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On the Graphs with Minimum Sombor Index

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Abstract

For a graph G the Sombor index of G is defined as $\sum_{uv\in E(G)} \sqrt{d(u)^2 + d(v)^2}$, where $d(u)$ is the degree of u in G. In the current paper, we study the structure of a graph with minimum Sombor index among all graphs with fixed order and fixed size. It is shown that in every graph with minimum Sombor index the difference between minimum and maximum degrees is at most 1.

1 Introduction

All graphs considered in this paper are finite and simple. Let $V(G)$ and $E(G)$ be the vertex set and the edge set of G, respectively, and let $|V(G)| = n$, $|E(G)| = m$. We use $\mathbb{CG}(n, m)$ to denote the set of all connected graphs with n vertices and m edges. The degree of the vertex v is denoted by $d(v)$. We use $N(v)$ to denote the set of all vertices adjacent to the vertex v. For a graph G, $\delta(G)$ and $\Delta(G)$ stand for the minimum degree and the maximum degree of G, respectively. Recall that a graph G is k-regular if $d(v) = k$ for all $v \in V(G)$; a regular graph is one that is k-regular for some k. A *unicyclic graph* is a connected graph containing exactly one cycle. Ivan Gutman in [8] proposed a geometric approach for interpreting degree-based graph invariants, and according to this approach, he introduced the Sombor index, defined as,

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d(u)^2 + d(v)^2}.$$

Recently, Sombor index has been studied extensively by many authors, for instance see [1-21]. We define $s(n, m)$ as follows:

$$s(n, m) = min\{SO(G) \,|\, G \in \mathbb{CG}(n, m)\}.$$

In the next result we show that the difference of the minimum and maximum degrees of a graph with the smallest Sombor index does not exceed 1.

Theorem 1. *Suppose that* $G \in \mathbb{CG}(n, m)$ *and for each* $H \in \mathbb{CG}(n, m)$, $SO(G) \leq SO(H)$, *then* $\Delta(G) - \delta(G) \leq 1$.

Proof. By contradiction, suppose that $\Delta(G) - \delta(G) \geq 2$. Let p and q be those vertices of graph G such that $d(p) = \delta$ and $d(q) = \Delta$. Since $d(p) < d(q)$, q has at least a neighbor w which is not adjacent to p. We obtain a graph G' by removing the edge qw of G and adding edge pw. Suppose $S \subseteq E(G)$ be the set of all incident edges with p or q. There are two cases:

Case 1. Two vertices p and q are not adjacent. Let $c_1, c_2, \ldots, c_\delta$ and $d_1, d_2, \ldots, d_\Delta$ be the degree of vertices in $N(p)$ and $N(q)$ in G, respectively. With no loss of generality, we assume that $d_\Delta = d(w)$. We have,

$$SO(G) = \sum_{uv\in E(G)\setminus S} \sqrt{d^2(u) + d^2(v)} + \sum_{i=1}^{\Delta} \sqrt{\Delta^2 + d_i^2} + \sum_{i=1}^{\delta} \sqrt{\delta^2 + c_i^2}$$

and also we have,

$$SO(G') = \sum_{uv\in E(G)\setminus S} \sqrt{d^2(u) + d^2(v)} + \sum_{i=1}^{\Delta-1} \sqrt{(\Delta-1)^2 + d_i^2}$$
$$+ \sum_{i=1}^{\delta} \sqrt{(\delta+1)^2 + c_i^2} + \sqrt{(\delta+1)^2 + d_\Delta^2}.$$

We claim that $SO(G) > SO(G')$ or equivalently,

$$\sum_{i=1}^{\Delta-1} \left(\sqrt{\Delta^2 + d_i^2} - \sqrt{(\Delta-1)^2 + d_i^2}\right) - \sum_{i=1}^{\delta} \left(\sqrt{(\delta+1)^2 + c_i^2} - \sqrt{\delta^2 + c_i^2}\right)$$

$$+\sqrt{\Delta^2+d_\Delta^2}-\sqrt{(\delta+1)^2+d_\Delta^2}>0.$$

Note that since $\delta+1\leq\Delta-1$, it is enough to show that,

$$\sum_{i=1}^{\Delta}\Big(\sqrt{\Delta^2+d_i^2}-\sqrt{(\Delta-1)^2+d_i^2}\Big)-\sum_{i=1}^{\delta}\Big(\sqrt{(\delta+1)^2+c_i^2}-\sqrt{\delta^2+c_i^2}\Big)>0. \qquad (1)$$

One can see that for $0\leq b\leq a$, the function $f(x)=\sqrt{a^2+x^2}-\sqrt{b^2+x^2}$ is decreasing on the interval $(0,+\infty)$. Hence, it is enough to show that the above inequality holds, where $c_i=\delta$ and $d_j=\Delta$ for $i=1,\dots,\delta$ and $j=1,\dots,\Delta$ which means that,

$$\sqrt{2}\delta^2-\delta\sqrt{\delta^2+(\delta+1)^2}+\sqrt{2}\Delta^2-\Delta\sqrt{(\Delta-1)^2+\Delta^2}>0. \qquad (2)$$

Note that the following inequality holds:

$$\frac{\Delta}{\sqrt{2}(\Delta+1)}\leq\sqrt{2}\Delta-\sqrt{(\Delta-1)^2+\Delta^2}. \qquad (3)$$

To see this by multiplying both sides to $\sqrt{2}(\Delta+1)$ we find,

$$\Delta\leq 2\Delta(\Delta+1)-\sqrt{2}(\Delta+1)\sqrt{(\Delta-1)^2+\Delta^2},$$

or equivalently,

$$\sqrt{2}(\Delta+1)\sqrt{(\Delta-1)^2+\Delta^2}\leq 2\Delta^2+\Delta,$$

and so we have,

$$2(\Delta+1)^2((\Delta-1)^2+\Delta^2)\leq 4\Delta^4+4\Delta^3+\Delta^2.$$

By calculations we have, $\Delta\geq\sqrt{\frac{2}{3}}$, as desired. Moreover, we have the following inequality,

$$-\frac{\delta+1}{\sqrt{2}\delta}\leq\sqrt{2}\delta-\sqrt{\delta^2+(\delta+1)^2} \qquad (4)$$

which has a similar proof. Hence, in order to prove the Inequality (2), one can use (3) and (4) to prove the following inequality.

$$\Delta\Big(\frac{\Delta}{\sqrt{2}(\Delta+1)}\Big)-\delta\Big(\frac{\delta+1}{\sqrt{2}\delta}\Big)>0$$

or equivalently,

$$\Delta^2-(\delta+1)(\Delta+1)>0.$$

Since $\delta+1\leq\Delta-1$ the above inequality is true. Therefore the Inequality (1) holds. Hence $SO(G)>SO(G')$, a contradiction.

Case 2. Two vertices p and q are adjacent. Let $c_1, c_2, \ldots, c_{\delta-1}$ and $d_1, d_2, \ldots, d_{\Delta-1}$ be the degree of vertices in $N(p)\backslash\{q\}$ and $N(q)\backslash\{p\}$, respectively. With no loss of generality, assume that $d_{\Delta-1} = d(w)$. Thus we have,

$$SO(G) = \sum_{uv\in E(G)\backslash S} \sqrt{d^2(u)+d^2(v)} + \sum_{i=1}^{\Delta-1} \sqrt{\Delta^2 + d_i^2} + \sum_{i=1}^{\delta-1} \sqrt{\delta^2 + c_i^2} + \sqrt{\Delta^2+\delta^2}.$$

Also, one can see that,

$$SO(G') = \sum_{uv\in E(G)\backslash S} \sqrt{d^2(u)+d^2(v)} + \sum_{i=1}^{\Delta-2} \sqrt{(\Delta-1)^2 + d_i^2} + \sum_{i=1}^{\delta-1} \sqrt{(\delta+1)^2 + c_i^2} +$$

$$\sqrt{(\delta+1)^2 + d_{\Delta-1}^2} + \sqrt{(\Delta-1)^2+(\delta+1)^2}.$$

Now, we show that $SO(G) > SO(G')$. Indeed we prove that,

$$\sum_{i=1}^{\Delta-2}\Big(\sqrt{\Delta^2+d_i^2} - \sqrt{(\Delta-1)^2+d_i^2}\Big) - \sum_{i=1}^{\delta-1}\Big(\sqrt{(\delta+1)^2+c_i^2} - \sqrt{\delta^2+c_i^2}\Big) +$$

$$\sqrt{\Delta^2+d_{\Delta-1}^2} - \sqrt{(\delta+1)^2+d_{\Delta-1}^2} + \sqrt{\Delta^2+\delta^2} - \sqrt{(\Delta-1)^2+(\delta+1)^2} > 0$$

Since $\delta+1 \le \Delta-1$, we obtain that,

$$\sum_{i=1}^{\Delta-1}\Big(\sqrt{\Delta^2+d_i^2} - \sqrt{(\Delta-1)^2+d_i^2}\Big) - \sum_{i=1}^{\delta-1}\Big(\sqrt{(\delta+1)^2+c_i^2} - \sqrt{\delta^2+c_i^2}\Big) +$$

$$\sqrt{\Delta^2+\delta^2} - \sqrt{(\Delta-1)^2+(\delta+1)^2} > 0.$$

Since for $0 \le b \le a$, the function $f(x) = \sqrt{a^2+x^2} - \sqrt{b^2+x^2}$ is decreasing on the interval $(0, +\infty)$, we need to show that the above inequality holds, where $c_i = \delta$ and $d_j = \Delta$ for $i = 1, \ldots, \delta-1$ and $j = 1, \ldots, \Delta-1$ that means we should prove,

$$(\Delta-1)\Big(\sqrt{2}\Delta - \sqrt{(\Delta-1)^2+\Delta^2}\Big) + (\delta-1)\Big(\sqrt{2}\delta - \sqrt{(\delta+1)^2+\delta^2}\Big) +$$

$$\sqrt{\Delta^2+\delta^2} - \sqrt{(\Delta-1)^2+(\delta+1)^2} > 0.$$

Notice that by (2), we have,

$$\Delta\Big(\sqrt{2}\Delta - \sqrt{(\Delta-1)^2+\Delta^2}\Big) + \delta\Big(\sqrt{2}\delta - \sqrt{(\delta+1)^2+\delta^2}\Big) > 0.$$

Therefore, we need to prove the following inequality,

$$\sqrt{(\Delta-1)^2+\Delta^2} - \sqrt{2}\Delta + \sqrt{(\delta+1)^2+\delta^2} - \sqrt{2}\delta + \sqrt{\Delta^2+\delta^2} - \sqrt{(\Delta-1)^2+(\delta+1)^2} > 0.$$

Now, to complete the proof, notice that if $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$, then we have,

$$\sqrt{a_1 + a_3} + \sqrt{a_2 + a_4} \geq \sqrt{a_1 + a_2} + \sqrt{a_3 + a_4}. \tag{5}$$

Let $a_1 = \Delta^2$, $a_2 = \Delta^2$, $a_3 = (\Delta - 1)^2$ and $a_4 = \delta^2$. We have,

$$\sqrt{(\Delta-1)^2+\Delta^2}+\sqrt{\Delta^2+\delta^2}+\sqrt{(\delta+1)^2+\delta^2} \geq \sqrt{2}\Delta+\sqrt{(\Delta-1)^2+\delta^2}+\sqrt{(\delta+1)^2+\delta^2}.$$

Next, let $a_1 = (\Delta - 1)^2$, $a_2 = (\delta + 1)^2$, $a_3 = \delta^2$ and $a_4 = \delta^2$ in Inequality (5). as a result,

$$\sqrt{2}\Delta + \sqrt{(\Delta-1)^2+\delta^2} + \sqrt{(\delta+1)^2+\delta^2} \geq \sqrt{2}\Delta + \sqrt{(\Delta-1)^2+(\delta+1)^2} + \sqrt{2}\delta.$$

Therefore $SO(G) > SO(G')$, a contradiction. The proof is complete. ■

Now, we have an immediate corollary which was first proved in [8].

Corollary 1. *Let $n \geq 3$ and $T \in \mathbb{CG}(n, n-1)$ be a tree and $SO(T) = s(n, n-1)$. Then T is a path.*

The following result was first proved in [11].

Corollary 2. *Let $G \in \mathbb{CG}(n, n)$ be a unicyclic graph and $SO(G) = s(n, n)$. Then G is a cycle.*

Proof. If G is a unicyclic graph and G is not a cycle, then $\delta(G) = 1$ and by Theorem 1, $\Delta(G) = 2$, a contradiction. ■

Corollary 3. *Let $G \in \mathbb{CG}(n, m)$ and $SO(G) = s(n, m)$. Then G has $2m - n\lfloor \frac{2m}{n} \rfloor$ vertices of degree $\lfloor \frac{2m}{n} \rfloor + 1$ and $n - n(\frac{2m}{n} - \lfloor \frac{2m}{n} \rfloor)$ vertices of degree $\lfloor \frac{2m}{n} \rfloor$.*

Proof. If $\Delta = \delta$, it is easy to see that the assertion holds. So we may assume that $\Delta \neq \delta$. By Theorem 1, $\Delta = \delta + 1$. Suppose that there are k vertices of degree δ and k' vertices of degree $\delta + 1$. We have, $k\delta + k'(\delta + 1) = 2m$, which means $(k + k')\delta + k' = 2m$. We know that $k + k' = n$, thus $2m = n\delta + k'$. By dividing both sides by n, we get $\frac{2m}{n} = \delta + \frac{k'}{n}$. Since $k' < n$ we have,

$$\delta = \left\lfloor \frac{2m}{n} \right\rfloor \quad , \quad \Delta = \left\lfloor \frac{2m}{n} \right\rfloor + 1.$$

Moreover, since $k' = 2m - n\delta$, we have,

$$k' = 2m - n\left\lfloor \frac{2m}{n} \right\rfloor \quad , \quad k = n - 2m + n\left\lfloor \frac{2m}{n} \right\rfloor.$$

■

We close the paper with the following result.

Corollary 4. *Let $G \in \mathbb{CG}(n,m)$ and $SO(G) = s(n,m)$. If $n \,|\, 2m$, then G is a regular graph and $s(n,m) = 2\sqrt{2}\frac{m^2}{n}$.*

Proof. By Corollary 3, it can be easily seen that G is a $\frac{2m}{n}$-regular graph. Thus we have,

$$s(n,m) = m\sqrt{(\frac{2m}{n})^2 + (\frac{2m}{n})^2} = 2\sqrt{2}\frac{m^2}{n}.$$

■

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