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# Lanzhou Index of Trees with Fixed Maximum and Second Maximum Degree

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**Abstract**

Let $G$ be a finite and simple graph with vertex set $V(G)$. The Lanzhou index $G$ is defined as

$$L_z(G) = \sum_{u \in V(G)} d_{\overline{G}}(u) d_G(u)^2;$$

where $d_G(u)$ denotes the degree of vertex $u$ in $G$. Dehgardi and Liu [MATCH Commun. Math. Comput. Chem. 86 (2021) 3–10] proved that for any tree $T$ of order $n \geq 11$ with maximum degree $\Delta$, $L_z(T) \geq (n - \Delta - 1)(4n + \Delta^2 - 12) + \Delta(n - 2)$. In this paper, we generalize the foregoing bound and show that for non-spider tree $T$ of order $n \geq 11$ $L_z(T) \geq (n-1)(\Delta^2 + \Delta'^2) - (\Delta^3 + \Delta'^3) - (3n - 10)(\Delta + \Delta') + (4n^2 - 14n + 4)$, where $\Delta$ and $\Delta'$ represents the maximum and second maximum degree of $T$. This result is an improvement of existing lower bounds. We also characterize the corresponding extremal trees.

## 1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The neighbourhood of a vertex $v$, denoted by $N_G(v)$, is the set of all vertices which are adjacent to $v$. The degree $d_G(u)$ of $u$

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in $G$ is the cardinality of $N_G(v)$. We use $\Delta = \Delta(G)$ for the maximum degree of the graph $G$. Let $v$ be a vertex of maximum degree in $G$. We define $\Delta' = \max\{d_G(u) : u \in V(G) \setminus \{v\}\}$ and call it the second maximum degree of $G$. The complement of graph $G$, denoted by $\overline{G}$, is the graph whose vertex set is same as of $G$ and two vertices are adjacent in $\overline{G}$ if they are not adjacent in $G$. The distance between two vertices is the length of a shortest path between them. We use $d(u, v)$ for the distance between two vertices $u$ and $v$. Given a tree, a vertex of degree at least 3 is called a core vertex or a core. A vertex of degree 2 is called a path vertex, and a vertex of degree 1 is called a leaf. For a core $v$, we often consider the subtrees created by removing $v$ from the tree, and call them the subtrees of its neighbors (one subtree for each neighbor). We sometimes consider the BFS tree that is created by rooting the tree at $v$. If $u$ is a vertex in $T$ other than the root, then the parent of $u$ is the vertex connected to $u$ on the path to the root. A subtree of a neighbor of a core $v$ that is a path (without any cores) is called a leg (or a standard leg) of $v$. If a leg consists of a single vertex (that is, $v$ is connected to a leaf), we call it a short leg, and otherwise it is called a long leg. A double spider is a tree having exactly two core vertices where as a spider is a tree with exactly one core vertex. We denote the set of all $n$-vertex double spider by $\mathcal{S}(n, p, q)$, where $p$ and $q$ represents the degrees of cores. A double star is a double spider that have adjacent cores and both the cores have short legs only. We use $\mathcal{T}(n, \Delta, \Delta')$ for the set of all $n$-vertex tree having maximum and second maximum degree $\Delta$ and $\Delta'$, respectively.

A graph invariant (also known as topological index) is a numerical value associated to a graph which is structurally invariant. A large variety of degree based topological indices has been defined in the mathematical and mathematico-chemical. Here, we consider a variant of the topological index, named the Lanzhou index. The Lanzhou index of $G$, denoted $L_z(G)$, is defined in [2] as the sum of weights $d_{\overline{G}}(u)d_G(u)^2$ of all vertices $u$ of $V(G)$, that is,

$$L_z(G) = \sum_{u \in V(G)} d_{\overline{G}}(u)d_G(u)^2.$$

Vukiĉević, Li, Sedlar and Doslic, in [2], proved the following lower bounds on the Lanzhou index of trees.

**Theorem 1.1** *[2] For any tree $T$ of order $n \geq 15$,*

$$L_z(T) \geq (n-1)(n-2)$$

*with equality if and only if $T = S_n$.*

**Theorem 1.2** *[2] For any tree $T$ of order $n$ with maximum degree at most* 4*,*

$$L_z(T) \geq 4n^2 - 18n + 20.$$

with equality if and only if $T = P_n$.

Recently, Dehgardi and Liu, in [1], proved the following lower bound on the Lanzhou index of trees.

**Theorem 1.3** *For any tree $T$ of order $n \geq 11$ with maximum degree $\Delta$,*

$$L_z(T) \geq (n - \Delta - 1)(4n + \Delta^2 - 12) + \Delta(n-2),$$

*with equality if and only if $T$ is a spider with the center of degree $\Delta$.*

The tree with only one core is a spider. Thus through out the article we consider all trees having at least two cores. In this paper, we establish a best possible lower bound for the Lanzhou index of trees in terms of their order, maximum and second maximum degree. Also we characterize all extremal trees, as a generalization of foregoing result.

# 2 Main Results

In this section, we present a sharp lower bound for the Lanzhou index of trees in terms of their order, maximum degree and second maximum degree. We also characterize all trees whose Lanzhou index achieves the lower bound. In the lemma below, we determine Lanzhou index for double spider $S(n, \Delta, \Delta')$.

**Lemma 2.1** *For a double spider $S(n,\Delta,\Delta')$ with $\Delta \geq \Delta' \geq 3$,*

$$\begin{aligned} L_z(S(n,\Delta,\Delta')) &= (n-1)(\Delta^2+\Delta'^2)-(\Delta^3+\Delta'^3) \\ &- (3n-10)(\Delta+\Delta')+(4n^2-14n+4)\ . \end{aligned}$$

Proof : Recall that the spider $S(n,\Delta,\Delta')$ has exactly two core vertices with degree $\Delta$ and $\Delta'$. Let $u$ and $v$ be these two cores with degrees $\Delta$ and $\Delta'$, respectively. Let $T = S(n,\Delta,\Delta')$. Since $S(n,\Delta,\Delta')$ does not contains any other core vertices (the vertices whose degree is at least three) except $u$ and $v$, each component of $T \setminus \{u\}$ (or $T \setminus \{v\}$) except one must be a leg. Hence there are $\Delta+\Delta'-2$ leafs of $S(n,\Delta,\Delta')$ which are came from $\Delta-1$ legs of $v$ and $\Delta'-1$ legs of $u$. The remaining vertices (except $u$, $v$ and leafs) are two degree vertices and hence they have $n-(\Delta+\Delta')$ in numbers. Thus the topological index $L_z(T)$ is given by

$$\begin{aligned} L_z(T) &= (n-1-\Delta)\Delta^2+(n-1-\Delta')\Delta'^2+4(n-3)\left(n-(\Delta+\Delta')\right) \\ &\quad +(n-2)(\Delta+\Delta'-2) \\ &= (n-1-\Delta)\Delta^2+(n-1-\Delta')\Delta'^2-(3n-10)(\Delta+\Delta') \\ &\quad +(4n^2-14n+4). \end{aligned}$$

Hence we obtain our result. ■

**Definition 2.1** (EDI operation) Let $T$ be a tree with at least one core. Choose a vertex $x$ adjacent to a core $v$ and a leaf $w$ such that they are not in same component of $T-v$. By an *edge-deletion-inclusion operation (simply, EDI operation)* on two vertices $x$ and $w$ we mean the deletion of the edge $\{v\,x\}$ and creation a new edge $\{x\,w\}$. Note that if $w$ and $x$ are in same component of $T-v$, then an EDI operation on these two vertices makes the tree as forest. So we avoid such type choices of $w$ and $x$. In Figure 1 we shows EDI operations for two trees $T_1$ and $T_2$.

**Lemma 2.2** *Let $T$ be a tree of order $n \geq 11$ with at least two cores. Let $v$ be a core such that $d_G(v) \leq \frac{n}{2}$. If $T_{wx}$ be a tree obtained from $T$ under EDI operation on a leaf $w$ and a vertex $x$ adjacent to $v$, then $L_z(T) > L_z(T_{wx})$.*

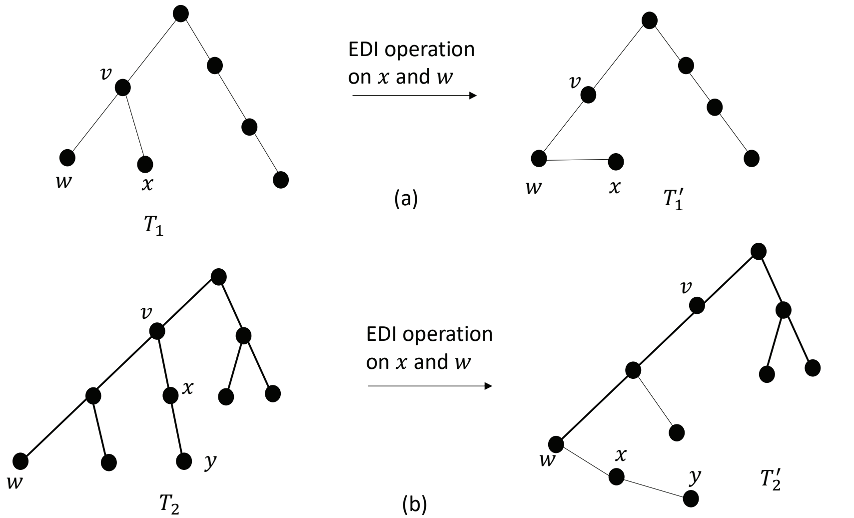


**Figure 1.** The tree $T_1$ has only one core $v$ where as the $T_2$ has three cores. The trees $T_1^{'}$ and $T_2^{'}$ represent the resulting tree under an EDI operation on the vertices $x$ and $w$. Note that the vertex $x$ is hanged on the core $v$.

Proof: Here the vertex $x$ is adjacent to the core $v$ and $w$ be a leaf which is not in $T_x$ (the component of $T - \{v\}$ that contains $x$). Let the parent of $w$ be $u$ and the degree of $v$ be $m \geq 3$ (as $v$ is a core). Let $S = \{v, w\}$. Under an EDI operation on the two vertices $x$ and $w$, the following changes occur in the degrees of vertices : $d_{T_{wx}}(w) = 2$, $d_{T_{wx}}(v) = m - 1$ and $d_T(y) = d_{T_{wx}}(y)$ for all $y \in V(T) \setminus S$. The Lanzhou indices for $T$ and $T_{wx}$ are given by

$$
\begin{aligned}
L_z(T) &= \sum_{y \in V(T)} d_{\overline{T}}(y) d_T^2(y) \\
&= \sum_{y \in V(T) \setminus S} d_{\overline{T}}(y) d_T^2(y) + (n-1-m)m^2 + (n-2) \qquad (1) \\
L_z(T_{wx}) &= \sum_{y \in V(T_{wx})} d_{\bar{T}_{wx}}(y) d_{T_{wx}}^2(y) \\
&= \sum_{y \in V(T) \setminus S} d_{\overline{T}}(y) d_T^2(y) + (n-m)(m-1)^2 + 4(n-3) \quad (2)
\end{aligned}
$$

Thus from (1) and (2), we get

$$\begin{aligned} L_z(T) - L_z(T_{wx}) &= (n-1-m)m^2 + (n-2) - (n-m)(m-1)^2 \\ &- 4(n-3) = 2mn - 3m^2 - 4n + m + 10 \\ &= 2n(m-2) - 3m^2 + m + 10. \end{aligned} \tag{3}$$

Since $n \geq 2m$, so we may take $n = 2m + \ell$ for some non-negative integer $\ell$. Then (3) reduces to

$$\begin{aligned} L_z(T) - L_z(T_{wx}) &= m^2 + (2\ell - 7)m - 4\ell + 10 \\ &= (m + 2\ell - 5)(m - 2). \end{aligned} \tag{4}$$

Since $m \geq 3$, so $m + 2\ell \geq 7$ when $\ell \geq 2$ and hence from (4), we have $L_z(T) - L_z(T_{wx}) > 0$ when $\ell \geq 2$. Again since $n \geq 11$ and $n = 2m + \ell$, so $m \geq 6$ and $m \geq 5$ according to $\ell = 0$ and $\ell = 1$. Thus in this case, we have $m + 2\ell \geq 6$ and hence we obtain $L_z(T) - L_z(T_{wx}) > 0$ in the case when $\ell = 0, 1$. This completes the proof. ■

**Corollary 2.1** *Let $T$ be a tree having at least two cores. Let $u$ be a vertex of maximum degree and $v \neq u$ be an another core. If $T'$ be a tree obtained by an EDI operation on a leaf $w$ and an adjacent vertex of $v$, then $L_z(T) > L_z(T')$. In particular, the spider $S(\Delta)$ has lower Lanzhou index than the same for double spider $S(n, \Delta, \Delta')$.*

Proof: Here $d_T(v) \leq d_T(u)$ and hence $d_T(v) \leq \frac{n}{2}$. Thus from applying Lemma 2.2, we obtained the result. ■

**Lemma 2.3** *Let $T$ be a tree with at least two cores and $x$ be a vertex of maximum degree. Then each component of $T - x$ can be be transform to a path by successive EDI operations. Moreover, an $n$-vertex tree with at least one core can be transform to a $n$-vertex spider.*

Proof : If $x$ be a vertex of maximum degree, then the degree of each core $u$ in each component of $T - x$ must be at most $\lfloor \frac{n}{2} \rfloor$. So we can apply EDI operation for each cores in the components of $T - x$. An EDI operation reduces the degree of a core vertex by 1, so every core vertex can be made a non-core vertex (a vertex with degree 1 or 2) by repeated implementation of EDI operation. Therefore, every component of $T-x$ can be transform to a path by applying successive EDI operations on it. Again since each component of $T-x$ is a path, so $T$ transform to a spider with center at $x$. In Figure 2, we have taken a component of a mother tree with a vertex $x$ of degree at least 4 and perform EDI operations to transform it into a path. ■

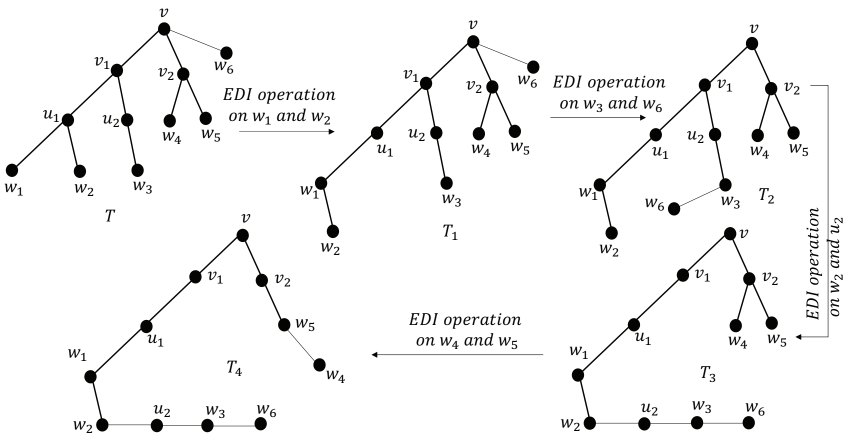


**Figure 2.** In this figure $T$ has taken as some component of a tree with respect to the maximum degree. The transformation of a component to a path on same vertices by successive EDI operations. The cores (the vertices of degree at least three) of $T$ are $u_1$, $v_1$, $v$ and $v_2$.

We are now in position to prove the main theorem of this section.

**Theorem 2.1** *For every tree $T \in \mathcal{T}(n, \Delta, \Delta')$ with $n \geq 11$,*

$$L_z(T) \geq (n-1)(\Delta^2+\Delta'^2)-(\Delta^3+\Delta'^3)-(3n-10)(\Delta+\Delta')+(4n^2-14n+4).$$

*The equality holds if and only if $T$ is a double spider with the degrees of cores $\Delta$ and $\Delta'$.*

Proof : The main idea behind the proof of this theorem is to transform the tree $T$ to $S(n, \Delta, \Delta')$ under successive EDI operations. Let $u$ and $v$ be two vertices in $T$ with degrees $\Delta$ and $\Delta'$, respectively, where $\Delta \geq \Delta' \geq 3$.

If we apply EDI operations successively for every core vertex except $u$ and $v$, then the degree of each vertex except $u$ and $v$ will be either 2 or 1. Algorithm 1 gives in details how the EDI operations has been performed to convert all cores vertices (except $u$ and $v$) to non-cores. After the execution of Algorithm 1, $T$ transform to a double spider $S(n, \Delta, \Delta')$ and consequently, by applying Lemma 2.2, we obtained $L_z(T) > L_z(S(n, \Delta, \Delta'))$. Hence we obtained our result by Lemma 2.1.

**Algorithm 1** An algorithm for the transformation of a tree with at least two cores to a double spider under successive EDI operations.

**Input:** A tree $T$ with maximum degree $\Delta \geq 3$ and second maximum degree $\Delta' \geq 3$.

**Output :** The double spider $S(n, \Delta, \Delta')$.

**Initialization :** Let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$ and $N(u) = \{u_1, u_2, \ldots, u_{\Delta'}\}$. Let $T$ be rooted at $v$. With no loss of generality, we assume $u_1$ is a parent of $u$ and $u$ lies in the component of $T - v$ that contains $v_1$. Let $P_{vu} : v_1 x_1 \ldots x_m u_1 u$ be the $vu$-path. A vertex $x$ is called a hanging vertex in a path $P$ if it is adjacent to some intermediate vertex of $P$. Let $H_{vu}$ be the set of all hanging vertices of the path $P_{vu}$. If $H_{vu}$ is empty, then go to Step-IV with $T'' = T$, otherwise go to Step-I.

**Step-I :** Do the EDI operation on a vertex $x \in H_{vu}$ and a leaf of a component of $T \backslash \{u\}$. Call the resulting tree be $T_x$. Set $H_{vu} = H_{vu} \backslash \{x\}$.

**Step-II :** Do the Step-I with $T = T_x$ and repeat these processes until $H_{vu}$ is empty set.

**Step-III :** Give the name of the resulting tree as $T''$ after doing all operations as mentioned in Step-II and Step-III. Note that in $T''$, all the intermediate vertices of $P_{vu}$ have degree 2 and all other vertices of the component $T''_{v_1}$ of $T'' - v$ are at deeper levels than $u$. Also the degrees of $u$ and $v$ in $T''$ remains unchanged.

**Step-IV :** Let $T''_{u_i}$ be the component of $T'' - u$ that contains $u_i$. For each $i$ do Step-V and Step-VI.

**Step-V :** Visit a core $w$ of a largest distance to $v$ (that is, a core of largest depth in the rooted tree) in the component $T''_{u_i}$. Let $N(w) = \{w_1, w_2, \ldots, w_m\}$. Each component of $T'' - w$ are legs except the component containing the parent of $w$ (since otherwise $w$ is not a core of maximum depth). Do EDI operations among the vertices of legs of $w$ such that these legs transformed into a single leg at $w$ (Lemma 2.3 gives the guarantees of such type of transformation). Let the resultant tree be $T_w$.

**Step-VI:** Repeat the Step-V until $T''_{u_i}$ reduces to a path.

**Step-VII:** Do similar computations as of Step-V and Step-VI for the components $T_{v_2}, T_{v_3}, \ldots, T_{v_\Delta}$ of $T - v$. ■

**Example 2.1** In this example, we explain the intermediate steps of Algorithm 1 for the tree $T$ as show in Figure 3. Applying EDI operations as described in our presented algorithm, we make the tree $T$ into a double spider $S(21, 4, 4)$ with centers $v$ and $u$ (green colored vertices). As $x$ is the only hanging vertices of $P_{vu} : v\, v_1\, u$, so under Step-I and Step-II, $T$ transform to $T''$ by only one EDI operation on $x$ and the leaf $w_1$. Now we transform the each component of $T'' - u_1$ to a leg at $u_1$. For this we visit a deepest core (the vertex $x$) in a component of $T'' - u$. For more details see operations described in Figure 4.

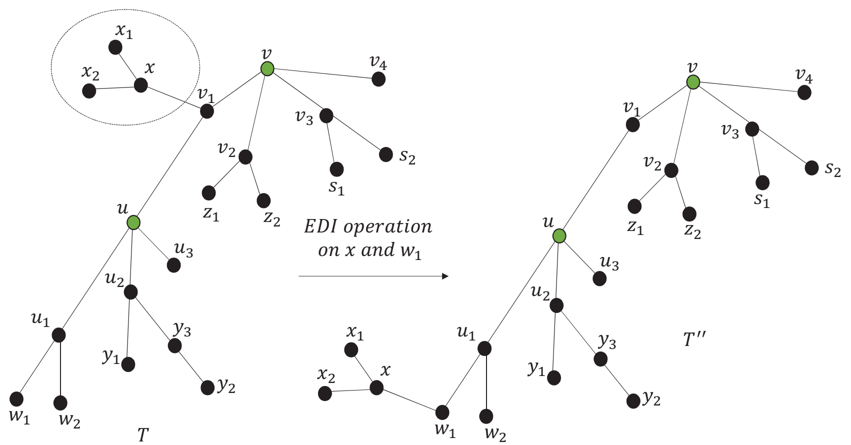


**Figure 3.** The path $P_{vu} = v\, v_1\, u$ has only one hanging vertices, namely $x$. The tree $T''$ is obtained from $T$ by EDI on $x$ and $w_1$.

In Figure 5, we transform each component of $T_1 - u$ into a path, i.e, a leg hooked at $u$.

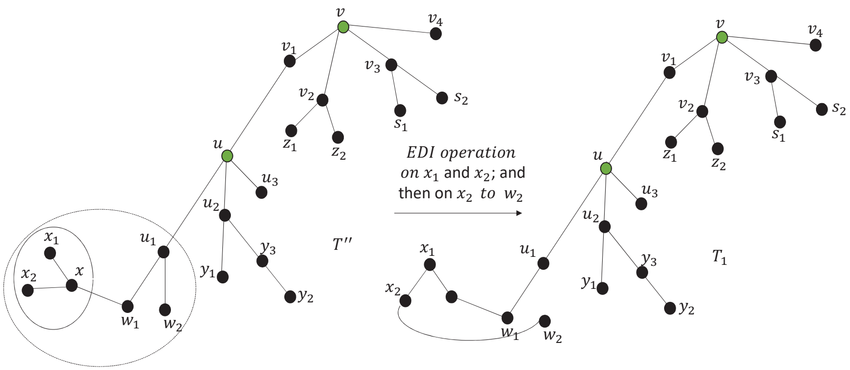


**Figure 4.** The transformation of $T''_{u_1}$ (the component of $T'' - u$ containing $u_1$) to a path.

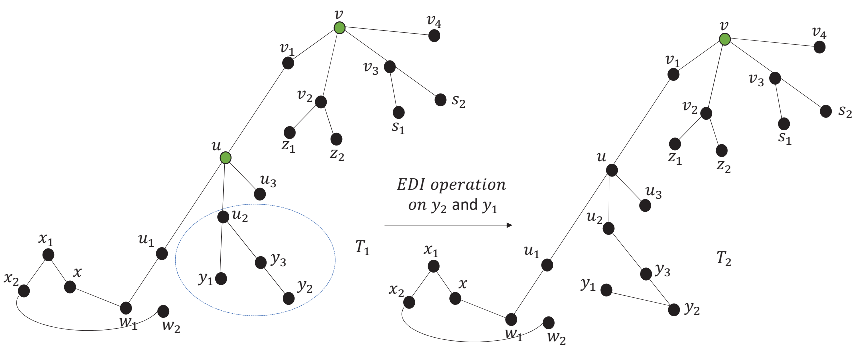


**Figure 5.** The transformations of each component for $T_1 - u$ into a path. The resulting tree is $T_2$.

Each component of $T_2 - u$ (except one) are legs of $u$. So we visit the components of $T_2 - v$ except the component that contains $u$ to make them as legs at $v$. In Figure 6 the steps have been shown.

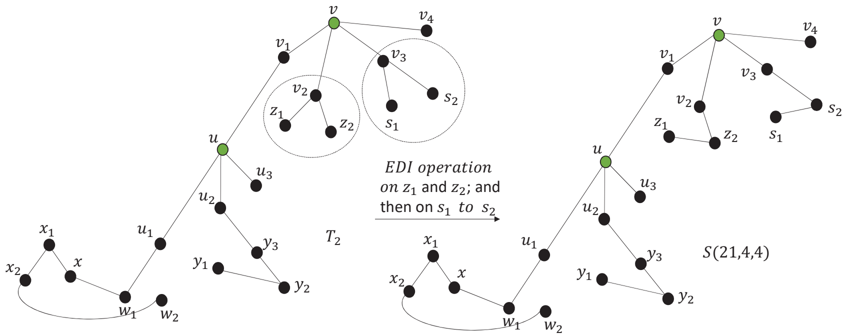


**Figure 6.** The transformations of each component of $T_2 - v$ except the component that contains $v_1$. The resulting tree is a double spider $S(21, 4, 4)$ with centres $u$ and $v$.

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[2] D. Vukičević, Q. Li, J. Sedlar, T. Došlić, Lanzhou index, *MATCH Commun. Math. Comput. Chem.* **80** (2018) 863–876.