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# The Proof of a Conjecture on the Reduced Sombor Index*

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**Abstract**

A new type of vertex-degree-based topological indices of a graph, called the reduced Sombor index, is proposed by Gutman very recently. Accurately, for a graph $G = (V(G), E(G))$, the reduced Sombor index of $G$, denoted by $SO_{red}(G)$, is defined as

$$SO_{red}(G) = \sum_{uv \in E(G)} \sqrt{(d(u)-1)^2 + (d(v)-1)^2},$$

where $d(u)$ denotes the degree of the vertex $u$ in $G$. In this note, we fix a flaw of a theorem on the upper bound for the reduced Sombor index of a bipartite graph and prove that Turán graph has the maximum reduced Sombor index among all $k$-chromatic graphs, which solves a conjecture on the reduced Sombor index proposed by Liu, You, Tang and Liu (On the reduced Sombor index and its Applications, MATCH Commun. Math. Comput. Chem. 86 (2021) 729-753).

## 1 Introduction

In this note, we are concerned with simple graphs, that is graphs without directed, weighted or multiple edges, and without self loops. We refer to [2] for undefined ter-

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minology and notation. For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set, respectively. We denote the number of vertices of $G$ by $n$. For a vertex $v \in V(G)$, the *degree* of $v$, denoted by $d(v)$, is the number of edges incident with $v$ in $G$. The maximum degree of $G$, denoted by $\Delta(G)$ (simply by $\Delta$), is $\max\{d(v) : v \in V(G)\}$. The *molecular graph* is the graph of maximum degree at most four.

Two new types of vertex-degree-based topological indices of a graph, called the Sombor index and the reduced Sombor index, were proposed by Gutman [6]. Precisely, for a graph $G$, its Sombor index $SO(G)$ and reduced Sombor index $SO_{red}(G)$ are defined respectively as follows:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2},$$

$$SO_{red}(G) = \sum_{uv \in E(G)} \sqrt{(d(u)-1)^2 + (d(v)-1)^2}.$$

A number of papers have appeared used to show the nature of the Sombor index after the first paper on the Sombor index was presented [6]. Gutman showed that for a tree of order $n$,

$$2\sqrt{5} + 2(n-3)\sqrt{2} = SO(P_n) \leq SO(T) \leq (n-1)SO(K_{1,n-1}) = \sqrt{n^2 - 2n + 2},$$

where $P_n$ is a path of order $n$. The left-side of equality holds if and only if $T \cong P_n$, and the right-side of equality holds if and only if $T \cong K_{1,n-1}$. Chen, Li and Wang [3] determined the extremal values of the Sombor index of trees with some given parameters and characterize completely the corresponding extremal trees. Horoldagva and Xu [7] obtained the sharp lower and upper bounds on the Sombor index of a connected graph, and characterize graphs for which these bounds are attained.

For the reduced Sombor index, Deng, Tang and Wu [5] gave the sharp upper bound and corresponding extremal graphs for all molecular trees of given order. The Sombor index and the reduced Sombor index are studied on molecular trees, molecular unicyclic graphs, molecular bicyclic graphs, molecular tricyclic graphs [9]. Moreover, they determined the first fourteen minimum molecular trees, the first four minimum molecular unicyclic graphs, the first three minimum molecular bicyclic graphs, the first seven minimum molecular tricyclic graphs. For more results for the Sombor index of graphs, we refer to [1,4,10].

Liu, You, Tang, Liu [8] obtained some bounds for reduced Sombor index of graphs with given several parameters and some special graphs. For instance, they established the following upper bound for the reduced Sombor index of a bipartite graph of order $n$.

**Theorem 1.1.** *( [8]) Let $G$ be a bipartite graph with $n$ vertices. Then*

$$SO_{red}(G) \leq \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor \sqrt{\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)^2 + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)^2}$$

*with equality if and only if $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

The statement of Theorem 1.1 is not complete since

$$SO_{red}(K_{1,3}) = 6 > 4\sqrt{2} = SO_{red}(K_{2,2}).$$

Indeed, for any bipartite graph $G$ of order $n \geq 5$, the statement of Theorem 1.1 is true, see the next section.

For a graph $G$, the *complement* $\overline{G}$ of $G$ is the graph with $V(\overline{G}) = V(G)$, in which two vertices $u$ and $v$ are adjacent if and only if $uv \notin E(G)$. A *complete $k$-partite graph* $K_{n_1,n_2,\ldots,n_k}$ is one whose vertex set can be partitioned into $k$ parts, in such a way that any two vertices in different parts are adjacent. The *Turán graph* $T_{n,k}$ is a special complete $k$-partite graph with $|n_i - n_j| \leq 1$ for $1 \leq i, j \leq k$.

Let $k$ be a positive integer. A $k$-coloring of a graph $G$ is a function $c : V(G) \mapsto \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E(G)$. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum $k$ for which $G$ has a $k$-coloring. Furthermore, $G$ is called $k$-chromatic if $\chi(G) = k$. Since $SO_{red}(G) < SO_{red}(G + e)$ for any $e \in E(\overline{G})$, for a $k$-chromatic graph $G$, $SO_{red}(G) \leq SO_{red}(H)$, where $H$ is a complete $k$-partite graph with the same order as that of $G$, see [8].

Based on the above theorem and observation, Liu, You, Tang, Liu [8] proposed the following conjecture.

**Conjecture 1.2.** *( [8]) If $\chi(G) \leq k$, then $SO_{red}(G) \leq SO_{red}(T_{n,k})$, with equality if and only if $G \cong T_{n,k}$.*

The aim of the note is to prove Conjecture 1.2.

## 2 The slight modification of Theorem 1.1

First we show the following theorem, which fix the flaw of Theorem 1.1.

**Theorem 2.1.** *Let $G$ be a bipartite graph with $n \geq 5$ vertices. Then*

$$SO_{red}(G) \leq SO_{red}(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}),$$

*with equality if and only if $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

*Proof.* Let $G$ be a bipartite graph with $n$ vertices. We suppose that $s \geq t$. Then by the definition of reduced Sombor index, $SO_{red}(G) \leq SO_{red}(K_{s,t}) = st\sqrt{(s-1)^2 + (t-1)^2} = s(n-s)\sqrt{(s-1)^2 + (n-s-1)^2}$.

Let $f(x) = x(n-x)\sqrt{(x-1)^2 + (n-x-1)^2}$, where $\lceil \frac{n}{2} \rceil \leq x \leq n-1$. It easily follows that $f'(x) = \frac{(n-2x)(3x^2-3nx+n^2-2n+2)}{\sqrt{(x-1)^2+(n-x-1)^2}}$. One can see that $n - 2x \leq 0$ for $x \geq \lceil \frac{n}{2} \rceil$. Next, we can let $g(x) = 3x^2 - 3nx + n^2 - 2n + 2$. Note that the $g(x)$ is monotone increasing in $x \geq \lceil \frac{n}{2} \rceil$, while the minimum is attained at $x = \lceil \frac{n}{2} \rceil$, where it turns into

$$g\left(\left\lceil \frac{n}{2} \right\rceil\right) \geq \frac{n^2}{4} - 2n + 2.$$

Rewrite $\frac{n^2}{4} - 2n + 2 \geq 0$ as $(n-4)^2 - 8 \geq 0$. Thus, $g(\lceil \frac{n}{2} \rceil) \geq 0$ for $n \geq 7$. Summing up the above, we conclude that $f'(x) \leq 0$ for $\lceil \frac{n}{2} \rceil \leq x \leq n-1$. It remains to consider the case when $5 \leq n \leq 6$:

(i) If $n = 6$, then $SO_{red}(K_{5,1}) = 20 < 8\sqrt{10} = SO_{red}(K_{4,2}) < 18\sqrt{2} = SO_{red}(K_{3,3})$.

(ii) If $n = 5$, then $SO_{red}(K_{4,1}) = 12 < 6\sqrt{5} = SO_{red}(K_{3,2})$. Thus,

$$SO_{red}(G) \leq SO_{red}(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}),$$

with equality if and only if $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. ■

# 3 The proof of Conjecture 1.2

In order to solve the Conjecture 1.2, we give the following two lemmas.

**Lemma 3.1.** *Let $a, b, c$ and $s$ be positive integers. If $a \geq b \geq c, a \geq c+2$ and $s \geq a+b+c$, then* $(a-1)b\sqrt{(s-a)^2+(s-b-1)^2}+b(c+1)\sqrt{(s-b-1)^2+(s-c-2)^2}$
$> ab\sqrt{(s-a-1)^2+(s-b-1)^2}+bc\sqrt{(s-b-1)^2+(s-c-1)^2}.$

*Proof.* One can see that the above formula can be divided into three parts:

$$-b\sqrt{(s-a)^2+(s-b-1)^2}+b\sqrt{(s-b-1)^2+(s-c-2)^2} \quad (1)$$
$$ab\sqrt{(s-a)^2+(s-b-1)^2}-ab\sqrt{(s-a-1)^2+(s-b-1)^2} \quad (2)$$
$$bc\sqrt{(s-b-1)^2+(s-c-2)^2}-bc\sqrt{(s-b-1)^2+(s-c-1)^2} \quad (3)$$

Thus,

$$\begin{aligned}(1) &= b\frac{(s-c-2)^2-(s-a)^2}{\sqrt{(s-b-1)^2+(s-c-2)^2}+\sqrt{(s-a)^2+(s-b-1)^2}}\\ (2) &= \frac{ab[2(s-a)-1]}{\sqrt{(s-a)^2+(s-b-1)^2}+\sqrt{(s-a-1)^2+(s-b-1)^2}}\\ (3) &= \frac{-bc[2(s-c)-3]}{\sqrt{(s-c-2)^2+(s-b-1)^2}+\sqrt{(s-c-1)^2+(s-b-1)^2}}.\end{aligned}$$

Since $s \geq a+b+c$, $a \geq c+2$, we can obtain $(1) \geq 0$. Since $a \geq b \geq c, a \geq c+2$ and $s \geq a+b+c$, we have the denominator of (2) is strictly less than that of (3). Moreover, Since $a \geq b \geq c \geq 1, a \geq c+2, s \geq a+b+c$, we have

$$\begin{aligned}a[2(s-a)-1]-c[2(s-c)-3] &= 2s(a-c)-2a^2-a+2c^2+3c\\ &\geq 2(a+b+c)(a-c)-2a^2-a+2c^2+3c\\ &= 2a^2+2ab-2bc-2c^2-2a^2-a+2c^2+3c\\ &= 2ab-2bc-a+3c=(2b-1)(a-c)+2c>0.\end{aligned}$$

Thus, $ab[2(s-a)-1] > bc[2(s-c)-3]$, implying $(2)+(3) > 0$.

Summing up the above, we arrive at our conclusion. ∎

**Corollary 3.2.** *Let $n_1, n_2, \ldots, n_k$ be positive integers with $n_1 \geq n_2 \geq \cdots \geq n_k$. If $n_1 \geq n_k+2$ and $n = n_1+n_2+\cdots+n_k$, then*

$$
\begin{aligned}
&\sum_{i=2}^{k-1}(n_1-1)n_i\sqrt{(n-n_1)^2+(n-n_i-1)^2}\\
&+\sum_{i=2}^{k-1}(n_k+1)n_i\sqrt{(n-n_k-2)^2+(n-n_i-1)^2}\\
&>\sum_{i=2}^{k-1}n_1n_i\sqrt{(n-n_1-1)^2+(n-n_i-1)^2}\\
&+\sum_{i=2}^{k-1}n_kn_i\sqrt{(n-n_k-1)^2+(n-n_i-1)^2}\,,
\end{aligned}
$$

*Proof.* The above formula can be expressed as the sum of the following $k-2$ terms: $(n_1-1)n_i\sqrt{(n-n_1)^2+(n-n_i-1)^2}+(n_k+1)n_i\sqrt{(n-n_k-2)^2+(n-n_i-1)^2}$ $-n_1n_i\sqrt{(n-n_1-1)^2+(n-n_i-1)^2}-n_kn_i\sqrt{(n-n_k-1)^2+(n-n_i-1)^2}$, where $i\in\{2,\ldots,k-1\}$.

Take $a=n_1$, $b=n_i$, $c=n_k$ and $s=n$. By Lemma 3.1, the value of the above formula is positive. The result then follows. ■

**Lemma 3.3.** *Let $s,t$ be two positive integers with $s>t$. Let $f(x)=x(s-x-t)\sqrt{(s-x-1)^2+(x+t-1)^2}$. Then $f(x)$ is monotone decreasing for $x\geq\frac{s-t}{2}$.*

*Proof.* First, one can obtain $f'(x)=\frac{(s-2x-t)[(s-x-1)^2+(x+t-1)^2-x(s-x-t)]}{\sqrt{(s-x-1)^2+(x+t-1)^2}}$. It is easy to see that $s-2x-t\leq 0$ for $x\geq\frac{s-t}{2}$. Next, let $g(x)=(s-x-1)^2+(x+t-1)^2-x(s-x-t)$. It can be rewritten as $g(x)=3x^2-x(3s-3t)+s^2-2s+t^2-2t+2$.

Note that the $g(x)$ is monotone increasing in $x\geq\frac{s-t}{2}$, while the minimum is attained at $x=\frac{s-t}{2}$, and

$$
g\left(\frac{s-t}{2}\right)=\frac{3(s+t-2)^2-(s-t)^2}{4}>0.
$$

So it follows that $f'(x)\leq 0$ for $x\geq\frac{s-t}{2}$, implying $f(x)$ is monotone decreasing for $x\geq\frac{s-t}{2}$.

■

**Corollary 3.4.** *Let $n_1, n_2, \ldots, n_k$ be positive integers with $n_1 \geq n_2 \geq \cdots \geq n_k$. If $n_1 \geq n_k + 2$ and $n = n_1 + n_2 + \cdots + n_k$, then*

$$(n_1 - 1)(n_k + 1)\sqrt{(n - n_1)^2 + (n - n_k - 2)^2} \geq n_1 n_k \sqrt{(n - n_1 - 1)^2 + (n - n_k - 1)^2}.$$

*Proof.* Let $f(x) = x(s - x - t)\sqrt{(s - x - 1)^2 + (x + t - 1)^2}$. We assume that $s = n = n_1 + n_2 + \cdots + n_k$ and $t = n_2 + n_3 + \cdots + n_{k-1}$. Since $n_1 \geq n_k + 2$, we have $n_1 - 1 \geq \frac{s-t}{2}$. Thus, by Lemma 3.3, $f(x)$ is monotone decreasing for $x \geq n_1 - 1$. Therefore, $f(n_1 - 1) \geq f(n_1)$, which implies that

$$(n_1 - 1)(n_k + 1)\sqrt{(n - n_1)^2 + (n - n_k - 2)^2} - n_1 n_k \sqrt{(n - n_1 - 1)^2 + (n - n_k - 1)^2} \geq 0.$$

■

**Lemma 3.5.** *Let $n_1, n_2, \ldots, n_k$ be positive integers with $n_1 \geq n_2 \geq \cdots \geq n_k$, $n_1 \geq n_k + 2$ and $n = n_1 + n_2 + \cdots + n_k$. If $G = K_{n_1, n_2, \ldots, n_k}$ and $G' = K_{n_1 - 1, n_2, \ldots, n_k + 1}$, then*

$$SO_{red}(G) < SO_{red}(G').$$

*Proof.* From the definition of reduced Sombor index, we have $SO_{red}(G') - SO_{red}(G) = \sum_{i=2}^{k-1}(n_1 - 1)n_i\sqrt{(n - n_1)^2 + (n - n_i - 1)^2} + (n_1 - 1)(n_k + 1)\sqrt{(n - n_1)^2 + (n - n_k - 2)^2} + \sum_{i=2}^{k-1}(n_k + 1)n_i\sqrt{(n - n_k - 2)^2 + (n - n_i - 1)^2} - \sum_{i=2}^{k-1} n_1 n_i \sqrt{(n - n_1 - 1)^2 + (n - n_i - 1)^2}$

$$-n_1 n_k \sqrt{(n - n_1 - 1)^2 + (n - n_k - 1)^2} - \sum_{i=2}^{k-1} n_k n_i \sqrt{(n - n_k - 1)^2 + (n - n_i - 1)^2}.$$

Thus, by Corollary 3.2 and Corollary 3.4, $SO_{red}(G') - SO_{red}(G) > 0$. ■

**Theorem 3.6.** *If $G = K_{n_1, \ldots, n_k}$ be a complete $k$-partite graph with $k \geq 2$, then*

$$SO_{red}(G) \leq SO_{red}(T_{n,k})$$

*with equality if and only if $G \cong T_{n,k}$.*

*Proof.* It is an immediate consequence of Theorem 3.5. ■

# 4 Concluding remarks

The reduced Sombor index is a new structure-descriptors whose properties await to examined. In this note, we show that among all $k$-chromatic graphs of order $n$, the Turán graph $T_{n,k}$ has the maximum reduced Sombor index. At prensent, the reduced Sombor indices of trees and molecular graphs were investigated. Exponential reduced Sombor index is defined as

$$e^{SO_{red}}(G) = \sum_{uv \in E(G)} e^{\sqrt{(d(u)-1)^2+(d(v)-1)^2}}.$$

Let $\mathcal{CT}_n$ be the set of molecular trees with $n$ vertices. An interesting conjecture on exponential reduced Sombor index was proposed in [8].

**Conjecture 4.1.** *Let $T \in \mathcal{CT}_n, n \geq 5$. Then*

$$e^{SO_{red}}(T) \leq \begin{cases} \frac{2}{3}(n+1)e^3 + \frac{1}{3}(n-5)e^{3\sqrt{2}}, & n \equiv 2(mod \quad 3) \\ \frac{1}{3}(2n+1)e^3 + \frac{1}{3}(n-13)e^{3\sqrt{2}} + 3e^{\sqrt{13}}, & n \equiv 1(mod \quad 3) \\ \frac{2}{3}ne^3 + \frac{1}{3}(n-9)e^{3\sqrt{2}} + 2e^{\sqrt{10}}, & n \equiv 0(mod \quad 3) \end{cases}$$

The conjecture is still open.

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