: 10.46793/match.88-3.605R

Sharp Bounds on the Sombor Energy of Graphs

Bilal Ahmad Rather, Muhammad Imran*

Department of Mathematical Sciences, College of Science,
United Arab Emirates University, Al Ain 15551, Abu Dhabi, UAE
bilalahmadrr@gmail.com, m.imran658@uaeu.ac.ae

(Received February 28, 2022)

Abstract

For a simple graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The Sombor matrix $S(G)$ of G is an $n \times n$ matrix, whose (i, j)-entry is equal is $\sqrt{d_i^2 + d_j^2}$, if i and j are adjacent and 0, otherwise. The multi-set of the eigenvalues of $S(G)$ is known as the Sombor spectrum of G, denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, where μ_1 is the Sombor spectral radius of G. The absolute sum of the Sombor eigenvalues if known as the Sombor energy. In this article, we find the bounds for the Sombor energy of G and characterize the corresponding extremal graphs. These bounds are better than already known results on Sombor energy.

1 Introduction

We consider only connected, simple and undirected graphs. A graph is denoted by $G = G(V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is its vertex set and E containing pairs of unordered vertices is its edge set. The cardinality of V is the order n and the cardinality of E is the size m of G. The *degree* of a vertex v in G is the number of edges incident with v and is denoted by d_v. A graph is called *regular*, if degree of each vertex is same number. We

*Corresponding author.

follow the standard terminology, K_n, $K_{a,b}$, and P_n, respectively, denote the compete graph, the complete bipartite graph, the path graph. The complete multipartite graph by $K_{n_1,n_2,\ldots,n_t}$, $t \geq 3$, the complete split graph with clique size ω and the independence number $\alpha = n-\omega$ by $CS(\omega, n-\omega)$. For other undefined notations, we follow [1].

The *adjacency matrix* $A(G)$ of a graph G is a $(0,1)$-square matrix of order $n \times n$, where (i,j)-entry is 1, if v_i is adjacent to v_j and 0, otherwise. The multi-set of eigenvalues of $A(G)$ is known as the spectrum of G. As $A(G)$ is real symmetric, so its eigenvalues can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For connected graphs, $A(G)$ is non-negative and irreducible, so by Perron Frobenius theorem, λ_1 is unique positive real number and its corresponding eigenvector has positive entries. Let M be a square matrix of order n and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ be the singular values of M. Nikiferov [19], defined the energy of matrix by

$$\mathcal{E}_M = \sum_{i=1}^{n} \sigma_i,$$

that is the energy of M is defined as the sum of the singular values of matrix M, which is known by trace norm of M in linear algebra. In case, matrix M is symmetric, then $\sigma = |\lambda_i|$ for $i = 1, 2, \ldots, n$, where λ_i's are the eigenvalues of matrix M. Gutman [10], defined the energy of $A(G)$ of G as the absolute sum of the eigenvalue of $A(G)$, that is

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The energy $\mathcal{E}(G)$ has its origin in theoretical chemistry and helps in approximating the π-electron energy of unsaturated hydrocarbons. There is a wealth of literature about the energy, more about $A(G)$ and its energy, see [3, 7, 15, 16, 18, 21–23].

In mathematical and chemical literature, there are several well known degree-based graph invariants (also known as topological indices) that have been studied extensively. The first Zagreb index Z_1 and the forgotten topological index F of a graph G are defined as

$$Z_1(G) = \sum_{i=1}^{n} d_i^2 = \sum_{v_i v_j \in E(G)} (d_i + d_j) \quad \text{and} \quad F(G) = \sum_{i=1}^{n} d_i^3 = \sum_{v_i v_j \in E(G)} (d_i^2 + d_j^2).$$

Recently, Gutman [11] introduced a new topological index known as Sombor index, denoted by $SO(G)$, defined as

$$SO(G) = \sum_{v_i v_j \in E(G)} \sqrt{d_i^2 + d_j^2}.$$

Das et. al [4] presented novel bounds for $SO(G)$ and deduced its relations with the Zagreb indices of G. In [25], extremal properties of Sombor index were discussed for several families of graphs and it was proved that the cycle C_n has the minimum Sombor index among all unicyclic graphs. Cruz, Gutman and Rada [2] investigated the Sombor index for chemical graphs and characterized extremal graphs for the classes of chemical graphs, chemical trees and hexagonal systems. Other chemical applicability and molecular properties of the Sombor index can be found in [5, 26].

The *Sombor matrix* of a graph G, denoted by $S(G)$ (or simply S if there is no ambiguity), is defined by

$$S(G) = (s_{ij}) = \begin{cases} \sqrt{d_i^2 + d_j^2} & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise,} \end{cases}$$

where d_i is the degree of vertex v_i. Evidently, this matrix is real symmetric. We denote its eigenvalues by μ_i's and order them as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The set of all eigenvalues (including multiplicities) of $S(G)$ is known as the Sombor spectrum of $S(G)$. The largest eigenvalue of $S(G)$ is called as the *Sombor spectral radius* of G, which for real symmetric matrices is same as the *spectral norm* of G. The Sombor energy of G, is defined by

$$\mathcal{E}_{SO}(G) = \sum_{i=1}^{n} |\mu_i|.$$

Various paper on spectral properties of Sombor matrix, like properties of Sombor eigenvalues, Sombor spectral radius, Sombor energy, Sombor Estrada index, relation of energy with Sombor energy and Sombor index and others can be found in [8, 9, 12, 13, 17, 18, 20, 27].

In Section 2, we establish the bounds on the Sombor energy of G, discuss extremal graphs, compare them with already known bounds on the Sombor energy. We close the article with some comments for further work.

2 Sombor energy of graphs

We start with statement of some already known results, which are later used in proving our results.

Lemma 1. *([17], Theorem 3.4) Let G be a connected graph with n vertices. Then*

$$\sqrt{\frac{2F}{n}} \leq \mu_1 \leq \sqrt{\frac{2F(n-1)}{n}},$$

with equality holding on left if and only if $G \cong K_2$ and equality holds on left if and only if G is the complete graph.

Lemma 2. *([17], Lemma 2.8) Let G be a connected graph with n vertices. Then*

$$|\mu_1| = |\mu_2| = \cdots = |\mu_n|$$

if and only if $G \cong K_2$.

Lemma 3. *([20]) Let G be a connected graph with order $n \geq 3$. Then G has two distinct Sombor eigenvalue if and only if G is a complete graph.*

Lemma 4. *([20]) A connected bipartite graph of order $n \geq 5$ has three distinct Sombor eigenvalues if and only if G is the complete bipartite graph.*

Recalling the definition of the Sombor matrix of G, $S(G) = r\sqrt{2}A(G)$ for r regular graphs. So, for regular graphs the Sombor energy is given by

$$\mathcal{E}_{SO}(G) = r\sqrt{2}\mathcal{E}(G).$$

Further, from the definition of the Sombor energy, we observe that

$$\mathcal{E}_{SO}(G) = \sum_{i=1}^{n} |\mu_i| = 2 \sum_{i=1,\mu_i \geq 0}^{n} \mu_i \geq 2\mu_1. \tag{1}$$

Also, we note that $\mu_1 \geq \frac{2SO(G)}{n}$ (see [12, 20]), with equality if and only if G is regular and $\mu_1 \geq \sqrt{\frac{2F}{n}}$ (see [17]), with equality if and only if $G \cong K_2$. Thus from (1), we have

$$\mathcal{E}_{SO}(G) \geq \frac{4SO(G)}{n}, \tag{2}$$

$$\mathcal{E}_{SO}(G) \geq 2\sqrt{\frac{2F}{n}}. \tag{3}$$

Equality holds in Inequalities (2) and (3), if and only G has only one positive Sombor eigenvalue. For (2), G is regular and in this case $S(G) = r\sqrt{2}A(G)$. Also $A(G)$ have one positive eigenvalue [24] if and only if G is the complete multipartite graph. Thus equality occurs in Inequality 2 if and only if G is the complete multipartite graph. Similarly, for (3), equality holds if and only if $G \cong K_2$.

The *Frobenious norm* of Matrix M is defined by

$$\|M\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2.$$

Similarly, the Frobenius norm (see [17, 20]) of $S(G)$ is

$$\mu_1^2 + \mu_2^2 + \cdots + \mu_n^2 = \|S(G)\|_F^2 = 2F,$$

where F is the forgotten topological index of G.

The following result gives the upper bound for the Sombor energy in terms of the forgotten topological index and the order of graph.

Theorem 1. *Let G be a graph of order n. Then*

$$\mathcal{E}_{SO}(G) \leq 2\sqrt{\frac{2F(n-1)}{n}},$$

equality occurs if and only if $G \cong K_n$.

Proof. By using Cauchy-Schwartz inequality, and the fact that $\sum_{i=1}^{n} \mu_i^2 = 2F$ we have

$$\sum_{i=2}^{n} |\mu_i| \leq \sqrt{(n-1)\sum_{i=2}^{n} \mu_i^2} = \sqrt{(n-1)(2F-\mu_1^2)}.$$

Using the above information and the definition of Sombor energy, we obtain

$$\mathcal{E}_{SO}(G) = \mu_1 + \sum_{i=2}^{n} |\mu_i| \leq \mu_1 + \sqrt{(n-1)(2F-\mu_1^2)}.$$

Now, we see that $f(x) = x + \sqrt{(n-1)(2F-x^2)}$ is decreasing in the interval $\left[\sqrt{\frac{1}{2n}}, \sqrt{2F}\right]$. From Lemma 1, $\mu_1 \leq \sqrt{\frac{2F(n-1)}{n}}$ with equality if and only if $G \cong K_n$. Now $\sqrt{\frac{2F(n-1)}{n}} \leq \sqrt{2F}$, implies that $F \geq 0$, which is true. So, $\mu_1 \leq \sqrt{\frac{2F(n-1)}{n}} \leq \sqrt{2F}$ and from $f(\mu_1) \leq f\left(\sqrt{\frac{2F(n-1)}{n}}\right)$, we obtain

$$\begin{aligned}\mathcal{E}_{SO}(G) \leq & \sqrt{\frac{2F(n-1)}{n}} + \sqrt{(n-1)\left(2F - \left(\sqrt{\frac{2F(n-1)}{n}}\right)^2\right)} \\ = & 2\sqrt{\frac{2F(n-1)}{n}}.\end{aligned} \tag{4}$$

If $G \cong K_n$, then $F = \frac{n(n-1)}{2}$, and $2\sqrt{\frac{2F(n-1)}{n}} = 2(n-1) = \mathcal{E}_{SO}(K_n)$, so equality occurs. Conversely, if equality occurs in (4), then according to above observation $\mu_1 = \sqrt{\frac{2F(n-1)}{n}}$ and for $i = 2, \ldots, n$, $|\mu_i| = \sqrt{\frac{2F-\mu_1^2}{n-1}}$. Thus, there are two possibilities : the absolute values of all Sombor eigenvalues are equal, and in this case G has exactly two distinct Sombor eigenvalues. By Lemma 3, equality holds in (4) if and only if $G \cong K_n$. For second case, the absolute value of all eigenvalues are not equal. So, G one positive eigenvalue μ_1 and all other Sombor eigenvalue are in absolute values. However, by Lemma $G \cong K_n$, $|\mu_2| = |\mu_3| = \cdots = |\mu_n| = 1$, and by recalling that $\sum_{i=1}^{n} \mu_i = 0$, we get $\mu_2 = \mu_3 = \cdots = \mu_n = -1$ and $\mu_1 = n-1$, which is the Sombor spectrum of K_n. Therefore, equality holds if and only if $G \cong K_n$. ■

Remark. Gutman ([12], Theorem 1), gave the following McClelland-type bound for Sombor energy

$$\mathcal{E}_{SO}(G) \leq \sqrt{2nF}.$$

Comparing it with the bound obtained in Theorem 1, we obtain

$$2\sqrt{\frac{2F(n-1)}{n}} \leq \sqrt{2nF},$$

which further gives $(n-2)^2 \geq 0$, which is always true. Thus, the Sombor energy upper bound $2\sqrt{\frac{2F(n-1)}{n}}$ is better then the McClelland-type bound $\sqrt{2nF}$.

Theorem 2. *Let G be a graph of order n and t be the positive integer such that μ_t is positive. Then*

$$\mathcal{E}_{SO} \leq \sqrt{2Fn - \frac{2n}{F}\Big(\mu_1^2 + \mu_2^2 + \cdots + \mu_t^2 - F\Big)^2}\,. \tag{5}$$

Proof. Since $\mu_1^2 + \mu_2^2 + \cdots + \mu_t^2 + \mu_{t+1}^2 + \cdots + \mu_n^2 = 2F$, so we have

$$\begin{aligned}\mu_1^2 + \mu_2^2 + \cdots + \mu_t^2 - F &= \frac{1}{2}\Big(\mu_1^2 + \mu_2^2 + \cdots + \mu_t^2 - \big(\mu_{t+1}^2 + \cdots + \mu_n^2\big)\Big)\\ &= \frac{1}{2}\Big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\Big).\end{aligned}$$

Now, using above information, Inequality (5) is equivalent to

$$\begin{aligned}\frac{n}{\mathcal{E}_{SO}^2} &\leq \frac{F}{2B^2 - 2\Big(\mu_1^2 + \mu_2^2 + \cdots + \mu_t^2 - F\Big)^2}\\ &= \frac{F}{2F^2 - 2\frac{1}{4}\Big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\Big)^2}.\end{aligned}$$

Again using $\sum_{i=1}^{n} \mu_i = 0$ and $\sum_{i=1}^{n} \mu_i^2 = 2F$, we have the following observation

$$\frac{2F}{4F^2 - \Big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\Big)^2}$$
$$= \sum_{i=1}^{n} \left(\frac{2F|\mu_i| - \big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\big)\mu_i}{4F^2 - \big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\big)^2} \right)^2$$
$$= \sum_{i=1}^{n} \frac{2F|\mu_i| - \big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\big)\mu_i}{\mathcal{E}_{AG}(G)\Big(4F^2 - \big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\big)^2\Big)}.$$

Finally, using all the above information, we have

$$\frac{n}{\mathcal{E}^2_{SO}(G)} - \frac{2F}{4F^2 - \big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\big)^2}$$
$$= \sum_{i=1}^{n} \frac{1}{\mathcal{E}^2_{SO}(G)} - \sum_{i=1}^{n} \left(\frac{2F}{4F^2 - \big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\big)^2} \right)^2$$
$$= \sum_{i=1}^{n} \left(\frac{1}{\mathcal{E}_{SO}(G)} - \frac{2F}{4F^2 - \big(\mu_1|\mu_1| + \mu_2|\mu_2| + \cdots + \mu_n|\mu_n|\big)^2} \right)^2 \geq 0,$$

which implies Inequality (5). ■

The following result is the immediate consequence of Theorem 2 and [[17], Theorem 4.2].

Corollary. *Let G be a graph with exactly one positive Sombor eigenvalue. Then*

$$\mathcal{E}_{AG}(G) \leq 2\sqrt{F + F\sqrt{\frac{n-2}{n}}}, \tag{6}$$

equality holding if and only if $G \cong K_2$.

Proof. From the proof of Theorem 4.3 of Lin and Miao [17], we have

$$\mathcal{E}_{SO}(G)^2 \geq 2\sqrt{F}, \tag{7}$$

with equality if and only if $G \cong K_2$. Also, for G having exactly one positive

Sombor eigenvalue, the Theorem 2 yields

$$\mathcal{E}_{SO}(G) \leq \sqrt{2Fn - \frac{2n}{F}(\mu_1^2 - F)^2}. \tag{8}$$

By comparing Inequalities (7) and (8), we obtain

$$4F \leq 2nF - \frac{2n}{F}\left(\mu_1^2 - F\right)^2,$$

which implies that

$$\mu_1 \leq \sqrt{F + F\sqrt{\frac{n-2}{2}}}.$$

Therefore, by definition of Sombor energy, we have

$$\mathcal{E}_{SO}(G) = 2\mu_1 \leq 2\sqrt{F + F\sqrt{\frac{n-2}{2}}}.$$

Since, for graphs with exactly one positive Sombor eigenvalue, $\mathcal{E}_{SO}(G) = 2\lambda_1$, which from above inequality is possible if $n = 2$ and $F = \lambda_1^2$, which is true if and only if G is K_2. ■

Remark. From the proof of Theorem 4.2 (see [17]), the upper bound of Sombor energy is

$$\mathcal{E}_{SO}(G) \leq \sqrt{2nF},$$

with equality if and only if $G \cong K_2$. Comparing it with Inequality (6), we have

$$\mathcal{E}_{SO}(G)^2 \leq 4F\left(1 + 1\sqrt{\frac{n-2}{n}}\right) \leq 2nF,$$

which implies that

$$n^2(n-2) > 4. \tag{9}$$

Inequality (9) is true for $n \geq 3$. Therefore, for graphs having exactly one positive Sombor eigenvalue, the bound (6) is better than the bounds of ([17], Theorem 4.2, [12] Theorem 1).

Similarly, from Theorem 1 and Corollary 2, if

$$2\sqrt{\frac{2F(n-1)}{n}} \geq 2\sqrt{F+F\sqrt{\frac{n-2}{n}}},$$

then we get $n-2 \geq n$, which is false. Hence

$$2\sqrt{\frac{2F(n-1)}{n}} \leq 2\sqrt{F+F\sqrt{\frac{n-2}{n}}} \leq \sqrt{2nF}.$$

Therefore, the upper bound of Theorem 1 gives the better bound for graphs with exactly one positive Sombor eigenvalue. The following table gives the values Sombor energy and the comparison of above bounds for some graphs having exactly one positive Sombor eigenvalue.

Graph G	$\mathcal{E}_{SO}(G)$	$2\sqrt{\frac{2F(n-1)}{n}}$	$2\sqrt{F+F\sqrt{\frac{n-2}{n}}}$	$\sqrt{2nF}$
$K_{2,3,3,3,4}$	401.344	434.376	441.792	870.689
$K_{1,14}$	105.033	143.503	145.953	287.646
$CS(5,3)$	117.109	120.955	124.9	182.866
$K_{3,9}$	98.5901	133.492	136.357	241.495

A graph is said to be non singular if all its adjacency eigenvalues are non zero. Likewise, G is Sombor non singular if the determinant of $S(G)$ is non zero.

The next result gives bounds on the Sombor energy for non singular Sombor graphs.

Theorem 3. *Let G be a non singular Sombor graph. Then following holds.*

(i) $\mathcal{E}_{SO}(G) \geq n|det(S)|^{\frac{1}{n}}$, *with equality if and only if* $G \cong K_2$.

(ii) $\mathcal{E}_{SO}(G) \geq \frac{2SO(G)}{n} + \frac{1}{\left(\frac{2SO(G)}{n}\right)^{\frac{1}{n-1}}}(n-1)|det(S)|^{\frac{1}{n-1}}$, *with equality if and only if* $G \cong K_n$.

Proof. By using Cauchy-Schwarz inequality for $\left(\sqrt{|\mu_1|}, \ldots, \sqrt{|\mu_n|}\right)$, we have

$$\sum_{i=1}^{n} \sqrt{|\mu_i|} \leq \sqrt{n \sum_{i=1}^{n} |\mu_i|}, \tag{10}$$

equality holding if and only if $\sqrt{|\mu_1|} = \sqrt{|\mu_2|} = \cdots = \sqrt{|\mu_n|}$. Inequality (10) is further equivalent to

$$\sqrt{\mathcal{E}_{SO}(G)} \geq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{|\mu_i|}.$$

Now, by arithmetic mean and geometric mean inequality, the above inequality can be written as

$$\sqrt{\mathcal{E}_{SO}(G)} \geq \frac{n\Big(\sqrt{|\mu_1\mu_2\ldots\mu_n|}\Big)^{\frac{1}{n}}}{\sqrt{n}} = \sqrt{n}\Big(\sqrt{|\mu_1\mu_2\ldots\mu_n|}\Big)^{\frac{1}{n}}.$$

Hence,

$$\mathcal{E}_{SO}(G) \geq n|det(S)|^{\frac{1}{n}}.$$

By (10), equality occurs if and only if $|\mu_1| = |\mu_2| = \cdots = |\mu_n|$, which by Lemma 2 holds if and only if $G \cong K_2$.

(ii). Again using Cauchy-Schwartz inequality $\Big(\sqrt{|\mu_2|}, \sqrt{|\mu_3|}, \ldots, \sqrt{|\mu_n|}\Big)$, we have

$$\sum_{i=2}^{n} \sqrt{|\mu_i|} \leq \sqrt{(n-1)\sum_{i=2}^{n} |\mu_i|} = \sqrt{(n-1)(\mathcal{E}_{SO}(G) - \mu_1)}, \tag{11}$$

equality occurs if and only if $\sqrt{|\mu_2|} = \sqrt{|\mu_3|} = \cdots = \sqrt{|\mu_n|}$. Inequality (11) can be written as

$$\sqrt{(\mathcal{E}_{SO}(G) - \mu_1)} \geq \frac{1}{\sqrt{n-1}}\sqrt{\sum_{i=2}^{n} |\mu_i|}.$$

Now, by using arithmetic mean and geometric mean inequality, the above inequality is equivalent to

$$\sqrt{(\mathcal{E}_{SO}(G)-\mu_1)} \geq \frac{1}{\sqrt{n-1}}\sqrt{\sum_{i=2}^{n}|\mu_i|} \geq \frac{1}{\sqrt{n-1}}\Big(\sqrt{|\mu_2\mu_3\ldots\mu_n|}\Big)^{\frac{1}{n}},$$

with equality on right if and only if $|\mu_2| = |\mu_3| = \cdots = |\mu_n|$. Thereby this implies that

$$\mathcal{E}_{SO}(G) \geq \mu_1 + (n-1)\Big(\sqrt{|\mu_2\mu_3\ldots\mu_n|}\Big)^{\frac{1}{n}} = \mu_1 + (n-1)\left(\frac{|det(S)|}{\mu_1}\right)^{\frac{1}{n-1}}.$$

Consider the function $f(x) = x + (n-1)\left(\frac{|det(S)|}{x}\right)^{\frac{1}{n-1}}$, then $f'(x) \geq 0$ implies that $f(x)$ in increasing for $x \geq |det(S)|^{\frac{1}{n}}$. Also, note that $\frac{2F}{n} = \frac{1}{n}\sum_{i=1}^{n}\mu_i^2 \geq \frac{2SO(G)}{n} \geq \frac{\sum_{i=1}^{n}|\mu_i|}{n} \geq (|\mu_1\mu_2\ldots\mu_n|)^{\frac{1}{n}}$. Therefore, $f(x)$ is increasing for $|det(S)|^{\frac{1}{n}} \leq \frac{2SO(G)}{n} \leq \mu_1 \leq \sqrt{\frac{2(n-1)F}{n}} \leq \sqrt{2F}$ and $f(\mu_1) \geq f\left(\frac{2SO(G)}{n}\right)$ implies that

$$\mathcal{E}_{SO}(G) \geq \frac{2SO(G)}{n} + \frac{1}{\left(\frac{2SO(G)}{n}\right)^{\frac{1}{n-1}}}(n-1)|det(S)|^{\frac{1}{n-1}}.$$

All above inequalities are equality if and only if $|\mu_2| = |\mu_3| = \cdots = |\mu_n|$ and $\mu_1 = \frac{2SO(G)}{n}$. For connected graphs, Perron-Frobenius theorem states that $\mu_1 > |\mu_i|,\ i = 2, \ldots, n$. Now, there are two possibilities $\mu_2 = \mu_3 = \cdots = \mu_n = c > 0$, which cannot happen since $\mu_1 + (n-1)c = 0$ for $n \geq 2$. The other possibility is $\mu_2 = \mu_3 = \cdots = \mu_n = -c$ and in this case G has exactly two distinct Sombor eigenvalues. Therefore by Lemma 3, G is the complete graph. Conversely, it can be easily verified that equality holds if $G \cong K_n$. ■

The graph obtained from K_ω and P_l by adding an edge between any vertex of K_ω and an end vertex of P_l is denoted by $PK_{\omega,l}$, is known as *path complete* graph of *kite graph*.

Remark. Based on the computations, the following table shows that for graphs with non zero Sombor eigenvalues, the second lower bound of Theorem 3 is better than the first bound.

Graph G	$\mathcal{E}_{SO}(G)$	Theorem 3 (i)	Theorem 3 (ii)
K_6	70.7107	55.4793	70.7107
P_{14}	46.4838	37.0265	37.7799
$PK_{3,9}$	44.6181	33.9839	35.0670

Next, we obtain the bounds for the Sombor energy in terms of the order, the independence number and the Forgotten topological index of graphs. For that, we need following result.

Theorem 4 (Interlacing Theorem, [14])**.** *Let $M \in \mathbb{M}_n$ be a real symmetric matrix. Let A be a principal submatrix of M of order m, $(m \leq n)$. Then the eigenvalues of M and A satisfy the following inequalities*

$$\lambda_{i+n-m}(M) \leq \lambda_i(A) \leq \lambda_i(M), \qquad \text{for } 1 \leq i \leq m.$$

A set of non adjacent vertices of G is known as the *independent* set and the cardinality of a largest independent set of is called the independence number α of G.

As the Sombor matrix $S(G)$ has principal submatrix $M' = \mathbf{0}_{\alpha\times\alpha}$, so by Theorem 4, we have

$$\mu_\alpha(S(G)) \geq \mu_\alpha(M') = 0 \qquad \text{and} \qquad \mu_{n-\alpha+1}(S(G)) \leq \mu_1(M') = 0.$$

Thus, it follows that

$$\alpha \leq \min\{n-p, n-q\} \tag{12}$$

Theorem 5. *Let G be a connected graph of order n and independence number α. Then*

$$\mathcal{E}_{SO}(G) \leq 2\sqrt{(n-\alpha)F}, \tag{13}$$

with equality holding in (13) *if and only if $G \cong K_{1,n-1}$.*

Proof. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ and $\mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_q$ be the positive and negative Sombor eigenvalues of G, respectively. Also, it is clear that

$\sum_{i=1}^{p} \mu_i = \sum_{i=1}^{q} |\mu_i'|$ and by definition of Sombor energy, we obtain

$$\mathcal{E}_{SO}(G) = 2\sum_{i=1}^{p} \mu_i = 2\sum_{i=1}^{q} |\mu_i'|.$$

Now, by using Cauchy-Schwartz inequality, we have

$$\mathcal{E}_{SO}(G) = 2\sum_{i=1}^{p} \mu_i \leq 2\sqrt{p\sum_{i=1}^{p} \mu_i^2},$$

with equality holding if and only if $\mu_1 = \mu_2 = \cdots = \mu_p$.

Similarly,

$$\mathcal{E}_{SO}(G) = 2\sum_{i=1}^{q} |\mu_i'| \leq 2\sqrt{q\sum_{i=1}^{q} (\mu_i')^2},$$

with equality holding if and only if $|\mu_1'| = |\mu_2'| = \cdots = |\mu_q'|$. Now, by applying (12), we have

$$\begin{aligned} \frac{\mathcal{E}_{SO}(G)}{2} &\leq p\sum_{i=1}^{p} \mu_i^2 + q\sum_{i=1}^{n} (\mu_i')^2 \\ &\leq (n-\alpha)\sum_{i=1}^{p} \mu_i^2 + (n-\alpha)\sum_{i=1}^{n} (\mu_i')^2 \\ &= (n-\alpha)\sum_{i=1}^{n} \mu_i^2 = (n-\alpha)2F, \end{aligned}$$

and the Inequality (13) follows.

If equality holds in (13), then $\mu_1 = \mu_2 = \cdots = \mu_p$, $|\mu_1'| = |\mu_2'| = \cdots = |\mu_q'|$ and $p = q = n - \alpha$. But by Perron-Frobenious theorem, μ_1 is simple eigenvalue of G, so $p = 1$ and it implies that $q = 1$, $\alpha = n - 1$ and the Sombor eigenvalue 0 has multiplicity $n-2$. By Lemma 4, G is the complete bipartite graph, thereby it follows that $G \cong K_{1,n-1}$, since its independence number is $n - 1$.

Also, the Sombor spectrum [[20], Proposition 2.2] of $K_{1,n-1}$ is

$$\left\{0^{[n-2]}, \pm\sqrt{(n-1)(n^2-2n+2)}\right\},$$

the independence number is $\alpha = n-1$, and

$$\begin{aligned}2F =&\left(\sqrt{(n-1)(n^2-2n+2)}\right)^2 + \left(-\sqrt{(n-1)(n^2-2n+2)}\right)^2\\ =&2(n-1)(n^2-2n+2).\end{aligned}$$

Therefore, $\mathcal{E}_{SO}(K_{1,n-1}) = 2\sqrt{(n-1)(n^2-2n+2)}$. ■

Clearly, the Sombor matrix of the bipartite graph G can be put as

$$S(G) = \begin{pmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{pmatrix}.$$

If η is an eigenvalue of $S(G)$ with eigenvector $X = (x_1, x_2)^T$, then it is clear that $S(G)X' = -\eta X'$, where $X' = (x_1, -x_2)^T$. It follows that Sombor eigenvalues of a bipartite graph are symmetric about the origin.

Theorem 6. *Let G be a connected bipartite graphs of order n. Then*

$$\mathcal{E}_{SO}(G) \leq 4\sqrt{\frac{F(p-1)}{p}},$$

with equality holding if and only if $\mu_1 = \sqrt{\frac{F(p-1)}{p}}$ and $\mu_2 = \cdots = \mu_p$.

Proof. Let $\mu_1 \geq \mu_2 \cdots \geq \mu_p$ be the p positive Sombor eigenvalues of G. By using above information, and applying Cauchy-Schwartz inequality, we have

$$\begin{aligned}\mathcal{E}_{SO}(G) = 2\sum_{i=1}^{p} \mu_i &= 2\mu_1 + 2\sum_{i=2}^{p} \mu_i\\ &\leq 2\left(\mu_1 + \sqrt{(p-1)\sum_{i=2}^{p}\mu_i^2}\right)\\ &= 2\left(\mu_1 + \sqrt{(p-1)(F-\mu_1^2)}\right).\end{aligned}$$

Clearly, the function $f(x) = x + \sqrt{(p-1)\big(F - x^2\big)}$ is decreasing for $x \geq \sqrt{\frac{F(p-1)}{p}}$. Hence, $f(x) \leq f\left(\sqrt{\frac{F(p-1)}{p}}\right)$ implies that

$$\mathcal{E}_{SO}(G) \leq 4\sqrt{\frac{F(p-1)}{p}},$$

with equality holding if and only if $\mu_1 = \sqrt{\frac{F(p-1)}{p}}$ and $\mu_2 = \mu_3 = \cdots = \mu_p$. ■

Finally, we establish the upper bound for the Sombor energy of the Path P_n and for this we need the following result is helpful.

Lemma 5. *[6] Let M_1 and M_2 be square matrices of order n. Then*

$$\sum_{i=1}^{n} \sigma_i(M_1 + M_2) \leq \sum_{i=1}^{n} \sigma_i(M_1) + \sum_{i=1}^{n} \sigma_i(M_2),$$

with equality holding if and only if there exists an orthogonal matrix $\mathcal{M}$, such that $\mathcal{M}M_1$ and $\mathcal{M}M_2$ are both positive semi-definite.

Proposition 7. *Let P_n be the path on $n \geq 4$ vertices. Then*

$$\mathcal{E}_{SO}(P_n) \leq 4\sqrt{2}\left(\csc\frac{\pi}{2(n+1)}\sin\frac{\Big(2\lfloor\frac{2}{2}\rfloor+1\Big)\pi}{2(n+1)}\right) + 2\Big(\sqrt{5} - 4\sqrt{2}\Big).$$

Proof. It is east to see that the Sombor matrix of P_n can be written as

$$\begin{aligned} S(P_n) =& 2\sqrt{2}\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \\ =& 2\sqrt{2}A(P_n) + B, \end{aligned}$$

where $\alpha = \sqrt{5} - 2\sqrt{2}$. The Sombor eigenvalues of B are $\big\{\sqrt{5} - 2\sqrt{2}, \sqrt{5} -$

$2\sqrt{2}\}$. Therefore by Lemma 5, we have

$$\begin{aligned}
\mathcal{E}_{SO}(P_n) &\le 2\sqrt{2}\mathcal{E}(P_n) + \mathcal{E}(B)\\
&= 2\sqrt{2}\left(4\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \cos\frac{\pi i}{n+1}\right) + 2(\sqrt{5} - 2\sqrt{2})\\
&= 2\sqrt{2}\left(2\csc\frac{\pi}{n+1}\sin\frac{\left(2\lfloor \frac{n}{2} \rfloor + 1\right)\pi}{2(n+1)} - 2\right) + 2(\sqrt{5} - 2\sqrt{2})\\
&= 2\sqrt{2}\left(2\csc\frac{\pi}{n+1}\sin\frac{\left(2\lfloor \frac{n}{2} \rfloor + 1\right)\pi}{2(n+1)}\right) + 2(\sqrt{5} - 4\sqrt{2}).
\end{aligned}$$

■

Example 1. The Sombor energy of P_{19} up to four decimal places is 64.2729, while by Proposition 7, the Sombor energy of P_{19} bounded above by 65.0355.

3 Comments

Let $S(G)(i) = \overline{s}_i$ be the sum of the i-th row of the matrix $S(G)$. Then it is clear that $SO(G) = \frac{1}{2}\sum_{i=1}^{n} \overline{s}_i$. We call $\overline{s}_i$ as the Sombor degree of G and say that G is r'-Sombor regular if and only if $\overline{s}_i = r'$, for every $i = 1, 2, \ldots, n$. Besides, it also follows that if G is r-regular, then G is also $r\sqrt{2}$-Sombor regular. Let $\overline{S} = diag(\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_1)$ be the diagonal matrix of Sombor degrees of G. Then the matrices $S_L(G) = \overline{S} - S(G)$ and $S_Q(G) = \overline{S} + S(G)$ are the Laplacian and the signless Laplacian matrices for the Sombor matrix of G. The Sombor Laplacian matrix $S_L(G)$ is the real symmetric and positive semi-definite matrix and we denote its eigenvalues by $\mu_1^L \ge \mu_2^L \ge \cdots \ge \mu_{n-1}^L \ge \mu_n^L = 0$. The multi-set of eigenvalues of $S_L(G)$ is know as the Sombor Lapacian spectrum of G and μ_1^L is the Sombor spectral radius of G. For $n \ge 3$, $S_Q(G)$ is real symmetric and positive definite matrix, we denotes its eigenvalues by $\mu_1^Q \ge \mu_2^Q \ge \cdots \ge \mu_n^Q$. Similarly, the multi-set of eigenvalues of $S_Q(G)$ is the Sombor signless Laplacian spectrum of G and μ_1^Q is the Sombor signless Laplacian spectral radius of G. Let $\overline{\mu} = \frac{1}{n}\sum_{i=1}^{n}\mu_i^L = \frac{1}{n}\sum_{i=1}^{n}\mu_i^Q = \frac{1}{n}\sum_{i=1}^{n}\overline{s}_i = \frac{2SO(G)}{n}$ be the average of the Sombor degrees of G. Then the trace norm of the real symmetric matrices $S_L(G)-\overline{\mu}I_n$ and $S_Q(G)-\overline{\mu}I_n$ is defined as the Sombor Laplacian energy and the Sombor signless Laplacian energy of G. That is, the Sombor Laplacian energy denoted by $S_LE(G)$, is defined by

$$S_LE(G) = \sum_{i=1}^{n}|\mu_i^L - \overline{\mu}| = \sum_{i=1}^{n}\left|\mu_i^L - \frac{2SO(G)}{n}\right|,$$

and the Sombor signless Laplacian energy denoted by $S_QE(G)$, is defined by

$$S_QE(G) = \sum_{i=1}^{n}|\mu_i^Q - \overline{\mu}| = \sum_{i=1}^{n}\left|\mu_i^L - \frac{2SO(G)}{n}\right|.$$

Various interesting properties about the Sombor Laplacian matrix and the Sombor signless Laplacian matrix can be elaborated. Extremal graphs can be characterized in terms of spectral radii, energies, spectral bounds, distribution of their eigenvalues and other graph/spectral invariants of these matrices.

***Acknowledgment*:** The authors are very thankful to the reviewers for their valuable comments that improved the manuscript. This research is supported by Asian Universities Alliance (AUA) grant of United Arab Emirates University via Grant No. G00003461.

References

[1] G. Chartrand, P. Zhang, *Introduction to Graph Theory*, Tata McGraw–Hill, New Delhi, 2006.

[2] R. Cruz, I. Gutman, J. Rada, Sombor index of graphs, *Appl. Math. Comp.* **399** (2021) #126018.

[3] D. M. Cvetković, P. Rowlison, S. Simić, *An Introduction to Theory of Graph Spectra*, Cambridge Univ. Press, Cambridge, 2010.

[4] K. C. Das, A. S. Çevik, I. N. Cangul, Y. Shang, On Sombor index, *Symmetry* **13** (2021) #140.

[5] H. Deng, Z. Tang, R. Wu, Molecular trees with extremal values of Sombor indices, *Int. J. Quantum Chem.* **121** (2021) #e26622..

[6] K. Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proc. Natl. Acad. Sci. USA* **37** (1951) 760–766.

[7] S. Filipovski, R. Jajcay, Bounds for the energy of graphs, *Mathematics* **9** (2021) #1687.

[8] N. Ghanbari, On the Sombor characteristic polynomial and Sombor energy of a graph, 2021, `https://arxiv.org/abs/2108.08552`.

[9] K. J. Gowtham, N. N. Swamy, On Sombor energy of graphs, *Nanosystems: Phys. Chem. Math.* **12** (2021) 411–417.

[10] Gutman, Ivan, The energy of a graph, *Ber. Math.-Statist. Sekt. Forsch. Graz* **103** (1978) 1–22

[11] I. Gutman, Geometric approach to degree–based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.* **86** (2021) 197–220.

[12] I. Gutman, Spectrum and energy of Sombor matrix, *Military Tech. Courier* **69** (2021) 551–561.

[13] I. Gutman, I. Redžepović, J. Rada, Relating energy and Sombor energy, *Contrib. Math.* **4** (2021) 41–44.

[14] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, 2013.

[15] A. Jahanbani, R. Zambrano, Koolen–Moulton–type upper bounds on the energy of a graph, *MATCH Commun. Math. Comput. Chem.* **83** (2020) 497–518.

[16] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[17] Z. Lin, L. Miao, On the spectral radius, energy and Estrada index of the Sombor matrix of graphs, 2021, `https://arxiv.org/abs/2102.03960`.

[18] Z. Lin, B. Deng, L. Miao, He Li, On the spectral radius, energy and Estrada index of the arithmetic-geometric matrix of a graph, *Discr. Math. Algor. Appl.* (2021) #2150108,

[19] V. Nikiforov, Beyond graph energy: norms of graphs and matrices, *Lin. Algebra Appl.* **506** (2016) 82–138.

[20] S. Pirzada, B. A. Rather, K. C. Das, Y. Shang, On Sombor eigenvalues and Sombor energy of graphs, preprint.

[21] S. Pirzada, H. A. Ganie, Bilal A. Rather, R. U. Shaban, On generalized distance energy of graphs, *Lin. Algebra Appl.* **603** (2020) 1–19.

[22] S. Pirzada, B. A. Rather, R. Ul Shaban, Merajuddin, On graphs with minimal distance signless Laplacian energy, *Acta Univ. Sap. Math.* **13** (2021) 450–467.

[23] B. A. Rather, On distribution of Laplacian eigenvalues of graphs, PhD thesis, 2020, `http://arxiv.org/abs/2107.09161`.

[24] J. H. Smith, Some properties of the spectrum of a graph, in: R. K. Guy (Ed.), *Combinatorial Structures and Their Applications*, Gordon & Breach, 1970, pp. 403–406.

[25] T. Réti, T. Došlić, A. Ali, On the Sombor index of graphs, *Contrib. Math.* **3** (2021) 11–18.

[26] I. Redžepović, Chemical applicability of Sombor indices, *J. Serb. Chem. Soc.* **86** (2021) 445–457.

[27] A. Ülker, A. Gürsoy, N. K. Gürsoy, The energy and Sombor index of graphs, *MATCH Commun. Math. Comput. Chem.* **87** (2022) 51–58.