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Bifurcation Caused by Delay in a Fractional–Order Coupled Oregonator Model in Chemistry

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Abstract

Establishing dynamical models to characterize the relation of different chemical compositions is an important topic in chemistry and mathematics. However, a lot of dynamical models are merely concerned with the integer-order dynamical models. The report on fractional-order chemical dynamical systems is quite few. In this current article, based on the earlier publications, we establish a new fractional-order coupled Oregonator model incorporating time delay. A set of sufficient conditions which ensure the stability and the onset of Hopf bifurcation of fractional-order coupled Oregonator model incorporating time delay are derived by regarding the time delay as bifurcation parameter. The exploration manifests that time delay has a vital influence on stabilizing system and controlling bifurcation of the investigated fractional-order coupled Oregonator model. At last, Matlab simulation results are adequately displayed to corroborate the derived theoretical achievements.

1 Introduction

Time delay plays a vital role in describing the dynamical peculiarity in many biological systems, chemical reaction, physical science, neural networks and so on [1-3]. Usually, time delay will make the system lose its stability, display periodic oscillation, generate chaos and so on. In particular, in a lot of chemical reaction systems, time delayed feedback has a very important effect on their dynamics. Thus the study on the impact of time delayed feedback on dynamical behavior of chemical reaction systems has attracted great attention from numerous scholars. Oregonator model, which is nonlinear differential system, is a vital model describing the Belousov Zhabotinsky (BZ) chemical reaction and attracts great interest from many researchers [4-6]. In 1999, Zhang et al. [7] established the following coupled Oregonator model:

$$\begin{cases} \epsilon\dfrac{dw_1(t)}{dt} = w_1(1-w_1) - gw_2\dfrac{w_1-v}{w_1+v} + A(w_3-w_1), \\ \dfrac{dw_2(t)}{dt} = w_1 - w_2, \\ \epsilon\dfrac{dw_3(t)}{dt} = w_3(1-w_3) - gw_4\dfrac{w_3-v}{w_3+v} + A(w_1-w_3), \\ \dfrac{dw_4(t)}{dt} = w_3 - w_4, \end{cases} \tag{1}$$

where w_1, w_3 represents the concentrations of $HBrO_2$, w_2, w_4 represent the concentrations of $Ce(IV)$, ϵ_0, A, v are all non-negative constants and g represents a controllable coefficient.

Considering the impact of electric current, then $Ce(IV)$ will be perturbed. To better depict the phenomenon, Wu and Zhang [8] set up the following coupled Oregonator model with the two perturbation terms $\vartheta w_2(t-\gamma)$ and $\vartheta w_4(t-\gamma)$:

$$\begin{cases} \epsilon\dfrac{dw_1(t)}{dt} = w_1(1-w_1) - gw_2\dfrac{w_1-v}{w_1+v} + A(w_3-w_1), \\ \dfrac{dw_2(t)}{dt} = w_1 - w_2 + \vartheta w_2(t-\gamma), \\ \epsilon\dfrac{dw_3(t)}{dt} = w_3(1-w_3) - gw_4\dfrac{w_3-v}{w_3+v} + A(w_1-w_3), \\ \dfrac{dw_4(t)}{dt} = w_3 - w_4 + \vartheta w_4(t-\gamma), \end{cases} \tag{2}$$

where γ stands for a time delay and ϑ is a governable coefficient. By applying the stability theorem and bifurcation theory of delayed differential equation, Wu and Zhang [8] set up a delay-independent condition to ensure the stability and the existence of Hopf bifurcation for integer-order delayed system (2).

In the past for a long time, many works on differential equation are only concerned with the integer-order case. The study on fractional-order differential systems is very few since the shortage of fundamental theory of fractional calculus and its own complexity. Only in recent decades, the work on fractional-order differential systems has attracted a great deal of interest from many scholars in mathematics and engineering due to its owned extensive application in numerous areas such as neural networks, biotechnology, chemical engineering, finance and monetary, viscoelasticity and all kinds of physical waves and so on [8-11]. Many scholars point out that in many case, it is more appropriate for us to depict the subsistent phenomenon in objective world by fractional-order differential model than by integer-order differential model since fractional calculus owns the memory and hereditary peculiarity of all sorts of materials and physical change processes [12]. Nowadays a number of scholars devote great effort to the investigation on various dynamics of fractional-order differential systems and fruitful results are constantly emerging. One can refer to [13-15]. In particular, the delay-induced Hopf bifurcation is a key topic in fractional-order differential equation. What is the influence of delay on Hopf bifurcation for the involved fractional-order systems? In order to explain this problem, some works on this aspect have been carried out and abundant fruits have been reported. For example, Huang et al. [16] discussed the Hopf bifurcation for fractional-order BAM neural networks involving multiple time delays; Xiao et al. [17] reported the fractional-order PD control technique of Hopf bifurcation for fractional-order small-world networks with time delays; Xu et al. [18] dealt with the Hopf bifurcation of fractional-order BAM neural networks concerning multiple time delays; In details, one can see [18-23].

Here it is worth pointing out that all the mentioned publications on Hopf bifurcation of fractional-order differential models are mainly concerned with biological population and neural network areas. By far, only very few works focus on chemical reaction systems. In order to display the application of fractional-order dynamical system in chemistry, we think that it is of importance to explore the Hopf bifurcation caused by time dealy for the fractional-order coupled Oregonator model. Stimulated by the analysis above and on the basis of work of Wu and Zhang [8], we modify coupled Oregonator model (2) as the following fractional-order coupled Oregonator model:

$$\begin{cases} \epsilon\dfrac{dw_1^{\varrho}(t)}{dt^{\varrho}} = w_1(1-w_1) - gw_2\dfrac{w_1-v}{w_1+v} + A(w_3-w_1), \\ \dfrac{dw_2^{\varrho}(t)}{dt^{\varrho}} = w_1 - w_2 + \vartheta w_2(t-\gamma), \\ \epsilon\dfrac{dw_3(t)}{dt} = w_3(1-w_3) - gw_4\dfrac{w_3-v}{w_3+v} + A(w_1-w_3), \\ \dfrac{dw_4^{\varrho}(t)}{dt^{\varrho}} = w_3 - w_4 + \vartheta w_4(t-\gamma), \end{cases} \tag{3}$$

where $0 < \varrho \leq 1$ and all the coefficients have the same meaning as those in coupled Oregonator model (2). In details, one can see [8].

The key aim of this work can be stated as follows: ① Explore the stability and onset of Hopf bifurcation of the fractional-order coupled Oregonator model (3); ② Fully reveal the role of delay in Hopf bifurcation of fractional-order coupled Oregonator model (3).

The primary highlights of the current research are given as follows:

ⓐ On the basis of the earlier works, a new fractional-order coupled Oregonator model is established to better portray the memory and hereditary property of the concentrations of chemical composition $HBrO_2$ and $Ce(IV)$.

ⓑ A new sufficient condition to ensure the stability and the onset of Hopf bifurcation for fractional-order coupled Oregonator model is set up. The role of delay of the fractional-order coupled Oregonator model is adequately embodied.

ⓒ Until now, the study on Hopf bifurcation of fractional-order coupled Oregonator model is rare.

The current study is planned as follows. Segment 2 lists some needful principle on fractional-order differential equation. Segment 3 establishes a delay-independent criterion guaranteeing the stability and the emergence of Hopf bifurcation of fractional-order delayed Oregonator model. Furthermore, the role of delay in the stability and bifurcation behavior for fractional-order coupled Oregonator model (3) is displayed. Segment 4 executes software simulation to verify the derived analysis results. Segment 4 completes this study.

2 Preliminaries

In this part, several essential definitions and lemmas on fractional-order differential equation are prepared.

Definition 2.1. [24] *The fractional-type integral of order ϱ for the function $g(\zeta)$ is given by*

$$\mathcal{I}^{\varrho}g(\tau)=\frac{1}{\Gamma(\varrho)}\int_{\tau_0}^{\tau}(\tau-s)^{\varrho-1}g(s)ds,$$

where $\tau\geq\tau_0,\varrho>0,$ and $\Gamma(s)=\int_0^{\infty}\tau^{s-1}e^{-\tau}d\tau$ stands for Gamma function.

Definition 2.2. [24] *Suppose that $g(\tau)\in C([\tau_0,\infty),R)$. Define the Caputo fractional-order derivative of order ϱ of $g(\varrho)$ as follows:*

$$\mathcal{D}^{\varrho}g(\zeta)=\frac{1}{\Gamma(m-\varrho)}\int_{\tau_0}^{\tau}\frac{g^{(m)}(s)}{(\zeta-s)^{\varrho-m+1}}ds,$$

where $\tau\geq\tau_0$ and m presents a positive integer that satisfies $m-1\leq\varrho<1$. In particular, When $0<\varrho<1$, then

$$\mathcal{D}^{\varrho}g(\tau)=\frac{1}{\Gamma(1-\varrho)}\int_{\tau_0}^{\tau}\frac{g^{'}(s)}{(\tau-s)^{\varrho}}ds.$$

Definition 2.3. [25] *Consider the fractional-order system:*

$$\mathcal{D}^{\varrho}u_i(t)=f_i(u_i(t)),i=1,2,\cdots,h, \tag{4}$$

where $\varrho\in(0,1],u_i(t)=(u_1(t),u_2(t),\cdots,u_h(t)),f_i(t)=(f_1(t),f_2(t),\cdots,f_h(t))$. Then $(u_1^*,u_2^*,\cdots,u_h^*)$ is the equilibrium point if $f_i(u_i^*)=0$.

Lemma 2.1. [26] *For the fractional-order model $\mathcal{D}^{\varrho}w=\mathcal{Q}w,w(0)=w_0$ where $0<\varrho<1,w\in R^h,\mathcal{Q}\in R^{h\times h}$. Denote $\lambda_l(i=1,2,\cdots,h)$ the root of the characteristic equation of $\mathcal{D}^{\varrho}w=\mathcal{Q}w$. Then system $\mathcal{D}^{\varrho}w=\mathcal{Q}w$ is asymptotically stable $\Leftrightarrow|arg(\lambda_l)|>\frac{\varrho\pi}{2}(l=1,2,\cdots,h)$. Moreover, this system is stable $\Leftrightarrow|arg(\lambda_l)|>\frac{\varrho\pi}{2}(l=1,2,\cdots,h)$ and those critical eigenvalues which satisfy $|arg(\lambda_l)|=\frac{\varrho\pi}{2}(l=1,2,\cdots,h)$ possess geometric multiplicity one.*

Lemma 2.2. [27] *For the fractional-order model $\mathcal{D}^{\varrho}u(t)=\mathcal{Q}_1u(t)+\mathcal{Q}_2u(t-\gamma)$, where $u(t)=\omega(t),t\in[-\gamma,0],\varrho\in(0,1],u\in R^n,\mathcal{Q}_1,\mathcal{Q}_2\in R^{n\times n},\varrho\in R^{+(n\times n)}$. Then the characteristic equation of the model can be expressed as the form:* $\det|s^{\varrho}\mathcal{I}-\mathcal{Q}_1-\mathcal{Q}_2e^{-s\gamma}|=0$. *Then the zero solution of the model is asymptotically stable if all roots of the equation* $\det|s^{\varrho}\mathcal{I}-\mathcal{Q}_1-\mathcal{Q}_2e^{-s\gamma}|=0$ *possess negative real roots.*

3 Bifurcation study of model (3)

In this part, we are to explore the influence of time delay on Hopf bifurcation for fractional-order coupled Oregonator model (3).

Let $(w_{1*}, w_{2*}, w_{3*}, w_{4*})$ be the equilibrium point of the coupled Oregonator model (3), then

$$\begin{cases} w_{1*}(1-w_{1*}) - g w_{2*}\dfrac{w_{1*}-v}{w_{1*}+v} + A(w_{3*}-w_{1*}) = 0, \\ w_{1*} - w_{2*} + \vartheta w_{2*} = 0, \\ w_{3*}(1-w_{3*}) - g w_{4*}\dfrac{w_{3*}-v}{w_{3*}+v} + A(w_{1*}-w_{3*}) = 0, \\ w_{3*} - w_{4*} + \vartheta w_{4*} = 0. \end{cases} \tag{5}$$

Then system (3) has three equilibrium point $\mathcal{W}_1(0,0,0,0), \mathcal{W}_2(w_{1*+}, w_{2*+}, w_{3*+}, w_{4*+})$, $\mathcal{W}_3(w_{1*-}, w_{2*-}, w_{3*-}, w_{4*-})$, where

$$\begin{cases} w_{1*+} = \dfrac{1-\frac{g}{1-\vartheta}-v+\sqrt{\left(1-\frac{g}{1-\vartheta}-v\right)^2+4v\left(1+\frac{g}{1-\vartheta}\right)}}{2}, \\ w_{2*+} = \dfrac{1-\frac{g}{1-\vartheta}-v+\sqrt{\left(1-\frac{g}{1-\vartheta}-v\right)^2+4v\left(1+\frac{g}{1-\vartheta}\right)}}{2(1-\vartheta)}, \\ w_{3*+} = \dfrac{1-\frac{g}{1-\vartheta}-v+\sqrt{\left(1-\frac{g}{1-\vartheta}-v\right)^2+4v\left(1+\frac{g}{1-\vartheta}\right)}}{2}, \\ w_{4*+} = \dfrac{1-\frac{g}{1-\vartheta}-v+\sqrt{\left(1-\frac{g}{1-\vartheta}-v\right)^2+4v\left(1+\frac{g}{1-\vartheta}\right)}}{2(1-\vartheta)} \end{cases} \tag{6}$$

and

$$\begin{cases} w_{1*-} = \dfrac{1-\frac{g}{1-\vartheta}-v-\sqrt{\left(1-\frac{g}{1-\vartheta}-v\right)^2+4v\left(1+\frac{g}{1-\vartheta}\right)}}{2}, \\ w_{2*-} = \dfrac{1-\frac{g}{1-\vartheta}-v-\sqrt{\left(1-\frac{g}{1-\vartheta}-v\right)^2+4v\left(1+\frac{g}{1-\vartheta}\right)}}{2(1-\vartheta)}, \\ w_{3*-} = \dfrac{1-\frac{g}{1-\vartheta}-v-\sqrt{\left(1-\frac{g}{1-\vartheta}-v\right)^2+4v\left(1+\frac{g}{1-\vartheta}\right)}}{2}, \\ w_{4*-} = \dfrac{1-\frac{g}{1-\vartheta}-v-\sqrt{\left(1-\frac{g}{1-\vartheta}-v\right)^2+4v\left(1+\frac{g}{1-\vartheta}\right)}}{2(1-\vartheta)}. \end{cases} \tag{7}$$

It is not difficult to obtain that if the following inequality

$$(Q_1) \quad \vartheta < 1$$

holds, then system (3) possesses the positive equilibrium point $\mathcal{W}_3(w_{1*+}, w_{2*+}, w_{3*+}, w_{4*+})$. Considering the practical implication of model (3) in chemistry, we only deal with the positive equilibrium point. Let

$$\begin{cases} \bar{w}_1(t) = w_1(t) - w_{1*+}, \\ \bar{w}_2(t) = w_2(t) - w_{2*+}, \\ \bar{w}_3(t) = w_3(t) - w_{3*+}, \\ \bar{w}_4(t) = w_4(t) - w_{4*+}, \end{cases} \tag{8}$$

then system (3) can be rewritten as

$$\begin{cases} \frac{d\bar{w}_1^\varrho(t)}{dt^\varrho} = \frac{1}{\epsilon}\Big[(\bar{w}_1 + w_{1*+})(1 - (\bar{w}_1 + w_{1*+})) - g(\bar{w}_2 + w_{2*+})\frac{(\bar{w}_1 + w_{1*+}) - v}{(\bar{w}_1 + w_{1*+}) + v} \\ \qquad + A((\bar{w}_3 + w_{3*+}) - (\bar{w}_1 + w_{1*+}))\Big], \\ \frac{d\bar{w}_2^\varrho(t)}{dt^\varrho} = \bar{w}_1 - \bar{w}_2 + \vartheta\bar{w}_2(t-\gamma), \\ \frac{d\bar{w}_3(t)}{dt} = \frac{1}{\epsilon}\Big[(\bar{w}_3 + w_{3*+})(1 - (\bar{w}_3 + w_{3*+})) - g(\bar{w}_4 + w_{4*+})\frac{(\bar{w}_3 + w_{3*+}) - v}{(\bar{w}_3 + w_{3*+}) + v} \\ \qquad + A((\bar{w}_1 + w_{1*+}) - (\bar{w}_3 + w_{3*+}))\Big], \\ \frac{d\bar{w}_4^\varrho(t)}{dt^\varrho} = \bar{w}_3 - \bar{w}_4 + \vartheta\bar{w}_4(t-\gamma). \end{cases} \tag{9}$$

The linear system of Eq. (9) at $(0,0,0,0)$ takes the following form:

$$\begin{cases} \frac{d\bar{w}_1^\varrho(t)}{dt^\varrho} = \alpha_1\bar{w}_1 + \alpha_2\bar{w}_2 + \alpha_3(\bar{w}_3 - \bar{w}_1), \\ \frac{d\bar{w}_2^\varrho(t)}{dt^\varrho} = \bar{w}_1 - \bar{w}_2 + \vartheta\bar{w}_2(t-\gamma), \\ \frac{d\bar{w}_3(t)}{dt} = \alpha_1\bar{w}_3 + \alpha_2\bar{w}_4 + \alpha_3(\bar{w}_1 - \bar{w}_3), \\ \frac{d\bar{w}_4^\varrho(t)}{dt^\varrho} = \bar{w}_3 - \bar{w}_4 + \vartheta\bar{w}_4(t-\gamma). \end{cases} \tag{10}$$

where

$$\begin{cases} \alpha_1 = \frac{1}{\epsilon}\left[1 - 2w_{1*+} - \frac{2vgw_{3*+}}{(v + w_{1*+})^2}\right], \\ \alpha_2 = \frac{g(v - w_{1*+})}{\epsilon(v + w_{1*+})}, \\ \alpha_3 = \frac{A}{\epsilon}. \end{cases} \tag{11}$$

W denote $\bar{w}_i (i = 1, 2, 3, 4)$ by w_i in (10), then system (10) becomes

$$\begin{cases} \frac{dw_1^\varrho(t)}{dt^\varrho} = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3(w_3 - w_1), \\ \frac{dw_2^\varrho(t)}{dt^\varrho} = w_1 - w_2 + \vartheta w_2(t-\gamma), \\ \frac{dw_3(t)}{dt} = \alpha_1 w_3 + \alpha_2 w_4 + \alpha_3(w_1 - w_3), \\ \frac{dw_4^\varrho(t)}{dt^\varrho} = w_3 - w_4 + \vartheta w_4(t-\gamma). \end{cases} \tag{12}$$

The characteristic equation of system (12) owns the expression:

$$\det\begin{bmatrix} s^\varrho - (\alpha_1 - \alpha_3) & -\alpha_2 & -\alpha_3 & 0 \\ -1 & s^\varrho + 1 - \vartheta e^{-s\gamma} & 0 & 0 \\ -\alpha_3 & 0 & s^\varrho - (\alpha_1 - \alpha_3) & -\alpha_2 \\ 0 & 0 & -1 & s^\varrho + 1 - \vartheta e^{-s\gamma} \end{bmatrix} = 0, \tag{13}$$

which leads to

$$s^{4\varrho}+\beta_1 s^{3\varrho}+\beta_2 s^{2\varrho}+\beta_3 s^{\varrho}+\beta_4+(\varsigma_1 s^{3\varrho}+\varsigma_2 s^{2\varrho}+\varsigma_3 s^{\varrho}+\varsigma_4)e^{-s\gamma}+(\delta_1 s^{\varrho}+\delta_2)e^{-2s\gamma}=0, \quad (14)$$

where

$$\begin{cases} \beta_1=2[1-(\alpha_1-\alpha_3)], \\ \beta_2=1-\alpha_2-4(\alpha_1-\alpha_3)+(\alpha_1-\alpha_3)^2-\alpha_3^2-\alpha_2, \\ \beta_3=2(\alpha_1-\alpha_3)^2-2(\alpha_1-\alpha_3)^{-}\alpha_2+2\alpha_2(\alpha_1-\alpha_3)-2\alpha_3^2-\alpha_2, \\ \beta_4=\alpha_2(\alpha_1-\alpha_3)+(\alpha_1-\alpha_3)^{-}\alpha_3^2+\alpha_2, \\ \varsigma_1=-2\vartheta, \\ \varsigma_2=-2\vartheta+4\vartheta(\alpha_1-\alpha_3)-2\vartheta(\alpha_1-\alpha_3)^2, \\ \varsigma_3=4\vartheta(\alpha_1-\alpha_3)+2\vartheta(\alpha_1-\alpha_3)^2+2\vartheta\alpha_3^2+\alpha_2\vartheta-\alpha_2\vartheta(\alpha_1-\alpha_3), \\ \varsigma_4=\vartheta\alpha_2(\alpha_1-\alpha_3), \\ \delta_1=2\vartheta^2(\alpha_1-\alpha_3)-\vartheta^2, \\ \delta_2=-\vartheta^2(\alpha_1-\alpha_3)^2. \end{cases} \quad (15)$$

The following necessary hypothesis is prepared:

$$(Q_2)\quad \begin{cases} \Lambda_1=\beta_1+\varsigma_1>0, \\ \Lambda_2=\det\begin{bmatrix} \beta_1+\varsigma_1 & 1 \\ \beta_3+\varsigma_3+\delta_1 & \beta_2+\varsigma_2 \end{bmatrix}>0, \\ \Lambda_3=\det\begin{bmatrix} \beta_1+\varsigma_1 & 1 & 0 \\ \beta_3+\varsigma_3+\delta_1 & \beta_2+\varsigma_2 & \beta_1+\varsigma_1 \\ 0 & \beta_4+\varsigma_4+\delta_2 & \beta_3+\varsigma_3+\delta_1 \end{bmatrix}>0, \\ \Lambda_4=(\beta_4+\varsigma_4+\delta_2)\Lambda_3>0. \end{cases}$$

Lemma 3.1. *Assume that (Q_1) and (Q_2) hold true, then the positive equilibrium point $\mathcal{W}_3(w_{1*+}, w_{2*+}, w_{3*+}, w_{4*+})$ is locally asymptotically stable.*

Proof By (Q_1), we know that system (3) possesses the positive equilibrium point $\mathcal{W}_3(w_{1*+}, w_{2*+}, w_{3*+}, w_{4*+})$. When $\gamma=0$. Then (14) becomes

$$\lambda^4+(\beta_1+\varsigma_1)\lambda^3+(\beta_2+\varsigma_2)\lambda^2+(\beta_3+\varsigma_3+\delta_1)\lambda+\beta_4+\varsigma_4+\delta_2=0. \quad (16)$$

In view of (Q_2), we derive that each root λ_l of (14) satisfies $|\arg(\lambda_l)|>\frac{\varrho\pi}{2}(l=1,2,3,4)$. It follows from Lemma 3.1 that $\mathcal{W}_3(w_{1*+}, w_{2*+}, w_{3*+}, w_{4*+})$ is locally asymptotically stable. The proof finishes. ■

It follows from (14) that

$$(s^{4\varrho}+\beta_1 s^{3\varrho}+\beta_2 s^{2\varrho}+\beta_3 s^{\varrho}+\beta_4)e^{s\gamma}+(\varsigma_1 s^{3\varrho}+\varsigma_2 s^{2\varrho}+\varsigma_3 s^{\varrho}+\varsigma_4)+(\delta_1 s^{\varrho}+\delta_2)e^{-s\gamma}=0, \quad (17)$$

Let $s=i\rho=\rho\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$ be a root of (14). Then

$$\begin{cases} a_1\cos\rho\gamma+a_2\sin\rho\gamma=a_3, \\ b_1\cos\rho\gamma+b_2\sin\rho\gamma=b_3, \end{cases} \quad (18)$$

where

$$\begin{cases} a_1 = \rho^{4\varrho}\cos 2\varrho\pi + \beta_1\rho^{3\varrho}\cos\dfrac{3\varrho\pi}{2} + \beta_2\rho^{2\varrho}\cos\varrho\pi + (\beta_3+\delta_1)\rho^{\varrho}\cos\dfrac{\varrho\pi}{2} + \beta_4 + \delta_2, \\ a_2 = -\rho^{4\varrho}\sin 2\varrho\pi - \beta_1\rho^{3\varrho}\sin\dfrac{3\varrho\pi}{2} - \beta_2\rho^{2\varrho}\sin\varrho\pi - (\beta_3-\delta_1)\rho^{\varrho}\sin\dfrac{\varrho\pi}{2}, \\ a_3 = -\varsigma_1\rho^{3\varrho}\cos\dfrac{3\varrho\pi}{2} - \varsigma_2\rho^{2\varrho}\cos\varrho\pi - \varsigma_3\rho^{\varrho}\cos\dfrac{\varrho\pi}{2} - \varsigma_4, \\ b_1 = \rho^{4\varrho}\sin 2\varrho\pi + \beta_1\rho^{3\varrho}\sin\dfrac{3\varrho\pi}{2} + \beta_2\rho^{2\varrho}\sin\varrho\pi + (\beta_3+\delta_1)\rho^{\varrho}\sin\dfrac{\varrho\pi}{2}, \\ b_2 = \rho^{4\varrho}\cos 2\varrho\pi + \beta_1\rho^{3\varrho}\cos\dfrac{3\varrho\pi}{2} + \beta_2\rho^{2\varrho}\cos\varrho\pi + (\beta_3-\delta_1)\rho^{\varrho}\cos\dfrac{\varrho\pi}{2} + \beta_4 - \delta_2, \\ b_3 = -\varsigma_1\rho^{3\varrho}\sin\dfrac{3\varrho\pi}{2} - \varsigma_2\rho^{2\varrho}\sin\varrho\pi - \varsigma_3\rho^{\varrho}\sin\dfrac{\varrho\pi}{2}. \end{cases} \tag{19}$$

According to (18), we get

$$\begin{cases} \cos\rho\gamma = \dfrac{a_3b_2 - a_2b_3}{a_1b_2 - a_2b_1}, \\ \sin\rho\gamma = \dfrac{a_3b_1 - a_1b_3}{a_1b_2 - a_2b_1}. \end{cases} \tag{20}$$

It follows from (20) that

$$(a_3b_2 - a_2b_3)^2 + (a_3b_1 - a_1b_3)^2 = (a_1b_2 - a_2b_1)^2. \tag{21}$$

In (19), we let

$$\begin{cases} e_1 = \cos 2\varrho\pi, \\ e_2 = \beta_1\cos\dfrac{3\varrho\pi}{2}, \\ e_3 = \beta_2\cos\varrho\pi, \\ e_4 = (\beta_3+\delta_1)\cos\dfrac{\varrho\pi}{2}, \\ e_5 = \beta_4 + \delta_2, \\ e_6 = -\sin 2\varrho\pi, \\ e_7 = -\beta_1\sin\dfrac{3\varrho\pi}{2}, \\ e_8 = -\beta_2\sin\varrho\pi, \\ e_9 = -(\beta_3-\delta_1)\sin\dfrac{\varrho\pi}{2}, \\ e_{10} = -\varsigma_1\cos\dfrac{3\varrho\pi}{2}, \\ e_{11} = -\varsigma_2\cos\varrho\pi, \\ e_{12} = -\varsigma_3\cos\dfrac{\varrho\pi}{2}, \\ e_{13} = -\varsigma_4, \\ e_{14} = \sin 2\varrho\pi, \\ e_{15} = \beta_1\sin\dfrac{3\varrho\pi}{2}, \\ e_{16} = \beta_2\sin\varrho\pi, \end{cases} \tag{22}$$

and

$$\begin{cases} e_{17} = (\beta_3 + \delta_1)\sin\dfrac{\varrho\pi}{2}, \\ e_{18} = (\beta_3 - \delta_1)\cos\dfrac{\varrho\pi}{2}, \\ e_{19} = \beta_4 - \delta_2, \\ e_{20} = -\varsigma_1\sin\dfrac{3\varrho\pi}{2}, \\ e_{21} = -\varsigma_2\sin\varrho\pi, \\ e_{22} = -\varsigma_3\sin\dfrac{\varrho\pi}{2}, \end{cases} \tag{23}$$

then (19) can be rewritten as

$$\begin{cases} a_1 = e_1\rho^{4\varrho} + e_2\rho^{3\varrho} + e_3\rho^{2\varrho} + e_4\rho^{\varrho} + e_5, \\ a_2 = e_6\rho^{4\varrho} + e_7\rho^{3\varrho} + e_8\rho^{2\varrho} + e_9\rho^{\varrho}, \\ a_3 = e_{10}\rho^{3\varrho} + e_{11}\rho^{2\varrho} + e_{12}\rho^{\varrho} + e_{13}, \\ b_1 = e_{14}\rho^{4\varrho} + e_{15}\rho^{3\varrho} + e_{16}\rho^{2\varrho} + e_{17}\rho^{\varrho}, \\ b_2 = e_1\rho^{4\varrho} + e_2\rho^{3\varrho} + e_3\rho^{2\varrho} + e_{18}\rho^{\varrho} + e_{19}, \\ b_3 = e_{20}\rho^{3\varrho} + e_{21}\rho^{2\varrho} + e_{22}\rho^{\varrho}. \end{cases} \tag{24}$$

By virtue of (21) and (24), one gets

$$\begin{aligned} &\varepsilon_1\rho^{16\varrho} + \varepsilon_2\rho^{15\varrho} + \varepsilon_3\rho^{14\varrho} + \varepsilon_4\rho^{13\varrho} + \varepsilon_5\rho^{12\varrho} + \varepsilon_6\rho^{11\varrho} \\ &+\varepsilon_7\rho^{10\varrho} + \varepsilon_8\rho^{9\varrho} + \varepsilon_9\rho^{8\varrho} + \varepsilon_{10}\rho^{7\varrho} + \varepsilon_{11}\rho^{6\varrho} + \varepsilon_{12}\rho^{5\varrho} \\ &+\varepsilon_{13}\rho^{4\varrho} + \varepsilon_{14}\rho^{3\varrho} + \varepsilon_{15}\rho^{2\varrho} + \varepsilon_{16}\rho^{\varrho} + \varepsilon_{17} = 0, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \varepsilon_1 &= (e_1^2 - e_6e_{14})^2, \\ \varepsilon_2 &= 2(e_1^2 - e_6e_{14})(2e_1e_2 - e_6e_{15} - e_7e_{14}), \\ \varepsilon_3 &= 2(e_1^2 - e_6e_{14})(e_1e_3 + e_2^2 + e_1e_3 - e_6e_{16} \\ &\quad - e_7e_{15} - e_8e_{14}) + (2e_1e_2 - e_6e_{15} - e_7e_{14})^2 \\ &\quad - (e_{10}e_{14} - e_1e_{20})^2 - (e_1e_{10} - e_6e_{20})^2, \\ \varepsilon_4 &= 2(e_1^2 - e_6e_{14})(e_1e_{18} + 2e_2e_3 + e_1e_4 - e_6e_{17} \\ &\quad - e_7e_{16} - e_8e_{15} - e_9e_{14}) + 2(2e_1e_2 - e_6e_{15} - e_7e_{14}) \\ &\quad \times (e_1e_3 + e_2^2 + e_1e_3 - e_6e_{16} - e_7e_{15} - e_8e_{14}) \\ &\quad - 2(e_1e_{10} - e_6e_{20})(e_2e_{10} + e_1e_{11} - e_1e_{21} - e_7e_{20}) \\ &\quad - 2(e_{10}e_{14} - e_1e_{20})(e_{10}e_{15} + e_{11}e_{14} - e_1e_{21} - e_2e_{20}), \\ \varepsilon_5 &= (e_1e_3 + e_2^2 + e_1e_3 - e_6e_{16} - e_7e_{15} - e_8e_{14})^2 \\ &\quad + 2(e_1^2 - e_6e_{14})(e_1e_{19} + e_2e_{18} + e_3^2 + e_2e_4 + e_1e_5 \\ &\quad - e_7e_{17} - e_8e_{16} - e_9e_{15}) + 2(2e_1e_2 - e_6e_{15} - e_7e_{14}) \end{aligned}$$

$$\times (e_1e_{18} + 2e_2e_3 + e_1e_4 - e_6e_{17} - e_7e_{16} - e_2e_{21} - e_7e_{20})$$
$$- 2(e_{10}e_{14} - e_1e_{20})(e_{10}e_{16} + e_{11}e_{15} + e_{12}e_{14} - e_2e_{21} - e_3e_{20})$$
$$- 2(e_{10}e_{14} - e_1e_{20})(e_{10}e_{16} + e_{11}e_{15} + e_{12}e_{14} - e_2e_{21} - e_3e_{20})$$
$$- 2(e_1e_{10} - e_6e_{20})(e_3e_{10} + e_2e_{11} + e_1e_{12} - e_6e_{22} - e_7e_{21} - e_8e_{22}),$$

$$\varepsilon_6 = 2(e_1^2 - e_6e_{14})(e_2e_{19} + e_3e_{18} + e_3e_4 + e_2e_5 - e_8e_{17} - e_9e_{16})$$
$$+ 2(2e_1e_2 - e_6e_{15} - e_7e_{14})(e_1e_{19} + e_2e_{18} + e_3^2 + e_2e_4 - e_7e_{17}$$
$$- e_8e_{16} - e_9e_{15}) + 2(e_1e_3 + e_2^2 + e_1e_3 - e_6e_{16} - e_7e_{15} - e_8e_{14})$$
$$\times (e_1e_{18} + 2e_2e_3 + e_1e_4 - e_6e_{17} - e_7e_{16} - e_8e_{15} - e_9e_{14})$$
$$- 2(e_1e_{10} - e_6e_{20})(e_{10}e_{18} + e_3e_{11} + e_2e_{12} + e_1e_{13} - e_7e_{22}$$
$$- e_8e_{21} - e_9e_{20}) - 2(e_2e_{10} + e_1e_{11} - e_1e_{21} - e_7e_{20})(e_3e_{10}$$
$$+ e_2e_{11} + e_1e_{12} - e_6e_{22} - e_7e_{21} - e_8e_{22}) - 2(e_{10}e_{14} - e_1e_{20})$$
$$\times (e_{10}e_{17} + e_{11}e_{16} + e_{12}e_{15} + e_{13}e_{14} - e_2e_{22} - e_3e_{21} - e_4e_{20})$$
$$- 2(e_{10}e_{15} + e_{11}e_{14} - e_1e_{21} - e_2e_{20})(e_{10}e_{16} + e_{11}e_{15} + e_{12}e_{14}$$
$$- e_2e_{21} - e_3e_{20}),$$

$$\varepsilon_7 = (e_1e_{18} + 2e_2e_3 + e_1e_4 - e_6e_{17} - e_7e_{16} - e_8e_{15} - e_9e_{14})^2$$
$$+ 2(e_1^2 - e_6e_{14})(e_3e_{19} + e_4e_{18} + e_5e_3 - e_9e_{17}) + 2(2e_1e_2$$
$$- e_6e_{15} - e_7e_{14})(e_2e_{19} + e_3e_{18} + e_3e_4 + e_2e_5 - e_8e_{17}$$
$$- e_9e_{16}) + 2(e_1e_3 + e_2^2 + e_1e_3 - e_6e_{16} - e_7e_{15} - e_8e_{14})$$
$$\times (e_1e_{19} + e_2e_{18} + e_3^2 + e_2e_4 - e_7e_{17} - e_8e_{16} - e_9e_{15})$$
$$- (e_{10}e_{16} + e_{11}e_{15} + e_{12}e_{14} - e_2e_{21} - e_3e_{20})^2$$
$$- (e_3e_{10} + e_2e_{11} + e_1e_{12} - e_6e_{22} - e_7e_{21} - e_8e_{22})^2$$
$$- 2(e_{10}e_{14} - e_1e_{20})(e_{11}e_{17} + e_{12}e_{16} + e_{13}e_{15} - e_3e_{22}$$
$$- e_4e_{21} - e_5e_{20}) - 2(e_{10}e_{15} + e_{11}e_{14} - e_1e_{21} - e_2e_{20})$$
$$\times (e_{10}e_{17} + e_{11}e_{16} + e_{12}e_{15} + e_{13}e_{14} - e_2e_{22} - e_3e_{21}$$
$$- e_4e_{20}) - 2(e_1e_{10} - e_6e_{20})(e_{10}e_{19} + e_{11}e_{18} + e_3e_{12}$$
$$+ e_2e_{13} - e_8e_{22} - e_9e_{21}) - 2(e_2e_{10} + e_1e_{11} - e_1e_{21} - e_7e_{20})$$
$$\times (e_{10}e_{19} + e_3e_{11} + e_2e_{12} + e_1e_{13} - e_7e_{22} - e_8e_{21} - e_9e_{20}),$$

$$\varepsilon_8 = 2(e_1^2 - e_6e_{14})(e_4e_{19} + e_5e_{18}) + 2(2e_1e_2 - e_6e_{15} - e_7e_{14})$$
$$\times (e_3e_{19} + e_4e_{18} + e_5e_3 - e_9e_{17}) + 2(e_1e_3 + e_2^2 + e_1e_3$$

$$
\begin{aligned}
&- e_6e_{16} - e_7e_{15} - e_8e_{14})(e_2e_{19} + e_3e_{18} + e_3e_4 + e_2e_5\\
&- e_8e_{17} - e_9e_{16}) + (e_1e_{18} + 2e_2e_3 + e_1e_4 - e_6e_{17} - e_7e_{16}\\
&- e_8e_{15} - e_9e_{14})(e_1e_{19} + e_2e_{18} + e_3^2 + e_2e_4 - e_7e_{17}\\
&- e_8e_{16} - e_9e_{15}) - 2(e_{10}e_{14} - e_1e_{20})(e_{12}e_{17} + e_{13}e_{16}\\
&- e_4e_{22} - e_5e_{21}) - 2(e_{10}e_{15} + e_{11}e_{14} - e_1e_{21} - e_2e_{20})\\
&\times (e_{11}e_{17} + e_{12}e_{16} + e_{13}e_{15} - e_3e_{22} - e_4e_{21} - e_5e_{20})\\
&- 2(e_{10}e_{16} + e_{11}e_{15} + e_{12}e_{14} - e_2e_{21} - e_3e_{20})(e_{10}e_{17}\\
&+ e_{11}e_{16} + e_{12}e_{15} + e_{13}e_{14} - e_2e_{22} - e_3e_{21} - e_4e_{20})\\
&- 2(e_1e_{10} - e_6e_{20})(e_{11}e_{19} + e_{12}e_{18} + e_3e_{13} - e_9e_{22})\\
&- 2(e_2e_{10} + e_1e_{11} - e_1e_{21} - e_7e_{20})\\
&\times (e_{10}e_{19} + e_{11}e_{18} + e_3e_{12} + e_2e_{13} - e_8e_{22} - e_9e_{21})\\
&- 2(e_3e_{10} + e_2e_{11} + e_1e_{12} - e_6e_{22} - e_7e_{21} - e_8e_{20})\\
&\times (e_{10}e_{18} + e_3e_{11} + e_2e_{12} + e_1e_{13} - e_7e_{22} - e_8e_{21} - e_9e_{20})
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_9 = {} & 2e_5e_{19}(e_1^2 - e_6e_{14}) + 2(2e_1e_2 - e_6e_{15} - e_7e_{14})(e_4e_{19} + e_5e_{18})\\
&+ 2(e_1e_3 + e_2^2 + e_1e_3 - e_6e_{16} - e_7e_{15} - e_8e_{14})(e_3e_{19} + e_4e_{18}\\
&+ e_5e_3 - e_9e_{17}) + 2(e_1e_{18} + 2e_2e_3 + e_1e_4 - e_6e_{17} - e_7e_{16}\\
&- e_8e_{15} - e_9e_{14})(e_2e_{19} + e_3e_{18} + e_3e_4 + e_2e_5 - e_8e_{17} - e_9e_{16})\\
&- 2(e_1e_{10} - e_6e_{20})(e_{12}e_{19} - e_{13}e_{18}) - 2(e_2e_{10} + e_1e_{11} - e_1e_{21}\\
&- e_7e_{20})(e_{11}e_{19} + e_{12}e_{18} + e_3e_{13} - e_9e_{22}) - 2(e_3e_{10} + e_2e_{11}\\
&+ e_1e_{12} - e_6e_{22} - e_7e_{21} - e_8e_{22})(e_{10}e_{19} + e_{11}e_{18} + e_3e_{12}\\
&+ e_2e_{13} - e_8e_{22} - e_9e_{21}) - (e_{10}e_{17} + e_{11}e_{16} + e_{12}e_{15} + e_{13}e_{14}\\
&- e_2e_{22} - e_3e_{21} - e_4e_{20})^2 - 2(e_{10}e_{14} - e_1e_{20})(e_{13}e_{17} - e_1e_{21}\\
&- e_5e_{22}) - 2(e_{10}e_{15} + e_{11}e_{14} - e_1e_{21} - e_2e_{20})(e_{12}e_{17} + e_{13}e_{16}\\
&- e_4e_{22} - e_5e_{21}) - 2(e_{10}e_{16} + e_{11}e_{15} + e_{12}e_{14} - e_2e_{21} - e_3e_{20})\\
&(e_{11}e_{17} + e_{12}e_{16} + e_{13}e_{15} - e_3e_{22} - e_4e_{21} - e_5e_{20}),
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{10} = {} & 2(2e_1e_2 - e_6e_{15} - e_7e_{14})e_5e_{19} + 2(e_1e_3 + e_2^2 + e_1e_3\\
&- e_6e_{16} - e_7e_{15} - e_8e_{14})^2 + 2(e_1^2 - e_6e_{14})(e_4e_{19} + e_5e_{18})\\
&+ 2(e_3e_{19} + e_4e_{18} + e_5e_3 - e_9e_{17}) + 2(e_1e_{19} + e_2e_{18}\\
&+ e_{11}e_{14} + e_3^2 + e_2e_4 - e_7e_{17} - e_8e_{16} - e_9e_{15})(e_2e_{19}
\end{aligned}
$$

$$
\begin{aligned}
&+ e_3e_{18} + e_3e_4 + e_2e_5 - e_8e_{17} - e_9e_{16}) - 2(e_{10}e_{15} - e_1e_{21} \\
&- e_2e_{20})(e_{13}e_{17} - e_1e_{21} - e_5e_{22}) - 2(e_{10}e_{16} + e_{11}e_{15} \\
&+ e_{12}e_{14} - e_2e_{21} - e_3e_{20})(e_{12}e_{17} + e_{13}e_{16} - e_4e_{22} \\
&e_5e_{21}) - 2(e_{10}e_{17} + e_{11}e_{16} + e_{12}e_{15} + e_{13}e_{14} \\
&- e_2e_{22} - e_3e_{21} - e_4e_{20})(e_{11}e_{17} + e_{12}e_{16} + e_{13}e_{15} \\
&- e_3e_{22} - e_4e_{21} - e_5e_{20}) - 2e_{13}e_{19}(e_1e_{10} - e_6e_{20}) \\
&- 2(e_2e_{20} + e_1e_{11} - e_2e_{21} - e_7e_{21})(e_{12}e_{19} + e_{18}e_{13}) \\
&- 2(e_3e_{10} + e_2e_{11} + e_1e_{12} - e_6e_{22} - e_7e_{21} - e_8e_{22}) \\
&\times (e_{11}e_{19} + e_{12}e_{18} + e_3e_{13} - e_9e_{22}) + 2(e_{10}e_{19} + e_{11}e_{18} \\
&+ e_3e_{12} + e_2e_{13} - e_8e_{22} - e_9e_{21}) - 2(e_{10}e_{18} + e_3e_{11} + e_2e_{12} \\
&+ e_1e_{13} - e_7e_{22} - e_8e_{21} - e_9e_{20})(e_{10}e_{19} + e_{11}e_{18} + e_3e_{12} \\
&+ e_2e_{13} - e_8e_{22} - e_9e_{21}),
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{11} = {}& (e_2e_{19} + e_3e_{18} + e_3e_4 + e_2e_5 - e_8e_{17} - e_9e_{16})^2 \\
&+ 2(e_1e_{19} + e_2e_{18} + e_3^2 + e_2e_4 - e_7e_{17} - e_8e_{16} \\
&- e_9e_{15})(e_3e_{19} + e_4e_{18} - e_9e_{17}) + 2(e_4e_{19} + e_5e_{18}) \\
&\times (e_1e_{18} + 2e_2e_3 + e_1e_4 - e_6e_{17} - e_7e_{16} - e_8e_{15} \\
&- e_9e_{14})e_5e_{19} - 2e_{13}e_{19}(e_2e_{10} + e_1e_{11} - e_1e_{21} - e_7e_{20}) \\
&- 2(e_{12}e_{19} + e_{13}e_{18})(e_3e_{10} + e_2e_{11} + e_1e_{12} - e_6e_{22} \\
&- e_7e_{21} - e_8e_{22}) - 2(e_{10}e_{18} + e_3e_{11} + e_2e_{12} + e_1e_{13} \\
&- e_7e_{22} - e_8e_{21} - e_9e_{20})(e_{10}e_{19} + e_{11}e_{18} + e_3e_{12} + e_2e_{13} \\
&- e_8e_{22} - e_9e_{21}) - (e_{11}e_{17} + e_{12}e_{16} + e_{13}e_{15} - e_3e_{22} \\
&- e_4e_{21} - e_5e_{20})^2 - 2(e_{10}e_{16} + e_{11}e_{15} + e_{12}e_{14} - e_2e_{21} \\
&- e_3e_{20})(e_{13}e_{17} - e_1e_{21} - e_5e_{22}) - 2(e_{12}e_{17} + e_{13}e_{16} \\
&- e_4e_{22} - e_5e_{21}),
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{12} = {}& 2(e_1e_{18} + 2e_2e_3 + e_1e_4 - e_6e_{17} - e_7e_{16} - e_8e_{15} - e_9e_{14})e_5e_{19} \\
&+ 2(e_1e_{19} + e_2e_{18} + e_3^2 + e_2e_4 - e_7e_{17} - e_8e_{16} - e_9e_{15}) \\
&\times (e_4e_{19} + e_5e_{18}) + 2(e_3e_{19} + e_4e_{18} + e_5e_3 - e_9e_{17}) \\
&\times (e_2e_{19} + e_3e_{18} + e_3e_4 + e_2e_5 - e_8e_{17} - e_9e_{16}) \\
&- (e_{10}e_{17} + e_{11}e_{16} + e_{12}e_{15} + e_{13}e_{14} - e_2e_{22} - e_3e_{21} - e_4e_{20})
\end{aligned}
$$

$$\times (e_{13}e_{17} - e_1e_{21} - e_5e_{22}) + 2(e_{13}e_{17} - e_1e_{21} - e_5e_{22})$$
$$\times (e_{11}e_{17} + e_{12}e_{16} + e_{13}e_{15} - e_3e_{22} - e_4e_{21} - e_5e_{20})$$
$$- 2e_{13}e_{19}(e_3e_{10} + e_2e_{11} + e_1e_{12} - e_6e_{22} - e_7e_{21} - e_8e_{22})$$
$$- 2(e_{10}e_{18} + e_3e_{11} + e_2e_{12} + e_1e_{13} - e_7e_{22} - e_8e_{21} - e_9e_{20})$$
$$\times (e_{12}e_{19} + e_{13}e_{18}) - 2(e_{11}e_{19} + e_{12}e_{18} + e_3e_{13} - e_9e_{22})$$
$$\times (e_{10}e_{19} + e_{11}e_{18} + e_3e_{12} + e_2e_{13} - e_8e_{22} - e_9e_{21}),$$

$$\varepsilon_{13} = 2e_5e_{19}(e_1e_{19} + e_2e_{18} + e_3^2 + e_2e_4 - e_7e_{17} - e_8e_{16}$$
$$- e_9e_{15}) + (e_3e_{19} + e_4e_{18} - e_9e_{17})^2 + 2(e_4e_{19} + e_5e_{18})$$
$$\times (e_2e_{19} + e_3e_{18} + e_3e_4 + e_2e_5 - e_8e_{17} - e_9e_{16})$$
$$- (e_{11}e_{19} + e_{12}e_{18} + e_3e_{13} - e_9e_{22})^2 - 2(e_{10}e_{19} + e_{11}e_{18}$$
$$+ e_3e_{12} + e_2e_{13} - e_8e_{22} - e_9e_{21})(e_{12}e_{19} + e_{13}e_{18})$$
$$- (e_{12}e_{17} + e_{13}e_{16} - e_4e_{22} - e_5e_{21})^2 - 2(e_{10}e_{17}$$
$$+ e_{11}e_{16} + e_{12}e_{15} + e_{13}e_{14} - e_2e_{22} - e_3e_{21} - e_4e_{20})$$
$$\times (e_{13}e_{17} - e_1e_{21} - e_5e_{22}),$$

$$\varepsilon_{14} = 2e_5e_{19}(e_2e_{19} + e_3e_{18} + e_3e_4 + e_2e_5 - e_8e_{17} - e_9e_{16})$$
$$+ 2(e_4e_{19} + e_5e_{18})(e_{11}e_{19} + e_{12}e_{18} + e_3e_{13} - e_9e_{22})$$
$$- 2(e_{12}e_{17} + e_{13}e_{16} - e_4e_{22} - e_5e_{21})(e_{13}e_{17} - e_1e_{21} - e_5e_{22})$$
$$- 2(e_{10}e_{19} + e_{11}e_{18} + e_3e_{12} + e_2e_{13} - e_8e_{22} - e_9e_{21})e_{13}e_{19}$$
$$- 2e_{12}e_{19} + e_{13}e_{18})(e_{11}e_{19} + e_{12}e_{18} + e_3e_{13} - e_9e_{22}),$$

$$\varepsilon_{15} = (e_4e_{19} + e_5e_{18})^2 + 2e_5e_{19}(e_3e_{19} + e_4e_{18} - e_9e_{17})$$
$$- (e_{12}e_{19} + e_{13}e_{18})^2 - 2e_{13}e_{19}(e_{11}e_{19} + e_{12}e_{18} + e_3e_{13}$$
$$- e_9e_{22}) - (e_{13}e_{17} - e_1e_{21} - e_5e_{21})^2,$$

$$\varepsilon_{16} = 2e_5e_{19}(e_4e_{19} + e_5e_{18}) - 22_{13}e_{19}(e_{12}e_{19} + e_{13}e_{18}),$$

$$\varepsilon_{17} = e_5^2e_{19}^2 - e_{13}^2e_{19}^2.$$

Let

$$\Pi(\rho) = \varepsilon_1\rho^{16\varrho} + \varepsilon_2\rho^{15\varrho} + \varepsilon_3\rho^{14\varrho} + \varepsilon_4\rho^{13\varrho} + \varepsilon_5\rho^{12\varrho} + \varepsilon_6\rho^{11\varrho}$$
$$+ \varepsilon_7\rho^{10\varrho} + \varepsilon_8\rho^{9\varrho} + \varepsilon_9\rho^{8\varrho} + \varepsilon_{10}\rho^{7\varrho} + \varepsilon_{11}\rho^{6\varrho} + \varepsilon_{12}\rho^{5\varrho}$$
$$+\varepsilon_{13}\rho^{4\varrho} + \varepsilon_{14}\rho^{3\varrho} + \varepsilon_{15}\rho^{2\varrho} + \varepsilon_{16}\rho^{\varrho} + \varepsilon_{17}. \tag{26}$$

Assume that

$$(Q_3)\ \ e_5^2e_{19}^2 < e_{13}^2e_{19}^2.$$

By (Q_3), we get $\varepsilon_{17} < 0$. Since $\frac{d\Pi(\rho)}{d\rho} > 0,\ \forall\ \rho > 0$, then Eq.(25) has at least one positive real root. So Eq.(17) has at least one pair of purely roots.

Suppose Eq.(26) has 16 real roots denoted by $\rho_j > 0 (j = 1, 2, \cdots, 16)$. By virtue of (15), one gets

$$\gamma_j^l = \frac{1}{\rho_j}\left[\arccos\left(\frac{a_3b_2 - a_2b_3}{a_1b_2 - a_2b_1}\right) + 2l\pi\right], \tag{27}$$

where $l = 0, 1, 2, \cdots, j = 1, 2, \cdots, 16$. Let

$$\gamma_0 = \min_{j=1,2,\cdots,16}\{\gamma_j^0\}, \rho_0 = \rho|_{\gamma=\gamma_0}. \tag{28}$$

Now the following assumption is needed:

(Q_4) $\mathcal{S}_1\mathcal{Z}_1 + \mathcal{S}_2\mathcal{Z}_2 > 0$, where

$$\begin{cases} \mathcal{S}_1 = \Big[4\varrho\rho_0^{4\varrho-1}\cos\frac{(4\varrho-1)\pi}{2} + 3\varrho\beta_1\rho_0^{3\varrho-1}\cos\frac{(3\varrho-1)\pi}{2} \\ \quad +2\varrho\beta_2\rho_0^{2\varrho-1}\cos\frac{(2\varrho-1)\pi}{2} + \varrho\beta_3\rho_0^{\varrho-1}\cos\frac{(\varrho-1)\pi}{2}\Big]\cos\rho_0\gamma_0 \\ \quad - \Big[4\varrho\rho_0^{4\varrho-1}\sin\frac{(4\varrho-1)\pi}{2} + 3\varrho\beta_1\rho_0^{3\varrho-1}\sin\frac{(3\varrho-1)\pi}{2} \\ \quad +2\varrho\beta_2\rho_0^{2\varrho-1}\sin\frac{(2\varrho-1)\pi}{2} + \varrho\beta_3\rho_0^{\varrho-1}\sin\frac{(\varrho-1)\pi}{2}\Big]\sin\rho_0\gamma_0 \\ \quad + 3\varrho\varsigma_1\rho_0^{3\varrho-1}\cos\frac{(3\varrho-1)\pi}{2} + 2\varrho\varsigma_2\rho_0^{2\varrho-1}\cos\frac{(2\varrho-1)\pi}{2} \\ \quad + \varrho\varsigma_3\rho_0^{\varrho-1}\cos\frac{(\varrho-1)\pi}{2} + \delta_1\varrho\rho_0^{\varrho-1}\cos\frac{(\varrho-1)\pi}{2}\cos\rho_0\gamma_0 \\ \quad + \delta_1\varrho\rho_0^{\varrho-1}\sin\frac{(\varrho-1)\pi}{2}\sin\rho_0\gamma_0, \\ \mathcal{S}_2 = \Big[4\varrho\rho_0^{4\varrho-1}\cos\frac{(4\varrho-1)\pi}{2} + 3\varrho\beta_1\rho_0^{3\varrho-1}\cos\frac{(3\varrho-1)\pi}{2} \\ \quad +2\varrho\beta_2\rho_0^{2\varrho-1}\cos\frac{(2\varrho-1)\pi}{2} + \varrho\beta_3\rho_0^{\varrho-1}\cos\frac{(\varrho-1)\pi}{2}\Big]\sin\rho_0\gamma_0 \\ \quad + \Big[4\varrho\rho_0^{4\varrho-1}\sin\frac{(4\varrho-1)\pi}{2} + 3\varrho\beta_1\rho_0^{3\varrho-1}\sin\frac{(3\varrho-1)\pi}{2} \\ \quad +2\varrho\beta_2\rho_0^{2\varrho-1}\sin\frac{(2\varrho-1)\pi}{2} + \varrho\beta_3\rho_0^{\varrho-1}\sin\frac{(\varrho-1)\pi}{2}\Big]\cos\rho_0\gamma_0 \\ \quad + 3\varrho\varsigma_1\rho_0^{3\varrho-1}\sin\frac{(3\varrho-1)\pi}{2} + 2\varrho\varsigma_2\rho_0^{2\varrho-1}\sin\frac{(2\varrho-1)\pi}{2} \\ \quad + \varrho\varsigma_3\rho_0^{\varrho-1}\sin\frac{(\varrho-1)\pi}{2} - \delta_1\varrho\rho_0^{\varrho-1}\cos\frac{(\varrho-1)\pi}{2}\sin\rho_0\gamma_0 \\ \quad + \delta_1\varrho\rho_0^{\varrho-1}\sin\frac{(\varrho-1)\pi}{2}\cos\rho_0\gamma_0, \end{cases} \tag{29}$$

$$\begin{cases} \mathcal{Z}_1 = \left(\rho_0^{4\varrho}\cos 2\varrho\pi + \beta_1\rho_0^{3\varrho}\cos\frac{3\varrho\pi}{2} + \beta_2\rho_0^{2\varrho}\cos\varrho\pi + \beta_3\rho_0^{\varrho}\frac{\varrho\pi}{2} + \beta_4\right)\gamma_0\rho_0\sin\rho_0\gamma_0 \\ \qquad - \left(\rho_0^{4\varrho}\sin 2\varrho\pi + \beta_1\rho_0^{3\varrho}\sin\frac{3\varrho\pi}{2} + \beta_2\rho_0^{2\varrho}\sin\varrho\pi + \beta_3\rho_0^{\varrho}\frac{\varrho\pi}{2}\right)\gamma_0\rho_0\cos\rho_0\gamma_0, \\ \mathcal{Z}_2 = -\left(\rho_0^{4\varrho}\cos 2\varrho\pi + \beta_1\rho_0^{3\varrho}\cos\frac{3\varrho\pi}{2} + \beta_2\rho_0^{2\varrho}\cos\varrho\pi + \beta_3\rho_0^{\varrho}\frac{\varrho\pi}{2} + \beta_4\right)\gamma_0\rho_0\cos\rho_0\gamma_0 \\ \qquad + \left(\rho_0^{4\varrho}\sin 2\varrho\pi + \beta_1\rho_0^{3\varrho}\sin\frac{3\varrho\pi}{2} + \beta_2\rho_0^{2\varrho}\sin\varrho\pi + \beta_3\rho_0^{\varrho}\frac{\varrho\pi}{2}\right)\gamma_0\rho_0\sin\rho_0\gamma_0. \end{cases} \tag{30}$$

Lemma 3.2. *If $s(\gamma) = \alpha_1(\gamma) + i\alpha_2(\gamma)$ is the root of (17) near $\gamma = \gamma_0$ such that $\alpha_1(\gamma_0) = 0, \alpha_2(\gamma_0) = \rho_0$, then Re $\left[\frac{ds}{d\gamma}\right]_{\gamma=\gamma_0,\rho=\rho_0} > 0$.*

Proof By virtue of (17), one derives

$$\begin{aligned} &(4\varrho s^{4\varrho-1} + 3\varrho\beta_1 s^{3\varrho-1} + 2\varrho\beta_2 s^{2\varrho-1} + \varrho\beta_3 s^{\varrho-1})e^{s\gamma}\frac{ds}{d\gamma} \\ &+e^{s\gamma}\left(\frac{ds}{d\gamma}\gamma + s\right)(s^{4\varrho} + \beta_1 s^{3\varrho} + \beta_2 s^{2\varrho} + \beta_3 s^{\varrho} + \beta_4) \\ &+ (3\varrho\varsigma_1 s^{3\varrho-1} + 2\varrho\varsigma_2 s^{2\varrho-1} + \varrho\varsigma_3 s^{\varrho-1})\frac{ds}{d\gamma} + \delta_1\varrho s^{\varrho-1}\frac{ds}{d\gamma}e^{-s\gamma} \\ &-\left(\frac{ds}{d\gamma}\gamma + s\right)(\delta_1 s^{\varrho} + \delta_2) = 0. \end{aligned} \tag{31}$$

It follows from (31) that

$$\left(\frac{ds}{d\gamma}\right)^{-1} = \frac{\mathcal{S}}{\mathcal{Z}} - \frac{\gamma}{s}, \tag{32}$$

where

$$\begin{cases} \mathcal{S} = e^{s\gamma}(4\varrho s^{4\varrho-1} + 3\varrho\beta_1 s^{3\varrho-1} + 2\varrho\beta_2 s^{2\varrho-1} + \varrho\beta_3 s^{\varrho-1}) \\ \qquad + (3\varrho\varsigma_1 s^{3\varrho-1} + 2\varrho\varsigma_2 s^{2\varrho-1} + \varrho\varsigma_3 s^{\varrho-1}) + \delta_1\varrho s^{\varrho-1}e^{-s\gamma}, \\ \mathcal{Z} = -\gamma e^{s\gamma}s(s^{4\varrho} + \beta_1 s^{3\varrho} + \beta_2 s^{2\varrho} + \beta_3 s^{\varrho} + \beta_4) + s(\delta_1 s^{\varrho} + \delta_2). \end{cases} \tag{33}$$

Then

$$\text{Re}\left[\left(\frac{ds}{d\gamma}\right)^{-1}\right] = \text{Re}\left[\left(\frac{\mathcal{S}}{\mathcal{Z}}\right)^{-1}\right]. \tag{34}$$

Thus

$$\text{Re}\left[\left(\frac{ds}{d\gamma}\right)^{-1}\right]_{\gamma=\gamma_0,\rho=\rho_0} = \frac{\mathcal{S}_1\mathcal{Z}_1 + \mathcal{S}_2\mathcal{Z}_2}{\mathcal{Z}_1^2 + \mathcal{Z}_2^2}. \tag{35}$$

Applying (G_4), we have

$$\text{Re}\left[\left(\frac{ds}{d\gamma}\right)^{-1}\right]_{\gamma=\gamma_0,\rho=\rho_0} > 0. \tag{36}$$

The proof of Lemma 3.2 finishes. ■

Utilizing the investigation above, the following assertion is derived.

Theorem 3.1. *Under the assumptions (G_1)-(G_4), the positive equilibrium point $\mathcal{W}_3(w_{1*+}, w_{2*+}, w_{3*+}, w_{4*+})$ of system (3) is locally asymptotically stable if γ falls into the range*

$[0, \gamma_0)$ and a Hopf bifurcation will take place near $\mathcal{W}_3(w_{1*+}, w_{2*+}, w_{3*+}, w_{4*+})$ if γ passes the critical value γ_0.

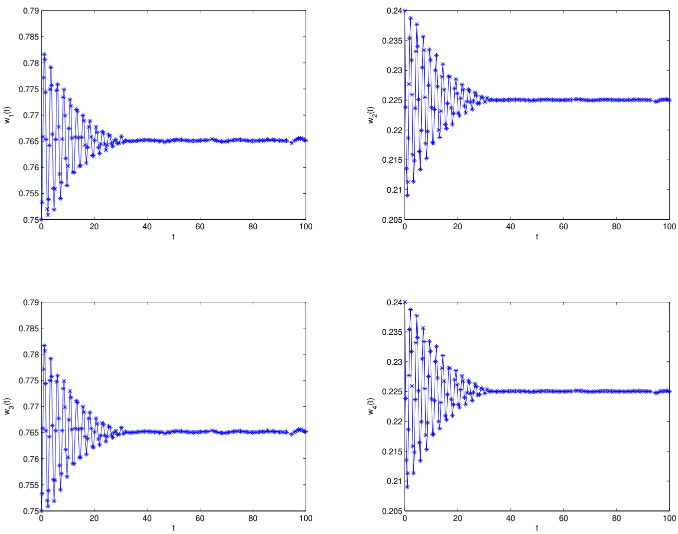
Remark 3.1. *In 2013, Wu and Zhang [8] probed deeply into the bifurcation issue for integer-order delayed coupled Oregonator model, they are not concerned with the fractional-order situation. In this current study, we build a new fractional-order delayed coupled Oregonator model which can effectively characterize memory peculiarity and hereditary influence for diverse chemical compositions. A novel delay-independent stability and Hopf bifurcation condition is set up via laplace transform, stability and Hopf bifurcation theory, which are different from that in integer-order version in [8]. Based on this viewpoint, We strongly believe that the established results of this study on the stability and Hopf bifurcation problem for the fractional-order delayed coupled Oregonator model completely innovative and complement the research of [8] to some extent.*

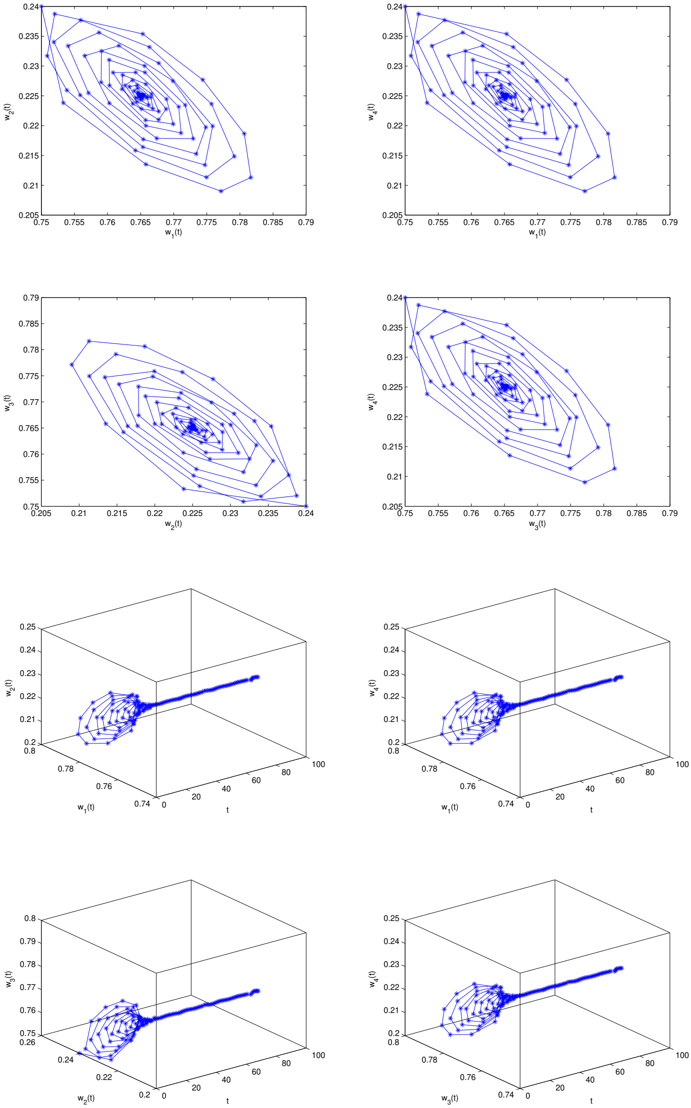
4 Matlab simulation results

Consider the following fractional-order coupled Oregonator model:

$$\begin{cases} \epsilon\dfrac{dw_1^{\varrho}(t)}{dt^{\varrho}} = w_1(1-w_1) - gw_2\dfrac{w_1 - v}{w_1 + v} + A(w_3 - w_1), \\ \dfrac{dw_2^{\varrho}(t)}{dt^{\varrho}} = w_1 - w_2 + \vartheta w_2(t-\gamma), \\ \epsilon\dfrac{dw_3(t)}{dt} = w_3(1-w_3) - gw_4\dfrac{w_3 - v}{w_3 + v} + A(w_1 - w_3), \\ \dfrac{dw_4^{\varrho}(t)}{dt^{\varrho}} = w_3 - w_4 + \vartheta w_4(t-\gamma), \end{cases} \tag{37}$$

where $\epsilon = 0.01, g = 0.8, v = 0.0007, \vartheta = 2.4$. One can easily derive that the coupled Oregonator model (37) possesses the unique positive equilibrium point $(0.7651, 0.2250, 0.7651, 0.2250)$. Fix $\varrho = 0.56$. Utilizing Matlab, one can derive $\rho_0 = 0.8875$ and $\gamma_0 = 0.12$. The assumptions (Q_1)-(Q_4) of Theorem 3.1 are fulfilled. In order to verify the stability of the positive equilibrium point $(0.7651, 0.2250, 0.7651, 0.2250)$ and bifurcation phenomenon for the coupled Oregonator model (37), we give both unequal delay numbers. Let $\gamma = 0.10 < \gamma_0 = 0.12$, we obtain the matlab simulation results which are displayed in Figure 1. From Figure 1, one can clearly see that the positive equilibrium point $(0.7651, 0.2250, 0.7651, 0.2250)$ keeps locally asymptotically stable level. Figure 1 contains 16 subfigures. Subfigure 1 in Figure 1 implies that with the increase of time t, the variable $w_1 \to 0.7651$. Subfigures 2-4 in Figure 1 imply that with the increase of time t, the variables $w_2 \to 0.2250$, $w_3 \to 0.7651$, $w_4 \to 0.2250$. Subfigures 5-8 in Figure 1 reveal the quantitative relation of w_1 and w_2, w_1 and w_4, w_2 and w_3, w_3 and w_4, respectively. Subfigures 9-12 in Figure 1 display the change relation of t-w_1-w_2, t-w_1-w_4, t-w_2-w_3 and t-w_3-w_4, respectively. Subfigures 13-16 in Figure 1 shows the change relation of w_1-w_2-w_3, w_1-w_2-w_4, w_1-w_3-w_4 and w_2-w_3-w_4, respectively. Let $\gamma = 0.21 > \gamma_0 = 0.12$, we obtain the matlab simulation results which are displayed in Figure 2. From Figure 2, one can clearly see that Hopf bifurcation occurs in the vicinity of the positive equilibrium point $(0.7651, 0.2250, 0.7651, 0.2250)$. Figure 2 contains 16 subfigures. Subfigure 1 in Figure 2 implies that with the increase of time t, the variable w_1 will remain a periodic oscillation around the value 0.7651. Subfigures 2-4 in Figure 2 imply that with the increase of time t, the variables w_2, w_3, w_4 will remain a periodic oscillation around the values $0.2250, 0.7651, 0.2250$, respectively. Subfigures 5-8 in Figure 2 reveal the quantitative relation of w_1 and w_2, w_1 and w_4, w_2 and w_3, w_3 and w_4, respectively. Subfigures 9-12 in Figure 2 display the change relation of t-w_1-w_2, t-w_1-w_4, t-w_2-w_3 and t-w_3-w_4, respectively. Subfigures 13-16 in Figure 2 shows the change relation of w_1-w_2-w_3, w_1-w_2-w_4, w_1-w_3-w_4 and w_2-w_3-w_4, respectively. The relation of ϱ, ρ_0 and γ_0 is presented. Additionally, the bifurcation diagrams are presented to verify the bifurcation value is 0.12 (see Figures 3-6).




w2(t)
w1(t)
w4(t)
w1(t)
w3(t)
w2(t)
w4(t)
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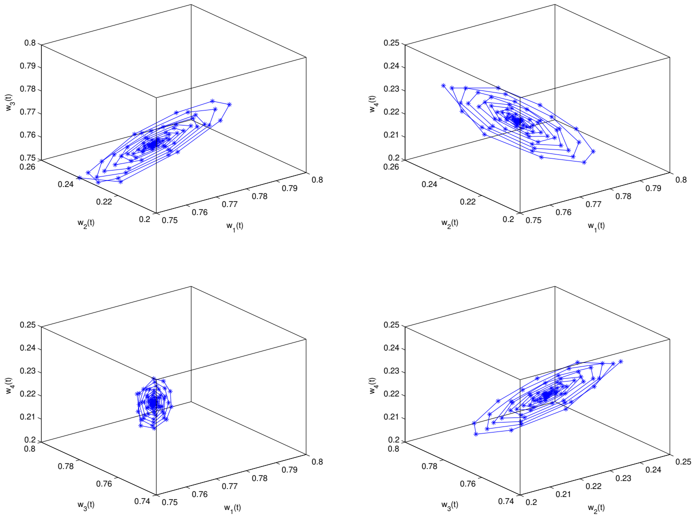
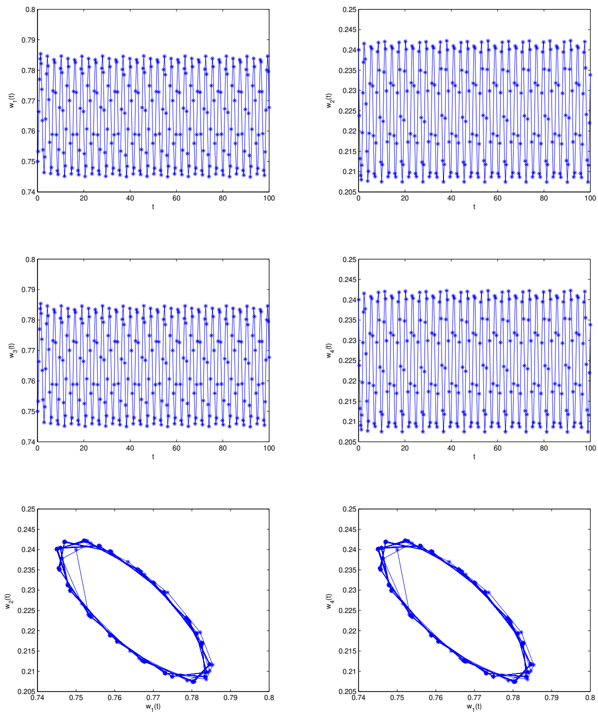


Figure 1. $\gamma = 0.10 < \gamma_0 = 0.12$.



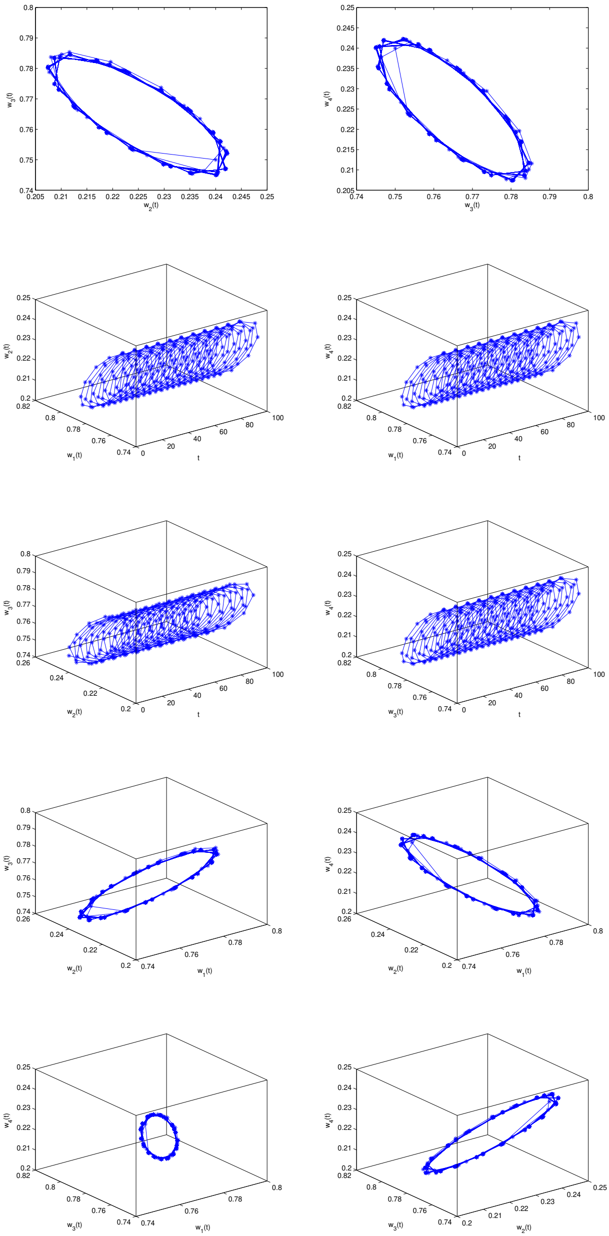


Figure 2. $\gamma = 0.21 > \gamma_0 = 0.12$.

Table 1. The magnitude relation for ϱ, ρ_0 and γ_0 of coupled Oregonator model (37).

ϱ	ρ_0	γ_0
0.12	1.4033	0.006
0.23	1.2356	0.008
0.32	1.1209	0.009
0.45	0.9123	0.010
0.56	0.8875	0.120
0.64	0.7534	0.140
0.76	0.6901	0.152
0.82	0.5823	0.167
0.91	0.4805	0.213

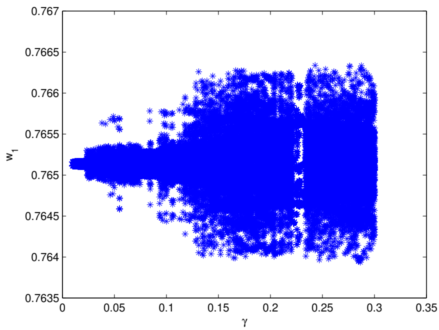


Figure 3. Bifurcation diagram for coupled Oregonator model (37): γ-w_1.

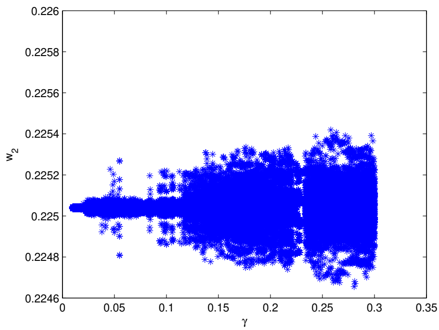


Figure 4. Bifurcation diagram for coupled Oregonator model (37): γ-w_2.

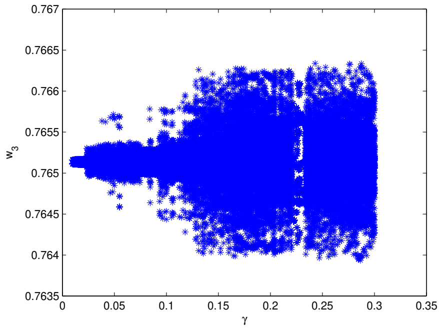


Figure 5. Bifurcation diagram for coupled Oregonator model (37): γ-w_3.

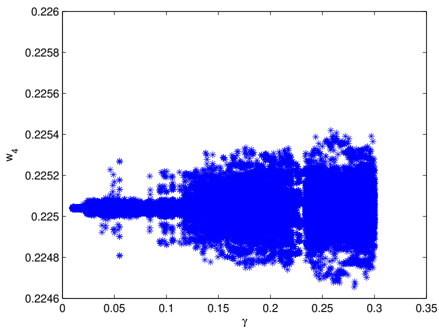


Figure 6. Bifurcation diagram for coupled Oregonator model (37): γ-w_4.

5 Conclusions

Fractional-order differential equation has displayed tremendous application prospect in chemistry. Based on the previous studies, we establish a new fractional-order coupled Oregonator model. By virtue of laplace transform, the characteristic equation of the established fractional-order coupled Oregonator model is derived. Taking advantage of the stability criteria and bifurcation knowledge on fractional-order dynamical system, a new sufficient condition that ensures the stability and the emergence of Hopf bifurcation for the established fractional-order coupled Oregonator model is presented. The research shows that time delay acts as a significant role in describing the stability and bifurcation behavior in the involved fractional-order coupled Oregonator model. The research fruits tell us that we can control the time delay in a suitable range of value to remain the stability of the coupled Oregonator system. By adjusting the value of time delay, we can postpone or advance the onset of Hopf bifurcation for the coupled Oregonator system. The obtained results can be applied to control the concentrations of the chemical compositions $HBrO_2$ and $Ce(IV)$.

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