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Non–Hypoenergetic Graphs with Nullity 2

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Abstract

The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of absolute values of all eigenvalues of G. A graph of order n, whose energy is less than n, i.e., $\mathcal{E}(G) < n$, is said to be hypoenergetic. Graphs for which $\mathcal{E}(G) \geq n$ are called non-hypoenergetic. A graph of order n is said to be orderenergetic, if its energy and its order are equal, i.e., $\mathcal{E}(G) = n$. In this paper, we characterize non-hypoenergetic graphs with nullity 2. It is proved that except two graphs, every connected graph with nullity 2 is non-hypoenergetic.

1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. By *order* and *size* of G, we mean the number of vertices and the number of edges of G, respectively. We denote the order of G by $|V(G)|$. For any vertex $v \in V(G)$, the open neighborhood of v in G is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Also the degree of v in G is just $d(v) = |N_G(v)|$. Let $S \subseteq V(G)$. By $\langle S \rangle$, we mean the subgraph of G induced by S. The path and the cycle of order n are denoted by P_n and C_n, respectively. A

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graph is *claw-free*, if it has no induced subgraph isomorphic to $K_{1,3}$. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. If $m = n$, then we say that $K_{m,m}$ is balanced. A $\{1,2\}$-*subgraph* of G is a subgraph which is a disjoint union of a matching and a 2-regular subgraph of G. A $\{1,2\}$-*subgraph* which is a spanning subgraph, is called a $\{1,2\}$-*factor*.

The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$, otherwise. Thus $A(G)$ is a symmetric matrix and all eigenvalues of $A(G)$ are real. Let $\eta(G)$, the *nullity* of G, denote the number of zero eigenvalues of $A(G)$. The *energy* of a graph G, $\mathcal{E}(G)$, is defined as the sum of absolute values of eigenvalues of $A(G)$, see [7].

Graphs of order n, satisfying the condition $\mathcal{E}(G) < n$ are named *hypoenergetic* and their properties were studied in [9–11]. Graphs for which $\mathcal{E}(G) \geq n$ are said to be *non-hypoenergetic*. A graph is called *orderenergetic*, if its energy and its order are equal, i.e., $\mathcal{E}(G) = n$. Some basic properties of orderenergetic graphs were studied in [3]. The authors showed that there are infinitely many connected orderenergetic graphs. They proved that a graph having a $\{1,2\}$-factor, is orderenergetic if and only if it is a disjoint union of balanced complete bipartite graphs. Also it was established that there is no orderenergetic graph with nullity 1. The concept of energy was extended by Nikiforov to arbirary complex matrices and digraphs, [13]. In particular, in [2], the authors studied hypoenergetic and non-hypoenergetic digraphs.

In this paper, we investigate some graphs whose energies exceed the number of vertices. We prove that if a graph G, has a $\{1,2\}$-subgraph of order $n-1$, then $\mathcal{E}(G) > n$, except for $K_{r,r+1}$, $r \geq 1$. It is shown that if $K_{2,4}$ is an induced subgraph of a graph G such that $G \setminus V(K_{2,4})$ has a perfect matching and G has no component isomorphic to $K_{r,r+2}$, for each r, then $\mathcal{E}(G) > n$. It is also proved that if P_3 is an induced subgraph of a graph G, such that $G \setminus V(P_3)$ has a perfect matching and G has no component isomorphic to $K_{r,r+1}$, for each r, then $\mathcal{E}(G) > n$. Using these results we show that if G is a claw-free graph of order n, then $\mathcal{E}(G) > n$, except for C_4 and $K_{1,2}$. Furthermore, it is proved that if the nullity of a graph G of order n, is 1, then $\mathcal{E}(G) > n$, except for $K_{1,2}$. Also, we show that except two graphs, every graph with nullity 2 is non-hypoenergetic. In particular, there exist only two connected orderenergetic graphs with nullity 2. The following lemmas are needed in the sequel.

Lemma 1. *[4] Let G be a graph and $H_1, \dots, H_k$ be its k vertex-disjoint induced subgraphs. Then*

$$\mathcal{E}(G) \geq \sum_{i=1}^{k} \mathcal{E}(H_i).$$

Lemma 2. *[12] Let H be an induced subgraph of a graph G. Then $\mathcal{E}(H) \leq \mathcal{E}(G)$ and equality holds if and only if $E(G) = E(H)$,*

Lemma 3. *[3] Let G be a graph of order n. If G has a $\{1,2\}$-factor, then $\mathcal{E}(G) \geq n$. Equality holds if and only if G is a disjoint union of balanced complete bipartite graphs.*

Lemma 4. *[1, 3] If n is an odd integer, then $\mathcal{E}(C_n) \geq n+1$. Moreover, for $n \geq 9$, $\mathcal{E}(C_n) \geq n+2$.*

Lemma 5. *[3] There is no connected orderenergetic graph with $\eta = 1$.*

Lemma 6. *[12] Let u be a pendent vertex of a graph G and v the vertex in G adjacent to u. Then $\eta(G) = \eta(G - u - v)$.*

Lemma 7. *[5] Let G be a connected claw-free graph. Then it contains a matching which avoids at most one vertex.*

2 Graphs whose energies exceed the number of vertices

In this section, we investigate some graphs whose energies exceed the number of vertices. Furthermore, we show that, except two graphs, every graph with nullity 2 is non-hypoenergetic

Lemma 8. *Let G be a graph of order n such that $K_{2,4}$ is an induced subgraph of G and $G \setminus V(K_{2,4})$ has a perfect matching. If G has no component isomorphic to $K_{r,r+2}$ (for each r), then $\mathcal{E}(G) > n$.*

Proof. Let M be a perfect matching of $G \setminus V(K_{2,4})$. Let r be the maximum number of P_2-components of M, such that the subgraph induced on $V(K_{2,4})$ and these P_2-components is complete bipartite. With no loss of generality, one can assume that these P_2-components are $e_1, \dots, e_r$. Let H be the subgraph induced on $V(K_{2,4})$ and vertices of $e_1, \dots, e_r$. (In fact, $H = K_{2+r,4+r}$.) Since H is not a component of G, there exists a P_2-component of

M, different from e_i, $1 \leq i \leq r$, say e, such that H is connected to e. We consider two cases:

Case 1. The edge e is connected to $K_{2,4}$. Let K be the subgraph induced on $V(K_{2,4})$ and endpoints of e. If $K \neq K_{3,5}$, then a computer search shows that $\mathcal{E}(K) > 8$. Now, since $G \setminus V(K)$ has a perfect matching, by Lemmas 1 and 3, we find that

$$\mathcal{E}(G) \geq \mathcal{E}(K) + \mathcal{E}(G \setminus V(K)) > 8 + |V(G \setminus V(K))| > n,$$

as desired. If $K = K_{3,5}$, then there exists some e_j, $1 \leq j \leq r$, such that the subgraph induced on endpoints of e and e_j is not C_4. Now, let K' be the subgraph induced on $V(K_{2,4})$ and endpoints of e and e_j. By a computer search one can find that $\mathcal{E}(K') > 10$. Now, a similar argument as above implies that $\mathcal{E}(G) > n$.

Case 2. The edge e is connected to some e_i, $1 \leq i \leq r$ and it is not connected to $K_{2,4}$. Let K'' be the subgraph induced on $V(K_{2,4})$ and endpoints of e and e_i. A computer search shows that $\mathcal{E}(K'') > 10$. Therefore as we did before $\mathcal{E}(G) > n$. This completes the proof. ■

Lemma 9. *Let G be a graph of order n such that P_3 is an induced subgraph of G and $G \setminus V(P_3)$ has a perfect matching. If G has no component isomorphic to $K_{r,r+1}$ (for each r), then $\mathcal{E}(G) > n$.*

Proof. Let M be a perfect matching of $G \setminus V(P_3)$. Let $e_1, \ldots, e_r$, $r \geq 1$ be the maximum number of P_2-components of M, such that the subgraph induced on $V(P_3)$ and the vertices of these P_2-components is complete bipartite. Let H be the subgraph induced on $V(P_3)$ and vertices of $e_1, \ldots, e_r$. (In fact, $H = K_{r+1,r+2}$.) Since H is not a component of G, there exists a P_2-components of M, different from e_i, $1 \leq i \leq r$, say e, such that H is connected to e. We consider two cases:

Case 1. The edge e is connected to P_3. Let K be the subgraph induced on $V(P_3)$ and endpoints of e. If $K \neq K_{2,3}$, then a computer search shows that $\mathcal{E}(K) \geq 5.22$. Now, since $G \setminus V(K)$ has a perfect matching, by Lemmas 1 and 3, we find that

$$\mathcal{E}(G) \geq \mathcal{E}(K) + \mathcal{E}(G \setminus V(K)) > 5 + |V(G \setminus V(K))| > n,$$

as desired. If $K = K_{2,3}$, then there exists some e_j, $1 \leq j \leq r$, such that the subgraph induced on endpoints of e and e_j is not C_4. Now, let K' be the subgraph induced on

$V(P_3)$ and endpoints of e and e_j. By a computer search one can find that $\mathcal{E}(K') \geq 7.94$. Now, a similar argument as above implies that $\mathcal{E}(G) > n$.

Case 2. The edge e is connected to some e_i, $1 \leq i \leq r$ and it is not connected to P_3. Let K'' be the subgraph induced on $V(P_3)$ and end points of e and e_i. A computer search shows that $\mathcal{E}(K'') > 7.45$. Hence as we did before $\mathcal{E}(G) > n$. ∎

Lemma 10. *Let G be a graph of order n such that $2P_3 = P_3 \cup P_3$ is an induced subgraph of G and $G \setminus V(2P_3)$ has a perfect matching. If G has no component isomorphic to $K_{r,r+1}$ (for each r), then $\mathcal{E}(G) > n$.*

Proof. Assume that W_1 and W_2 are P_3-components of G. Let M be a perfect matching of $G \setminus V(W_1 \cup W_2)$. Let $e_1, \dots, e_r$, $r \geq 1$ be the maximum number of P_2-components of M, such that the subgraph H induced on $V(W_1)$ and vertices of $e_1, \dots, e_r$ is complete bipartite. Since H is not a component of G, either H is connected to some P_2-component of M, say e, where $e \neq e_i$, $1 \leq i \leq r$, or H is connected to W_2. Two cases can be considered:

Case 1. Assume that the subgraph H is connected to some P_2-component e of M, where e is different from e_i, $1 \leq i \leq r$. Then the argument used in the proof of Lemma 9, implies that W_1 is contained in a connected subgraph K of G, where K is of order 5 or 7 and $\mathcal{E}(K) > |V(K)|$. As we stated in the proof of Lemma 9, if K is of order 5, then $\mathcal{E}(K) \geq 5.22$ and if K is of order 7, then $\mathcal{E}(K) \geq 7.45$. First, suppose that K is of order 5. Now, since $\mathcal{E}(W_2) = 2.82$ and $G \setminus [V(K) \cup V(W_2)]$ has a perfect matching, we obtain

$$\mathcal{E}(G) \geq \mathcal{E}(K) + \mathcal{E}(W_2) + \mathcal{E}(G \setminus [V(K) \cup V(W_2)]) \geq$$

$$5.22 + 2.82 + |V(G \setminus [V(K) \cup V(W_2)])| > n.$$

Next, assume that the order of K is 7. We have

$$\mathcal{E}(G) \geq \mathcal{E}(K) + \mathcal{E}(W_2) + \mathcal{E}(G \setminus [V(K) \cup V(W_2)]) \geq$$

$$7.45 + 2.82 + |V(G \setminus [V(K) \cup V(W_2)])| > n,$$

as desired.

Case 2. Assume that the subgraph H is connected to W_2. Then there exists some e_j, $1 \leq j \leq r$, such that W_2 is connected to e_j. Let K' be the subgraph induced on $V(W_1) \cup V(W_2)$ and endpoints of e_j. A computer search shows that $\mathcal{E}(K') > 8$. Now, since $G \setminus V(K')$ has a perfect matching, we find that $\mathcal{E}(G) > n$. This completes the proof. ■

Corollary 11. *Let G be a connected graph of order n. If G has a $\{1,2\}$-subgraph of order $n-1$, then $\mathcal{E}(G) > n$, except for $G = K_{r,r+1}$, $r \geq 1$. In particular, if the nullity of G is 1, then $\mathcal{E}(G) > n$, except for $K_{1,2}$.*

Proof. Let H be a $\{1,2\}$-subgraph of order $n-1$. Let $V(G) \setminus V(H) = \{u\}$. First assume that H contains at least one odd cycle, say C. One may assume that every odd cycle in H is an induced odd cycle, because if we have an odd cycle with a chord, then there is a chord which partitions the vertices of odd cycle into an induced odd cycle and some paths of order 2. Furthermore, we can assume that H has no even cycle, because the vertex set of every even cycle can be partitioned into disjoint copies of P_2. Now, since $\langle V(H) \rangle$ is a proper subgraph of G, by Lemmas 1, 2 and 4 we obtain

$$\mathcal{E}(G) > \mathcal{E}(\langle V(H) \rangle) \geq \mathcal{E}(C) + \mathcal{E}(\langle V(H) \setminus V(C) \rangle) \geq (|V(C)| + 1) + |\langle V(H) \setminus V(C) \rangle| \geq n,$$

so we are done.

Next, suppose that every component of H is P_2. Let vw be the P_2-component of H, such that $v \in N(u)$. Let $K = \langle u, v, w \rangle$. If $K = C_3$, then G has a $\{1,2\}$-factor. Hence, by Lemma 3, $\mathcal{E}(G) \geq n$. Note, that in the equality case, G should a balanced complete bipartite graph, but this contradicts the existence of C_3 in G. Thus $\mathcal{E}(G) > n$. If $K = P_3$, then Lemma 9, implies that $\mathcal{E}(G) > n$, as desired. In order to prove the last assertion, let $\phi(G,x) = \sum_{i=0}^{n} c_i x^{n-i}$ be the characteristic polynomial of G. Since $\eta(G) = 1$, $c_{n-1} \neq 0$. Now, by Sachs Theorem [12, p. 7], G contains a $\{1,2\}$-subgraph of order $n-1$. Hence the result follows from the previous part. ■

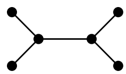
Remark 12. Note that Corollary 11, implies Lemma 5.

Corollary 13. *If G is a connected claw-free graph of order n, then $\mathcal{E}(G) > n$, except for $K_{1,2}$ and C_4.*

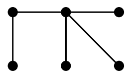
Proof. Let G be a claw-free graph of order n. Then by Lemma 7, either G has a $\{1,2\}$-factor, or it has a $\{1,2\}$-subgraph of order $n-1$. First, assume that G has a $\{1,2\}$-factor, then by Lemma 3, $\mathcal{E}(G) \geq n$. If the equality holds, then G is a balanced complete bipartite graph. However, since G is claw-free, this is only possible if $G = C_4$. Hence $\mathcal{E}(G) > n$, except for C_4. Next, suppose that G has a $\{1,2\}$-subgraph of order $n-1$. Then by Corollary 11, $\mathcal{E}(G) > n$, except for $K_{1,2}$. So we are done. ■

Now, we are in a position to state the main theorem of this section.

Theorem 14. *Let G be a connected graph of order n with nullity 2. If $G \notin \{K_{1,3}, C_4, G_1, G_2\}$, then $\mathcal{E}(G) > n$. In particular, C_4 and Graph G_1 are the only orderenergetic graphs with nullity 2.*



G_1



G_2

Proof. Let $\phi(G,x) = \sum_{i=0}^{n} c_i x^{n-i}$ be the characteristic polynomial of G. Since $\eta(G) = 2$, $c_{n-2} \neq 0$. Now, by Sachs Theorem, G contains a $\{1,2\}$-subgraph of order $n-2$, say H. Let $V(G) \setminus V(H) = \{u, v\}$. Note that if u and v are adjacent, then G has a $\{1,2\}$-factor. Hence by Lemma 3, $\mathcal{E}(G) \geq n$. However, if $\mathcal{E}(G) = n$, then G is a balanced complete bipartite graph. But the only balanced complete bipartite graph with nullity 2 is C_4, which contradicts the assumption. Therefore $\mathcal{E}(G) > n$. Hence in the following we may assume that u and v are not adjacent.

Case 1. Assume that H contains at least one induced odd cycle. Let C be a cycle-component of H. First, suppose that at least one of the u or v is adjacent to C. Then it is easy to see that G has a $\{1,2\}$-subgraph of order $n-1$. Now, by Corollary 11, $\mathcal{E}(G) > n$. (Note that since $\eta = 2$, G is not isomorphic to $K_{r,r+1}$, for some r.) Next assume that both u and v are adjacent to the same P_2 component of H, say xy. Let $K = \langle u, v, x, y \rangle$. Note that if K has a perfect matching, then G has a $\{1,2\}$-factor and a similar augment as above yields that $\mathcal{E}(G) > n$. So we may exclude subgraphs K with a perfect matching. It follows that $K = K_{1,3}$. Now, since $G \setminus (V(K) \cup V(C))$ has a perfect matching and $\mathcal{E}(K) = 3.46$, using Lemmas 1, 3 and 4 we have:

$$\mathcal{E}(G) \geq \mathcal{E}(K) + \mathcal{E}(C) + \mathcal{E}(G \setminus [V(K) \cup V(C)]) \geq$$

$$(|V(K)| - 0.54) + (|V(C)| + 1) + |V(G \setminus [V(K) \cup V(C)]| > n.$$

Finally, suppose that u and v are adjacent to different P_2-components of H, say wz and rs, respectively. Let $u \in N(w)$ and $v \in N(r)$. Let $N_1 = \langle u, w, z \rangle$ and $N_2 = \langle v, r, s \rangle$. We note that if N_1 or N_2 is C_3, then G has a $\{1,2\}$-subgraph of order $n-1$. Hence by Corollary 11, $\mathcal{E}(G) > n$. So we let $N_1 = N_2 = P_3$. We have $\mathcal{E}(P_3) = 2.82$. Now, by Lemmas 1, 3 and 4 we obtain:

$$\mathcal{E}(G) \geq \mathcal{E}(C) + \mathcal{E}(N_1) + \mathcal{E}(N_2) + \mathcal{E}(G \setminus [V(C) \cup V(N_1) \cup V(N_2)]) \geq$$

$$(|V(C)|+1) + (|V(N_1)| - 0.18) + (|V(N_2)| - 0.18) + |V(G \setminus [V(C) \cup V(N_1) \cup V(N_2)])| > n,$$

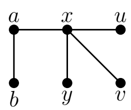
as desired.

Case 2. Assume that every component of H is P_2. Note that this is only possible if n is even. We consider three cases:

Subcase 2.1. If u and v are adjacent to different vertices of a P_2-component of H, then G has a $\{1,2\}$-factor. So $\mathcal{E}(G) > n$.

Subcase 2.2. If both u and v are adjacent to one vertex of a P_2-component of H, say xy, then let $K = \langle u, v, x, y \rangle$. We have $K = K_{1,3}$. Let x be the central vertex of $K = K_{1,3}$ and $L = G \setminus V(K)$. Clearly, L has a perfect matching, say M. Two cases can be considered:

(a) Assume that u, v and y are pendent vertices of G. First note that, by Lemma 6, $\eta(G-u-x) = 2$. Clearly, $G-u-x = 2K_1 \cup L$. Hence $\eta(L) = 0$. Now, since G is connected, there exists a P_2-component of M, say ab, such that $a \in N(x)$. Let $W = \langle a, b, u, v, x, y \rangle$. A computer search shows that if W is not Graph (a), then $\mathcal{E}(W) > 6$. Note that the nullity of Graph (a) is 2 and its energy is 5.81.

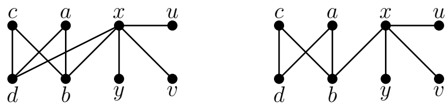


(a)

So if W is not Graph (a), then since $G \setminus V(W)$ has a perfect matching, we obtain

$$\mathcal{E}(G) \geq \mathcal{E}(W) + \mathcal{E}(G \setminus V(W)) > 6 + (n-6) > n.$$

If W is Graph (a), then there exists a P_2-component of M, say cd such that c or d is adjacent to some vertex of W different from u, v and y. Let $W' = \langle V(W), c, d\rangle$. We observed that except Graphs (b) and (c), $\mathcal{E}(W') > 8$. This implies that $\mathcal{E}(G) > n$.



If W' is one of the Graphs (b) or (c), then we note that C_4 is a subgraph of L. But since $\eta(L) = 0$, C_4 is not a component of L. Let N be the component of L which contains C_4. Clearly, N is not balanced complete bipartite, otherwise $\eta(L) \geq 2$. Moreover, N has a perfect matching, say M_1. Let M' be a proper subset of M_1, such that the subgraph $N' = \langle V(M')\rangle$ is the balanced complete bipartite subgraph of N of maximum order. Since C_4 is a subgraph of N, the order of N' is at least 4. Now, since N is connected, there exists a P_2-component of $M_1 \setminus M'$, say rs, such that N' is connected to rs. This means that there is a P_2-component of M', say ef such that the subgraph $\langle e, f, r, s\rangle$ is connected. Now, if $\langle e, f, r, s\rangle \neq K_{2,2}$, then let $gh \in M' \setminus \{ef\}$ be arbitrary and consider $N_1 = \langle e, f, g, h, r, s\rangle$ (Note that since the order of N' is at least 4, the edge gh exists). A computer search shows that $\mathcal{E}(N_1) \geq 6.63$. If $\langle e, f, r, s\rangle = K_{2,2}$, then let $gh \in M' \setminus \{ef\}$ such that $\langle g, h, r, s\rangle \neq K_{2,2}$. Note that gh exists, otherwise $\langle V(N'), r, s\rangle$ is a balanced complete bipartite subgraph of N whose order is greater than N', a contradiction. Now, consider the subgraph $N_2 = \langle e, f, g, h, r, s\rangle$. We observed that $\mathcal{E}(N_2) \geq 6.92$. Thus for $i = 1, 2$ we have

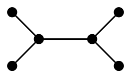
$$\mathcal{E}(G) \geq \mathcal{E}(K) + \mathcal{E}(N_i) + \mathcal{E}(G \setminus [V(K) \cup V(N_i)]).$$

Since $G \setminus [V(K) \cup V(N_i)]$ has a perfect matching, we obtain

$$\mathcal{E}(G) \geq 3.46 + 6.63 + |V(G \setminus [V(K) \cup V(N_i)])| > n.$$

(b) Suppose that at least one of the vertices u, v and y is not a pendent vertex of G, say u. We remind that the subgraph $K = \langle u, v, x, y \rangle$ is $K_{1,3}$ and $L = G \setminus V(K)$ has a perfect matching, M. Now, since $d(u) > 1$ and $v, y \notin N(u)$, there exists a P_2-component wz of M such that $w \in N(u)$. Let $W = \langle u, v, x, y, w, z \rangle$. Again we exclude subgraphs W with a perfect matching. Now, a computer search shows that except $W = K_{2,4}$, $\mathcal{E}(W) > 6$, which implies that $\mathcal{E}(G) > n$. So assume that $W = K_{2,4}$. Clearly, $G \setminus V(W)$ has a perfect matching. Now, Lemma 8, yields that $\mathcal{E}(G) > n$.

Subcase 2.3. Suppose that there exist P_2-components xy and wz of H, such that $x \in N(u)$ and $w \in N(v)$. Let $W = \langle u, v, x, y, w, z \rangle$. As we discussed in the previous subcase, we can exclude subgraphs W with a perfect matching. Now, a computer search shows that except $2P_3$, $K_{2,4}$ and the Graph (d), $\mathcal{E}(W) > 6$, which implies that $\mathcal{E}(G) > n$:



(d)

Note that Graph (d), is orderenergetic and has nullity 2. Assume that $n > 6$. We consider three cases:

(a) If W is Graph (d), then since G is connected, there exists a P_2-component of $H \setminus V(W)$, say rs, such that r or s is adjacent to some vertex of W. Now, consider the subgraph $\langle V(W), r, s \rangle$. A computer search shows that the energy of this subgraph is greater that 8. Consequently, $\mathcal{E}(G) > n$.

(b) If $W = K_{2,4}$, then by Lemma 8 we find that $\mathcal{E}(G) > n$.

(c) If $W = 2P_3$, then Lemma 10 implies that $\mathcal{E}(G) > n$. The proof is complete. ■

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