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Energy of Extended Bipartite Double Graphs

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Abstract

The energy of a graph is the sum of the absolute values of its eigenvalues. In this article, an exact relation between the energy of extended bipartite double graph and the energy of a graph together with some other graph parameters is given. As a consequence, equienergetic, borderenergetic, orderenergetic and non-hyperenergetic extended bipartite double graphs are presented. The obtained results generalize the existing results on equienergetic bipartite graphs.

1 Introduction

All graphs in this article are simple, finite and undirected. The order and the size of a graph G is the number of vertices and the number of edges in it. Let d_i denotes the degree of a vertex v_i of a graph G. The *eigenvalues* of a graph G are the eigenvalues of its adjacency matrix $A(G)$. The eigenvalues of a graph G of order n are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The number of positive, negative and zero eigenvalues of G are denoted by n^+, n^- and n^0 respectively. The *energy* [8] of a graph G is defined as $\mathcal{E}(G) = \sum_{j=1}^{n} |\lambda_j|$. An equivalent expression to the energy of a graph G is as follows:

$$\mathcal{E}(G) = \sum_{j=1}^{n} |\lambda_j| = 2\sum_{j=1}^{n^+} \lambda_j = -2\sum_{j=1}^{n^-} \lambda_{n-j+1}. \tag{1}$$

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This quantity is well studied and has applications in chemistry [9,12,13]. Two graphs of same order with equal energies are called *equienergetic* graphs. There are numerous papers which deals with equienergetic graphs available in the literature, see [2,14–18] and references therein. But very few papers deals with equienergetic bipartite graphs [11]. This motivates us to study the equienergetic bipartite graphs. A graph of order n is said to be *orderenergetic* if its energy is equal to its order n [1]. A graph of order n is said to be *non-hyperenergetic* if $\mathcal{E}(G) \leq 2(n-1)$ [19] and is called *borderenergetic* if $\mathcal{E}(G) = 2(n-1)$ [7]. The *line graph* $\mathcal{L}(G)$ of a graph G is the graph with vertex set same as the edge set of G and two vertices in $\mathcal{L}(G)$ are adjacent if the corresponding edges in G have a vertex in common. The *k-th iterated line graph* of G for $k = 0, 1, 2, \ldots$ is defined as $\mathcal{L}^k(G) \equiv \mathcal{L}(\mathcal{L}^{k-1}(G))$, where $\mathcal{L}^0(G) \equiv G$ and $\mathcal{L}^1(G) \equiv \mathcal{L}(G)$ [10]. As usual the graphs K_n and $K_{p,q}$ denote the complete graph of order n and the complete bipartite graph of order $p+q$ respectively. For other terminology and results related to the spectra of graphs, we follow [5].

Definition. [4] Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *extended bipartite double* graph $Ebd(G)$ of a graph G is the bipartite graph with its partite sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ in which two vertices x_i and y_j are adjacent if $i = j$ or v_i and v_j are adjacent in G.

The $Ebd(G)$ is also called as the *extended double cover* [3]. The extended bipartite double graph $Ebd(G)$ is one of the best tool to generate a bipartite graph from given any graph G. If G is a graph of order n then $Ebd(G)$ is of order $2n$ and, if G is an r-regular graph then $Ebd(G)$ is an $(r+1)$-regular graph. The energy of $Ebd(G)$ is studied in [6,11].

Theorem 1.1. [4] *Let G be a graph of order n with eigenvalues λ_i, $1 \leq i \leq n$. Then the eigenvalues of extended bipartite double graph $Ebd(G)$ are $\pm(\lambda_i + 1)$, $1 \leq i \leq n$.* ■

Theorem 1.2. [15] *Let G be a graph of order n_0 and size m_0 with $d_i + d_j \geq 6$ to each edge $e = v_i v_j$ in G then the iterated line graphs $\mathcal{L}^k(G)$ have all the negative eigenvalues equal to -2 with the multiplicity $m_{k-1} - n_{k-1}$ for $k \geq 2$. And all the iterated line graphs $\mathcal{L}^k(G)$ of such graphs G are mutually equienergetic with energy $4(n_k - n_{k-1})$ for $k \geq 2$, where n_k and m_k are the order and the size of $\mathcal{L}^k(G)$ respectively.* ■

2 Energy of extended bipartite double graphs

In the following, an exact relation between the energy of $Ebd(G)$ and the energy of G together with other graph parameters is given.

Theorem 2.1. *Let G be a graph of order n with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\mathcal{E}(Ebd(G)) = 2\Big(n + \mathcal{E}(G) - 2n^- + 2 \sum_{\lambda_j \in (-1,0)} (\lambda_j + 1)\Big).$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ be the eigenvalues of G and $Ebd(G)$ respectively. By definition of energy of a graph, we have

$$\begin{aligned}
\mathcal{E}(Ebd(G)) &= \sum_{j=1}^{n} |\lambda_j^*| = \sum_{j=1}^{n} |\lambda_j + 1| + \sum_{j=1}^{n} |-\lambda_j - 1| \qquad \text{by Theorem 1.1} \\
&= 2\sum_{j=1}^{n} |-1-\lambda_j| = 2\Big(\sum_{\lambda_j \leq -1} (-1-\lambda_j) + \sum_{\lambda_j > -1} (1+\lambda_j)\Big) \\
&= 2\Big(-n_\lambda([\lambda_n, -1]) + \sum_{\lambda_j \leq -1} |\lambda_j| + n_\lambda((-1, \lambda_1]) + \sum_{\lambda_j \in (-1,0)} \lambda_j + \sum_{\lambda_j \geq 0} \lambda_j\Big),
\end{aligned}$$

where $n_\lambda(\mathbf{I})$ denotes the number of eigenvalues of G which lies in the interval $\mathbf{I}$ and $n_\lambda([\lambda_n, -1]) = 0$ if $\lambda_n \geq -1$.

The energy of a graph G can be expressed as

$$\mathcal{E}(G) = \sum_{j=1}^{n} |\lambda_j| = \sum_{\lambda_j \leq -1} |\lambda_j| + \sum_{\lambda_j \in (-1,0)} |\lambda_j| + \sum_{\lambda_j \geq 0} \lambda_j.$$

With this we have,

$$\begin{aligned}
\mathcal{E}(Ebd(G)) &= -2n_\lambda([\lambda_n, -1]) + 2n_\lambda((-1, \lambda_1]) + 2\sum_{\lambda_j \in (-1,0)} \lambda_j + 2\Big(\mathcal{E}(G) - \sum_{\lambda_j \in (-1,0)} |\lambda_j|\Big) \\
&= 2\mathcal{E}(G) - 2n_\lambda([\lambda_n, -1]) + 2\Big(n - n_\lambda([\lambda_n, -1])\Big) + 4\sum_{\lambda_j \in (-1,0)} \lambda_j \\
&= 2\mathcal{E}(G) + 2n - 4n_\lambda([\lambda_n, -1]) + 4\sum_{\lambda_j \in (-1,0)} \lambda_j. \qquad (2)
\end{aligned}$$

The total number of eigenvalues n of a graph G can be expressed as

$$n = n_\lambda([\lambda_n, -1]) + n_\lambda((-1, 0)) + n^0 + n^+.$$

Therefore

$$n_\lambda([\lambda_n, -1]) = n - n^+ - n^0 - n_\lambda(-1, 0) = n^- - n_\lambda((-1, 0)). \qquad (3)$$

Also, we have

$$\sum_{\lambda_j \in (-1,0)} (\lambda_j + 1) = \sum_{\lambda_j \in (-1,0)} \lambda_j + n_\lambda((-1,0)). \tag{4}$$

Using (3) and (4) in (2), we get

$$\mathcal{E}(Ebd(G)) = 2\Big(n + \mathcal{E}(G) - 2n^- + 2 \sum_{\lambda_j \in (-1,0)} (\lambda_j + 1)\Big),$$

which completes the proof. ■

It is easy to observe that $n^- > \sum_{\lambda_j \in (-1,0)} (\lambda_j + 1) > 0$ for any graph G. With this fact we have the following.

Corollary 2.2. *Let G be a graph of order n. Then*

$$\mathcal{E}(Ebd(G)) < 2(n + \mathcal{E}(G)).$$

■

There are many graphs with no eigenvalues in the interval $(-1, 0)$, such as integral graphs, all iterated line graphs $\mathcal{L}^k(G)$ for $k \geq 2$ in Theorem 1.2 etc. If a graph has no eigenvalue in the interval $(-1, 0)$ then we have the following.

Corollary 2.3. *Let G be a graph of order n. Then G has no eigenvalue in the interval $(-1, 0)$ if and only if*

$$\mathcal{E}(Ebd(G)) = 2n + 2\mathcal{E}(G) - 4n^-.$$

Proof. Proof follows directly from the fact that $\sum_{\lambda \in (-1,0)} (\lambda + 1) = 0$ if and only if G has no eigenvalue λ in the interval $(-1, 0)$. ■

Now it is easy to construct equienergetic bipartite graphs by using Theorem 2.1 with the help of equienergetic graphs with no eigenvalues in the interval $(-1, 0)$ and having the same number of negative eigenvalues.

Corollary 2.4. *Let G_1 and G_2 be two equienergetic graphs of same order n with the eigenvalues $\lambda_1' \geq \lambda_2' \geq \cdots \geq \lambda_n'$ and $\lambda_1'' \geq \lambda_2'' \geq \cdots \geq \lambda_n''$, and the number of negative eigenvalues n_1^- and n_2^- respectively. Then $Ebd(G_1)$ and $Ebd(G_2)$ are equienergetic if and only if $n_1^- - \sum_{\lambda_j' \in (-1,0)} (\lambda_j' + 1) = n_2^- - \sum_{\lambda_j'' \in (-1,0)} (\lambda_j'' + 1)$.*
In particular, if G_1 and G_2 have no eigenvalues in the interval $(-1, 0)$, then $Ebd(G_1)$ and $Ebd(G_2)$ are equienergetic if and only if $n_1^- = n_2^-$.

Proof. Proof follows directly from Theorem 2.1 by taking two equienergetic graphs of same order and from the fact that $\sum_{\lambda\in(-1,0)}(\lambda+1)=0$ if and only if G has no eigenvalue λ in the interval $(-1,0)$. ■

Example. The graphs $\mathcal{L}^k(K_{n,n}\square K_{n-1})$ and $\mathcal{L}^k(K_{n-1,n-1}\square K_n)$ are integral equienergetic graphs with the same number of negative eigenvalues for all $n\geq 5, k\geq 0$ [17], where $\square$ denotes the Cartesian product. Therefore, by Corollary 2.4, $Ebd(\mathcal{L}^k(K_{n,n}\square K_{n-1}))$ and $Ebd(\mathcal{L}^k(K_{n-1,n-1}\square K_n))$ are equienergetic for all $n\geq 5, k\geq 0$.

Remark. Hou and Xu in [11] studied the spectra and the energy of $Ebd\big(\mathcal{L}^2(G)\big)$, where G is an r-regular graph of degree $r\geq 3$, and constructed a large pairs of non-trivial equienergetic bipartite regular graphs $Ebd\big(\mathcal{L}^k(G)\big)$ for $k\geq 2$. The iterated line graphs $\mathcal{L}^k(G)$ for $k\geq 2$ in [11] are the part of iterated line graphs in Theorem 1.2. Ramane et al. in [15] by using Theorem 1.2 constructed a large class of non-trivial equienergetic bipartite graphs $Ebd\big(\mathcal{L}^k(G)\big)$ for $k\geq 2$. It is noted that all these results become particular case of Corollary 2.4.

Corollary 2.5. *Let G be a graph of order n. If $\mathcal{E}(G)\leq n-1$, then $Ebd(G)$ is non-hyperenergetic graph.*

Proof. By Corollary 2.2 and the condition that $\mathcal{E}(G)\leq n-1$, we have

$$\mathcal{E}(Ebd(G))<2n+2\mathcal{E}(G)\leq 2n+2(n-1)=2(2n-1),$$

which shows that the graph $Ebd(G)$ is non-hyperenergetic. ■

In the following we give a necessary and sufficient condition for a graph $Ebd(G)$ to be borderenergetic.

Corollary 2.6. *Let G be a graph of order n. Then $Ebd(G)$ is borderenergetic if and only if $\mathcal{E}(G)=n-1+2\Big(n^- - \sum_{\lambda_j\in(-1,0)}(\lambda_j+1)\Big)$. In particular, if G has no eigenvalue in the interval $(-1,0)$, then $Ebd(G)$ is borderenergetic if and only if $\mathcal{E}(G)=n-1+2n^-$.*

Proof. If G is a graph of order n then the order of $Ebd(G)$ is $2n$. If $Ebd(G)$ borderenergetic then $\mathcal{E}(Ebd(G))=2(2n-1)$. Now proof follows directly from Theorem 2.1. ■

Corollary 2.7. *Let G be a graph of order n. Then $Ebd(G)$ is orderenergetic if and only if $\mathcal{E}(G)=2\Big(n^- - \sum_{\lambda_j\in(-1,0)}(\lambda_j+1)\Big)$. In particular, if G has no eigenvalue in the interval $(-1,0)$, then $Ebd(G)$ is orderenergetic if and only if G is union of complete graphs or* *empty graph.*

Proof. If G is a graph of order n then the order of $Ebd(G)$ is $2n$. Now by the definition of orderenergetic graph and the Theorem 2.1, $Ebd(G)$ is orderenergetic if and only if $\mathcal{E}(G) = 2\Big(n^- - \sum\limits_{\lambda_j\in(-1,0)} (\lambda_j+1)\Big)$. If G has no eigenvalue in the interval $(-1,0)$, then $Ebd(G)$ is orderenergetic if and only if $\mathcal{E}(G) = 2n^-$. But $\mathcal{E}(G) = 2n^-$ implies $-2\sum\limits_{j=1}^{n^-}\lambda_{n-j+1} = 2n^-$ by (1), which gives $-\sum\limits_{j=1}^{n^-}\lambda_{n-j+1} = n^-$. This equality $-\sum\limits_{j=1}^{n^-}\lambda_{n-j+1} = n^-$ holds true only if G is union of complete graphs or empty graph. ■

The *k-th iterated extended bipartite double* graph of G for $k = 0,1,2,\ldots$ is defined as $Ebd^k(G) \equiv Ebd(Ebd^{k-1}(G))$, where $Ebd^0(G) \equiv G$ and $Ebd^1(G) \equiv Ebd(G)$. In the following, an exact relation between the energy of $Ebd^2(G)$ and the energy of G together with other graph parameters is given.

Theorem 2.8. *Let G be a graph of order n with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$. Then*

$$\mathcal{E}(Ebd^2(G)) = 4\Big(n + \mathcal{E}(G) - 2n^- + \sum_{\lambda_j\in(-2,0)} (\lambda_j+2)\Big).$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of G then by Theorem 1.1 the eigenvalues of $Ebd^2(G)$ are $\pm(\lambda_j+2)$ and $\pm\lambda_j$, $1\leq j\leq n$. Now proof follows similar to that of proof of the Theorem 2.1. ■

It is easy to observe that to each negative eigenvalue $\lambda_j \in (-2,0)$ we have $0 < \lambda_j+2 < 2$, which gives $2n^- - \sum\limits_{\lambda_j\in(-2,0)}(\lambda_j+2) > 0$ for any graph G. Now with this fact we have the following.

Corollary 2.9. *Let G be a graph of order n. Then*

$$\mathcal{E}(Ebd^2(G)) < 4(n + \mathcal{E}(G)).$$

■

Theorem 1.2 ensures that there are many graphs with no eigenvalue in the interval $(-2,0)$. If a graph has no eigenvalue in the interval $(-2,0)$, then we have the following.

Corollary 2.10. *Let G be a graph of order n. If G has no eigenvalue in the interval $(-2,0)$, then*

$$\mathcal{E}(Ebd^2(G)) = 4(n + \mathcal{E}(G) - 2n^-) = 2\mathcal{E}(Ebd(G)).$$

Proof. Proof follows directly from Theorem 2.8 and Theorem 2.1. ■

Again it is easy to construct equienergetic bipartite graphs by using Theorem 2.8 with the help of equienergetic graphs with no eigenvalues in the interval $(-2, 0)$ and having the same number of negative eigenvalues.

Corollary 2.11. *Let G_1 and G_2 be two equienergetic graphs of same order n with the eigenvalues $\lambda_1^{'} \geq \lambda_2^{'} \geq \cdots \geq \lambda_n^{'}$ and $\lambda_1^{''} \geq \lambda_2^{''} \geq \cdots \geq \lambda_n^{''}$, and the number of negative eigenvalues n_1^- and n_2^- respectively. Then $Ebd^2(G_1)$ and $Ebd^2(G_2)$ are equienergetic if and only if $2n_1^- - \sum\limits_{\lambda_j^{'} \in (-2,0)} (\lambda_j^{'} + 2) = 2n_2^- - \sum\limits_{\lambda_j^{''} \in (-2,0)} (\lambda_j^{''} + 2)$.*
In particular, if G_1 and G_2 have no eigenvalues in the interval $(-2, 0)$, then $Ebd^2(G_1)$ and $Ebd^2(G_2)$ are equienergetic if and only if $n_1^- = n_2^-$.

Proof. Proof directly follows from Theorem 2.8 by taking two equienergetic graphs of same order. ■

In the following we present another large class of equienergetic bipartite graphs.

Proposition 2.12. *Let G be a graph of order n_0 and size m_0 with $d_i + d_j \geq 6$ to each edge $e = v_i v_j$ in G. Then the graphs $Ebd^2(\mathcal{L}^k(G))$ of such graphs G are mutually equienergetic for $k \geq 2$.*

Proof. If G is a graph of order n_0 and size m_0 with $d_i + d_j \geq 6$ to each edge $e = v_i v_j$ then by Theorem 1.2, the iterated line graphs $\mathcal{L}^k(G)$ of such graphs G have all the negative eigenvalues equal to -2 with the multiplicity $m_{k-1} - n_{k-1}$ for $k \geq 2$ and are mutually equienergetic. Thus proof follows by the Corollary 2.11. ■

3 Conclusion

In this article, the energy of extended bipartite double graphs is studied. It is characterized to construct a class of equienergetic bipartite graphs. Also the energy of second iterated extended bipartite double graphs is obtained. It is worthwhile to investigate the following problem.

- Find the energy relations between a graph G and its iterated extended bipartite double graphs and characterize various energy types.

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