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On the Borderenergeticity of Line Graphs

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Abstract

A graph G of order n is *borderenergetic* if it has the same energy as the complete graph K_n. In this paper, we obtain the result that for any connected graph G, except for the five graphs (one of order 5, three of order 6 and one of order 10), the line graph $L(G)$ of G is not borderenergetic. As a consequence, we get that if G is a borderenergetic graph, then the line graph $L(G)$ of G is not borderenergetic. In addition, we observe a relation between the lower bound of the energy of the line graph $L(G)$ of a borderenergetic graph G and the minimum degree $\delta(G)$ of G.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let G be a graph with n $(= n(G))$ vertices and degree sequence $d_1, d_2, \cdots, d_n$. The *line graph* $L(G)$ of a graph G is defined as the graph whose vertex set is the set of edges of G, where two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G have a common

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vertex. Obviously, the line graph of a regular graph is also regular. The *tensor product* of two graphs G_1 and G_2 is the graph $G_1 \bigotimes G_2$ with vertex set $V(G_1) \times V(G_2)$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if both $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. For natation and terminology not defined here, we refer to [22].

The graph energy [11,12] of a graph G, denoted by $E(G)$, is defined as the sum of the absolute values of all eigenvalues of its adjacency matrix, and is defined by

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For more information on the graph energy and its applications in chemistry, we refer to [13–15, 18].

Gong et al. in [10] proposed the concept of *borderenergetic* graphs, i.e., graphs of order n satisfying $E(G) = 2(n-1)$. More results on the borderenergetic graphs can be seen in [3–7, 9, 17, 19–21]. In particular, Li and Deng et al. in [5, 6, 8] studied the properties of structures of borderenergetic graphs related to their girths and complements.

The energy of the line graph of a graph G and its relations with other graph energies were early studied in [2, 16]. In this paper, we shall study the borderenergetic property or borderenergeticity of the line graphs of connected graphs. As a result, we obtain that for any connected graph G, except for the five graphs G_1, G_2, G_3, G_4 and G_5 (see Figure 1), the line graph $L(G)$ of G is not borderenergetic. As a consequence, we get that if G is a borderenergetic graph, then the line graph of G is not borderenergetic. In addition, we observe a relation between the lower bound of the energy of line graph $L(G)$ of a borderenergetic graph G and the minimum degree $\delta(G)$ of G.

2 Preliminary

We need some preparations before proceeding to our main results.

Theorem 2.1. *[1] Let G be a graph with n vertices and m edges. Then*

$$\sum_{i=1}^{n} d_i^2 \geq \Delta^2 + \delta^2 + \frac{(2m - \Delta - \delta)^2}{n-2}.$$

Moreover, the equality holds if and only if $d_2 = d_3 = \cdots = d_{n-1}$.

If G is a connected graph with $n(\geq 2)$ vertices, $1 \leq \Delta \leq n-1$ and $1 \leq \delta \leq n-1$, then we get the following results.

Corollary 2.2. *Let G be a connected graph with n vertices and m edges. Then*

$$m \leq \frac{1}{2}\sqrt{(3n-6)\left(\sum_{i=1}^{n} d_i^2 - 2\right)} + n - 1.$$

Theorem 2.3. *[16] Let G be a graph with n vertices and m edges. Then*

$$\sqrt{2\sum_{i=1}^{n} d_i^2 - 4m} \leq E(L(G)) \leq \sum_{i=1}^{n} d_i^2 - 2m.$$

Equality on the left-hand side is attained if and only if the components of G are P_1 and/or P_2 and a single copy of either P_3 or P_4. Equality on the right-hand side is attained if and only if the components of G are P_1 and/or P_2 and/or P_3.

From Theorem 2.3, we can get the following conclusion.

Corollary 2.4. *If the line graph of a connected graph G of order n is borderenergetic. Then*

$$\sum_{i=1}^{n} d_i^2 \geq 4n - 6.$$

Proof. Let m be the number of edges of G. Because the line graph of G is borderenergetic, we have $E(L(G)) = 2(m-1)$. From the right-hand inequality of Theorem 2.3, we have

$$2(m-1) \leq \sum_{i=1}^{n} d_i^2 - 2m,$$

i.e.,

$$\sum_{i=1}^{n} d_i^2 \geq 4m - 2.$$

Since the graph G is connected, we have $m \geq n-1$. So, eventually we get $\sum_{i=1}^{n} d_i^2 \geq 4n-6$. ■

Theorem 2.5. *[16] Let G be an (n,m)-graph with vertex degrees d_i, $i = 1, 2, \cdots, n$. Then the line graph $L(G)$ of G has m vertices and q edges, where q is given by*

$$q = -m + \frac{1}{2}\sum_{i=1}^{n} d_i^2.$$

3 Main Result

By exhaust computer searching, we can conclude that, in all connected graphs with $1 \leq n \leq 10$ vertices, there are only five graphs, i.e., G_1, G_2, G_3, G_4 and G_5 (See Figure 1), whose line graphs are borderenergetic, and their adjacency spectra are as follows:

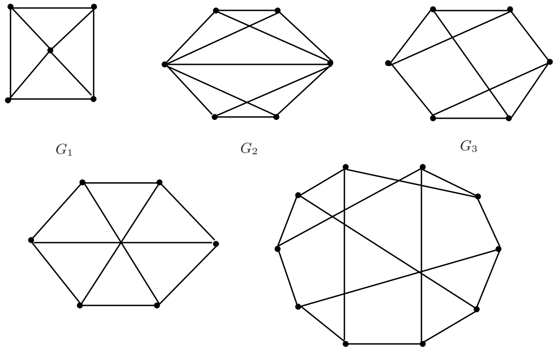

Figure 1. The connected graphs G_1, G_2, G_3, G_4 and G_5.

$$
\begin{aligned}
S_p(G_1) &= \{3.2361, 0^{(2)}, -1.2361, -2\};\\
S_p(G_2) &= \{3.8284, 1, -1^{(3)}, -1.8284\};\\
S_p(G_3) &= \{3, 1, 0^{(2)}, -2^{(2)}\};\\
S_p(G_4) &= \{3, 0^{(4)}, -3\};\\
S_p(G_5) &= \{3, 1^{(5)}, -2^{(4)}\};\\
S_p(L(G_1)) &= \{4.5616, 1^{(2)}, 0.4384, -1, -2^{(3)}\};\\
S_p(L(G_2)) &= \{6, 2^{(2)}, 0^{(3)}, -2^{(5)}\};\\
S_pL(G_3)) &= \{4, 2, 1^{(2)}, -1^{(2)}, -2^{(3)}\};\\
S_p(L(G_4)) &= \{4, 1^{(4)}, -2^{(4)}\};\\
S_p(L(G_5)) &= \{4, 2^{(5)}, -1^{(4)}, -2^{(5)}\}.
\end{aligned}
$$

Based on this, with some theoretical proof we obtain the following result.

Theorem 3.1. *For any connected graph G, except for the five graphs G_1, G_2, G_3, G_4 and G_5, the line graph $L(G)$ of G is not borderenergetic.*

Proof. For the case $1 \le n \le 10$, there are only five graphs, i.e., G_1, G_2, G_3, G_4 and G_5, whose line graphs are borderenergetic. Now we consider connected graphs with orders $n \ge 11$. Let G be a connected graph with $n \ge 11$ vertices and m edges. By contradiction, suppose the line graph $L(G)$ of G is borderenergetic. Then $E(L(G)) = 2(m-1)$. Since $E(G) \ge 2\sqrt{m}$, we have

$$E(L(G)) \ge 2\sqrt{q},$$

where q is the number of edges of the line graph $L(G)$ and $q = \frac{1}{2}\sum_{i=1}^{n} d_i^2 - m$. So,

$$E(L(G)) \ge 2\sqrt{q} = 2\sqrt{\frac{1}{2}\sum_{i=1}^{n} d_i^2 - m}.$$

Moreover, we have

$$2(m-1) \ge 2\sqrt{\frac{1}{2}\sum_{i=1}^{n} d_i^2 - m},$$

i.e.,

$$(m-1)^2 \ge \frac{1}{2}\sum_{i=1}^{n} d_i^2 - m.$$

By the above inequality, we obtain

$$m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2 + 1 \ge 0.$$

Assume $\sum_{i=1}^{n} d_i^2 = x$. Then we have

$$m^2 - m - \frac{1}{2}x + 1 \ge 0. \tag{1}$$

Since G is a connected graph, we have $m \ge n-1$. Combining Corollary 2.2 and (1), we get

$$\left[\frac{1}{2}\sqrt{(3n-6)(x-2)} + n - 1\right]^2 - n + 1 - \frac{1}{2}x + 1 \ge 0.$$

So,

$$\frac{(3n-8)^2}{16}x^2 - \frac{6n^3 - 5n^2 - 48n + 72}{4}x + n^4 - 3n^3 + \frac{33}{4}n^2 - 24n + 24 \le 0.$$

The left-hand expression of the above inequality can be seen as a function $f(x)$ on variable x. Then the above inequality can be written as

$$f(x) \le 0.$$

By solving the inequality, we get that the above inequality holds if $x_1 \leq x \leq x_2$, where

$$x_1 = \frac{2(6n^3 - 5n^2 - 48n + 72 - \sqrt{240n^5 - 1680n^4 + 4560n^3 - 6000n^2 + 3840n - 960})}{(3n-8)^2},$$

$$x_2 = \frac{2(6n^3 - 5n^2 - 48n + 72 + \sqrt{240n^5 - 1680n^4 + 4560n^3 - 6000n^2 + 3840n - 960})}{(3n-8)^2}.$$

From Corollary 2.4, we know that $x \geq 4n - 6$ when the line graph $L(G)$ of G is a borderenergetic graph. But we can check that when $n \geq 11$, it holds that

$$\begin{aligned} x_2 - (4n-6) &= \frac{2(6n^3 - 5n^2 - 48n + 72 + \sqrt{240n^5 - 1680n^4 + 4560n^3 - 6000n^2 + 3840n - 960})}{(3n-8)^2} \\ &- (4n-6) \\ &= \frac{-24n^3 + 236n^2 - 640n + 528}{(3n-8)^2} \\ &+ \frac{2\sqrt{240n^5 - 1680n^4 + 4560n^3 - 6000n^2 + 3840n - 960}}{(3n-8)^2} \\ &< 0. \end{aligned}$$

That is, $x_2 < 4n - 6$, which is a contradiction to $4n - 6 \leq x \leq x_2$. The result thus follows. ■

It is easy to check that the above five exceptional graphs G_1 through G_5 in Figure 1 are not borderenergetic graphs. So we get the following consequence.

Corollary 3.2. *If G is a connected borderenergetic graph, then the line graph $L(G)$ of G is not borderenergetic.*

4 A low bound for the energy of the line graphs of borderenergetic graphs

In this section, we are concerned about the lower bound of the graph energy of the line graph of a borderenergetic graph. Under the condition of minimum degree δ, the energy of the line graph $L(G)$ of a graph G has the following property.

Theorem 4.1. *[1] Let G be a graph of order $n > 2$ with m edges and minimum degree δ. If $\delta \geq \frac{n}{2} + 1$, then the energy of $L(G)$ is $4(m - n)$. Thus, the line graphs of all (n, m)-graphs under the condition $\delta \geq \frac{n}{2} + 1$ are equienergetic.*

Now we consider the energy of the line graph $L(G)$ of a graph G under the condition $\delta \leq \frac{n}{2}$. Indeed, for any graph G, there exits a relation between the eigenvalues of the line graph $L(G)$ and the signless Laplacian eigenvalues of G; see below.

Theorem 4.2. *[1] Let G be a graph of order n with $m \geq 1$ edges. Let q_i be the i-th greatest signless Laplacian eigenvalue of G and $\lambda_i(L(G))$ be the i-th greatest eigenvalue of the line graph $L(G)$ of G. Then*

$$q_i(G) = \lambda_i(L(G)) + 2, \ i = 1, 2, \cdots, k, \quad where\ k = min\{n, m\}.$$

In addition, if $m > n$, then $\lambda_i(L(G)) = -2$ for $i \geq n+1$ and if $n > m$, then $q_i(G) = 0$ for $i \geq m+1$.

Using Theorem 4.2, we can get the following result.

Theorem 4.3. *Let G be a non-complete borderenergetic graph with n vertices and m edges such that the minimum degree $\delta(G) \leq \frac{n}{2}$. Then*

$$E(L(G)) \geq 4(m - n).$$

Proof. Since $n\delta \leq \frac{n^2}{2}$ and $n\delta \leq 2m \leq n(n-1)$, we have

$$n\delta \leq min\{\frac{n^2}{2}, 2m\} \leq n(n-1).$$

Thus, there are the following two cases to be discussed. Note that $n \geq 7$ when G is a borderenergetic graph.

Case 1. $min\{\frac{n^2}{2}, 2m\} = \frac{n^2}{2}$.

Then we have $n\delta \leq \frac{n^2}{2} \leq 2m \leq n(n-1)$, i.e., $m \geq \frac{n^2}{4} > n$. By Theorem 4.2, we can obtain

$$\sum_{i=n+1}^{m} \lambda_i(L(G)) = -2(m-n),$$

$$\sum_{i=n+1}^{m} |\lambda_i(L(G))| = 2(m-n).$$

Due to $\sum_{i=1}^{m} \lambda_i(L(G)) = 0$, we get

$$\sum_{i=1}^{n} \lambda_i(L(G)) = \sum_{i=1}^{m} \lambda_i(L(G)) - \sum_{i=n+1}^{m} \lambda_i(L(G)) = 0 + 2(m-n) = 2(m-n).$$

So, we obtain

$$\begin{aligned} E(L(G)) &= \sum_{i=1}^{m} |\lambda_i(L(G))| \\ &= \sum_{i=1}^{n} |\lambda_i(L(G))| + \sum_{i=n+1}^{m} |\lambda_i(L(G))| \end{aligned}$$

$$\begin{aligned} &\geq \sum_{i=1}^{n} \lambda_i(L(G)) + 2(m-n) \\ &= 2(m-n) + 2(m-n) \\ &= 4(m-n). \end{aligned}$$

Case 2. $min\{\frac{n^2}{2}, 2m\} = 2m$.

Then we have $n\delta \leq 2m \leq \frac{n^2}{2} \leq n(n-1)$, i.e., $m \leq \frac{n^2}{4}$. If $m > n$, then it follows from Case 1 that $E(L(G)) \geq 4(m-n)$. For the case $m = n$, it is obvious that $E(L(G)) \geq 0$. As $m < n$, we have

$$E(L(G)) = \sum_{i=1}^{m} |\lambda_i(L(G))| \geq 4(m-n).$$

■

Next we find three graphs H_1, $H_2 = (K_3 \bigotimes K_4) \bigcup K_2$ and H_3, which are borderenergetic graphs with minimum degrees of 2, 1 and 0, respectively. The graphs H_1 and H_3 are shown in Figure 2.

H	δ	n	$S_p(L(H_i))$	$E(L(H_i))$	$4(m-n)$
H_1	2	10	$\{-2^{(22)}, -0.3567, 2, 2.5858, 3.3406, 4, 5^{(2)}, 5.0484, 5.4142, 11.9677\}$	88.7134	88
H_2	1	14	$\{-2^{(24)}, 0, 1^{(2)}, 2^{(3)}, 5^{(6)}, 10\}$	96	92
H_3	0	11	$\{-2^{(20)}, 2^{(5)}, 5^{(4)}, 10\}$	80	76

Table 2. The spectra and graph energy of the line graphs of graphs H_1, H_2 and H_3.

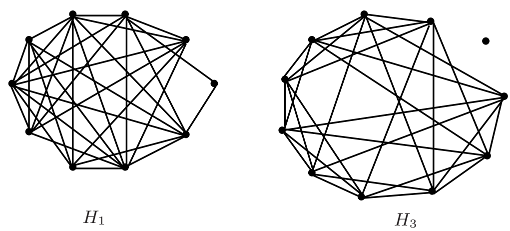

Figure 2. The borderenergetic graphs H_1 and H_3.

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