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Weakly Discriminating Vertex–Degree–Based Topological Indices

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Abstract

Let $\mathcal{G}_n$ be the set of graphs with n vertices and $\mathcal{H} \subseteq \mathcal{G}_n$. For each $H \in \mathcal{H}$, let $m(H) = \{m_{i,j}(H)\}$, where $m_{i,j}(H)$ is the number of edges in H that join a vertex of degree i with a vertex of degree j. A vertex-degree-based (VDB, for short) topological index φ is discriminating over $\mathcal{H}$ if non-isomorphic graphs in $\mathcal{H}$ have different values of φ. We say that φ is *weakly discriminating* over $\mathcal{H}$ if the following weaker condition is satisfied for every $H, H' \in \mathcal{H}$:

$$\varphi(H) = \varphi(H') \Longleftrightarrow m(H) = m(H').$$

Let $\mathcal{CT}_n$ be the set of chemical trees with n vertices. In this paper we show that many of the well-known VDB topological indices are *not* weakly discriminating over $\mathcal{CT}_n$. However, the recently introduced Sombor index is weakly discriminating over $\mathcal{CT}_n$. Also, we give conditions under which a VDB topological index φ is weakly discriminating over an arbitrary class $\mathcal{H} \subseteq \mathcal{G}_n$.

1 Introduction

Throughout this paper a graph $G = (V, E)$ is simple, undirected and unweighted, where V is the set of vertices and E the set of edges. If there is an edge from vertex u to vertex v we indicate this by writing uv (or vu). The degree of the vertex v of G is denoted by d_v. A vertex v is isolated if $d_v = 0$. We denote by $n_i = n_i(G)$ the number of vertices of G with degree i and $m_{i,j} = m_{i,j}(G)$ the number of edges in G joining vertices of degree i and j.

Topological indices are molecular descriptors which play an important role in theoretical chemistry, especially in QSPR/QSAR research [22, 23]. One important class of topological indices are the so-called vertex-degree based (VDB, for short) topological indices (see [10, 18], and for recent results [1, 3, 5–7, 15–17, 24]).

Let $\mathcal{G}_n$ be the set of graphs with n non-isolated vertices and $\mathcal{H} \subseteq \mathcal{G}_n$. Let $\Delta = \max\{\Delta(H) : H \in \mathcal{H}\}$. Consider the set

$$K = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq j \leq \Delta\},$$

lexicographically ordered. Let $h = |K| = \frac{\Delta(\Delta+1)}{2}$. Define the function $m : \mathcal{H} \longrightarrow \mathbb{R}^h$ by $m(H) = (m_{i,j}(H))_{(i,j)\in K}$, where $H \in \mathcal{H}$. Formally, a VDB topological index is a function $T_\varphi : \mathcal{H} \longrightarrow \mathbb{R}$ induced by a vector $\varphi \in \mathbb{R}^h$, defined for $H \in \mathcal{H}$ as

$$T_\varphi(H) = m(H) \cdot \varphi, \tag{1}$$

where "$\cdot$" denotes the dot product over $\mathbb{R}^h$. For brevity, we simply write

$$\varphi(H) = m(H) \cdot \varphi, \tag{2}$$

and say that φ is a VDB topological index defined over $\mathcal{H}$. Most of the well-known VDB topological indices studied in mathematical chemistry are induced by symmetric functions of the form $\varphi(x, y)$. In other words, the vector $\varphi \in \mathbb{R}^h$ has components $(\varphi(x, y))_{(x,y)\in K}$, where $\varphi(x, y)$ is a symmetric function.

From expression (2), we easily deduce that if $H, H' \in \mathcal{H}$, then [19]

$$\varphi(H) = \varphi(H') \Longleftrightarrow [m(H) - m(H')] \cdot \varphi = 0 \Longleftrightarrow [m(H) - m(H')] \perp \varphi.$$

For the well-known VDB topological indices $\varphi \in \mathbb{R}^h$, the perpendicularity condition above occurs very frequently in most of the important classes of sets $\mathcal{H}$ studied (connected graphs, trees, chemical graphs, etc ...). In other words, non-isomorphic graphs of $\mathcal{H}$ have equal value φ. This degeneracy property has been investigated extensively in mathematical chemistry [8, 9, 13].

In [19], the concept of discrimination of a VDB topological index was generalized to equivalence relations defined in $\mathcal{H}$. Specifically, if $\sim$ is an equivalence relation defined in $\mathcal{H}$, then φ discriminates on $\mathcal{H}$ with respect to $\sim$ if the following condition holds: for all $H, H' \in \mathcal{H}$,

$$\varphi(H) = \varphi(H') \Longleftrightarrow H \sim H'.$$

In this case we say that φ is discriminating over $\mathcal{H}$ with respect to the equivalence relation $\sim$. When we consider the isomorphism relation $\simeq$ on $\mathcal{H}$, then we recover the usual concept of discrimination. But there are other interesting equivalence relations, for example, in [19, Example 2.2] we consider the equivalence relation defined on $\mathcal{H}$,

$$H \sim H' \Longleftrightarrow m(H) = m(H'). \tag{3}$$

If a vertex-degree-based topological index φ is discriminating over $\mathcal{H}$ with respect to the relation (3), then we simply say that φ is *weakly discriminating* over $\mathcal{H}$. Since

$$H \simeq H' \Longrightarrow m(H) = m(H'),$$

it follows that if φ is discriminating over $\mathcal{H}$ (in the usual way), then φ is weakly discriminating over $\mathcal{H}$.

Naturally a question arises: which VDB topological indices are weakly discriminating over $\mathcal{H}$? In Section 2 we study the case $\mathcal{H} = \mathcal{CT}_n$, the set of chemical trees with n vertices. Specifically, we show in Examples 2.1 and 2.2 several well-known VDB topological indices which are *not* weakly discriminating over $\mathcal{CT}_n$: the Randić index [20], the first Zagreb index [12], the sum-connectivity index [25], the harmonic index [26], first hyper-Zagreb index [21], and the reciprocal sum-connectivity index [15]. On the other hand, we show in Theorem 2.3, that the recently introduced Sombor index [11] is weakly discriminating over $\mathcal{CT}_n$.

In Section 3 we give conditions under which a VDB topological index φ is weakly discriminating over an arbitrary class $\mathcal{H} \subseteq \mathcal{G}_n$. Concretely, we show that if the components $\{\varphi_{i,j}\}_{(i,j)\in K}$ of φ are linearly independent over the set of integers $\mathbb{Z}$, then φ is weakly discriminating over $\mathcal{H}$. As a byproduct, we show that if the components $\{\varphi_{i,j}\}_{(i,j)\in K}$ are distinct algebraic numbers, then the exponential e^{φ} is weakly discriminating over $\mathcal{H}$.

2 Weakly discriminating VDB topological indices over $\mathcal{CT}_n$

Which VDB topological indices are weakly discriminating over $\mathcal{H}$? Let us see some examples when $\mathcal{H} = \mathcal{CT}_n$, the set of chemical trees with n vertices. In this case

$$K = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq j \leq 4\},$$

and since $m_{1,1}(T) = 0$ for all $T \in \mathcal{CT}_n$ when $n \geq 3$, all VDB topological indices φ are vectors in $\mathbb{R}^9$:

$$\varphi = (\varphi_{1,2}, \varphi_{1,3}, \varphi_{1,4}, \varphi_{2,2}, \varphi_{2,3}, \varphi_{2,4}, \varphi_{3,3}, \varphi_{3,4}, \varphi_{4,4}).$$

Example 2.1 *The Randić index is induced by the symmetric function* $\varphi(x,y) = \frac{1}{\sqrt{xy}}$, *so its associated vector is*

$$\varphi = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{8}}, \frac{1}{3}, \frac{1}{\sqrt{12}}, \frac{1}{4}\right).$$

It turns out that the Randić index is not weakly discriminating over $\mathcal{CT}_n$. *In fact, consider the trees* S *and* T *depicted in Figure 1. Let* p *be a positive integer. In the tree* S *there are* $4p+1$ *vertices of degree* 4. *In the tree* T *there are* $3p+1$ *vertices of degree* 4, *there is a path with* $p \geq 1$ *edges between vertices* u *and* v, *and there is a path with* p *edges between vertices* w *and* z. *By a counting argument, one easily sees that* S *and* T *are trees with* $20p+9$ *vertices. Moreover,*

$$m_{1,2}(S) = 8p+4, \quad m_{2,4}(S) = 8p+4, \quad m_{4,4}(S) = 4p,$$

and $m_{i,j}(S) = 0$ *for all other values of* i,j. *On the other hand,*

$$m_{1,2}(T) = 6p+4, \quad m_{2,2}(T) = 2p, \quad m_{2,4}(T) = 12p+4,$$

and $m_{i,j}(T) = 0$ *for all other values of* i,j. *In particular,* $m(S) \neq m(T)$. *However,*

$$\varphi(S) = \varphi(T) = \left(6\sqrt{2}+1\right)p + 3\sqrt{2}.$$

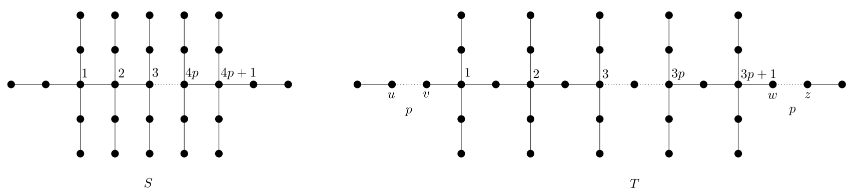

Figure 1. Trees with equal Randić index but different m-structure.

Example 2.2 *Consider the following VDB topological indices induced by the function* $\varphi(x,y)$*:*

$\varphi(x,y)$	*Name*
$x+y$	*first Zagreb index*
$(x+y)^2$	*first hyper-Zagreb index*
$\frac{1}{\sqrt{x+y}}$	*sum-connectivity index*
$\sqrt{x+y}$	*reciprocal sum-connectivity index*
$\frac{2}{x+y}$	*harmonic index*

All these VDB topological indices have the following property in common:

$$\varphi(1,3)=\varphi(2,2)\,;\quad \varphi(1,4)=\varphi(2,3)\,;\quad \varphi(2,4)=\varphi(3,3)\,. \tag{4}$$

Hence, for every $U\in\mathcal{CT}_n$*,*

$$\begin{aligned}\varphi(U) &= \varphi(1,2)\,m_{1,2}(U)+\varphi(1,3)\,(m_{1,3}(U)+m_{2,2}(U))+\\ &\quad \varphi(1,4)\,(m_{1,4}(U)+m_{2,3}(U))+\varphi(2,4)\,(m_{2,4}(U)+m_{3,3}(U))+\\ &\quad \varphi(3,4)\,m_{3,4}(U)+\varphi(4,4)\,m_{4,4}(U)\,.\end{aligned}$$

So a condition that implies that two trees $S,T\in\mathcal{CT}_n$ *have equal value of* φ *is the following:*

$$\begin{array}{c} m_{1,3}(S)+m_{2,2}(S)=m_{1,3}(T)+m_{2,2}(T)\,;\\ m_{1,4}(S)+m_{2,3}(S)=m_{1,4}(T)+m_{2,3}(T)\,;\\ m_{2,4}(S)+m_{3,3}(S)=m_{2,4}(T)+m_{3,3}(T)\,;\\ m_{i,j}(S)=m_{i,j}(T)\ \textit{for the other values of}\ (i,j)\in K. \end{array} \tag{5}$$

Consider the trees $S,T\in\mathcal{CT}_n$ *shown in Figure 2. There is a path with* $p\geq 1$ *edges between vertices* u *and* v *in the tree* S*, and there is a path with* $p+6$ *edges between vertices* w *and* z *in the tree* T*. Note that*

$$\begin{array}{l} m_{1,2}(S)=2;\quad m_{1,3}(S)=6;\quad m_{1,4}(S)=0;\quad m_{2,2}(S)=p;\quad m_{2,3}(S)=4;\\ m_{2,4}(S)=0;\quad m_{3,3}(S)=4;\quad m_{3,4}(S)=0;\quad m_{4,4}(S)=0, \end{array}$$

and

$$\begin{array}{l} m_{1,2}(T)=2;\quad m_{1,3}(T)=0;\quad m_{1,4}(T)=4;\quad m_{2,2}(T)=p+6;\quad m_{2,3}(T)=0;\\ m_{2,4}(T)=4;\quad m_{3,3}(T)=0;\quad m_{3,4}(T)=0;\quad m_{4,4}(T)=0. \end{array}$$

Clearly, conditions (5) hold so $\varphi(S)=\varphi(T)$*. However,* $m(S)\neq m(T)$*, which implies that none of these VDB topological indices are weakly discriminating over* $\mathcal{CT}_n$*.*

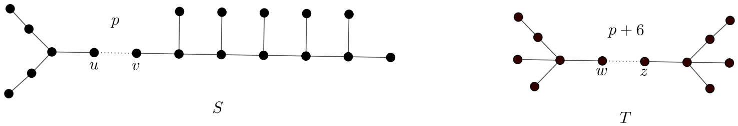

Figure 2. Trees with equal φ but different m-structure.

Hence, many of the well known VDB topological indices are *not* weakly discriminating over $\mathcal{CT}_n$, as we have seen in the previous examples. However, the Sombor index is weakly discriminating over $\mathcal{CT}_n$, as we shall see in our next result. Recall that the Sombor index is induced by the symmetric function $SO(x, y) = \sqrt{x^2 + y^2}$, so the SO vector is given by

$$SO = \left(\sqrt{5}, \sqrt{10}, \sqrt{17}, 2\sqrt{2}, \sqrt{13}, 2\sqrt{5}, 3\sqrt{2}, 5, 4\sqrt{2}\right). \tag{6}$$

If T is a chemical tree with n vertices, then

$$\begin{aligned} SO(T) &= \sqrt{5}\,(m_{1,2} + 2m_{2,4}) + \sqrt{2}\,(2m_{2,2} + 3m_{3,3} + 4m_{4,4}) \\ &\quad + \sqrt{10}m_{1,3} + \sqrt{17}m_{1,4} + \sqrt{13}m_{2,3} + 5m_{3,4}. \end{aligned} \tag{7}$$

We will also use the following well-known relations:

$$\begin{aligned} 2m_{1,1} + m_{1,2} + m_{1,3} + m_{1,4} &= n_1 \\ m_{1,2} + 2m_{2,2} + m_{2,3} + m_{2,4} &= 2n_2 \\ m_{1,3} + m_{2,3} + 2m_{3,3} + m_{3,4} &= 3n_3 \\ m_{1,4} + m_{2,4} + m_{3,4} + 2m_{4,4} &= 4n_4 \end{aligned} \tag{8}$$

Theorem 2.3 *The Sombor index SO is weakly discriminating over $\mathcal{CT}_n$.*

Proof. Let $T \in \mathcal{CT}_n$ and set $m_{i,j} = m_{i,j}(T)$ and $n_i = n_i(T)$. By (8),

$$\begin{aligned} 12n &= 12\,(n_1 + n_2 + n_3 + n_4) \\ &= 18p + 6q - 10m_{3,3} - 18m_{4,4} - 27m_{2,4} \\ &\quad + 16m_{1,3} + 15m_{1,4} + 10m_{2,3} + 7m_{3,4}, \end{aligned} \tag{9}$$

where

$$p = m_{1,2} + 2m_{2,4} \tag{10}$$

and

$$q = 2m_{2,2} + 3m_{3,3} + 4m_{4,4}. \tag{11}$$

Assume now that $T' \in \mathcal{CT}_n$ is such that $SO(T) = SO(T')$. Set $m'_{i,j} = m_{i,j}(T')$ and $n'_i = n_i(T')$. Since

$$\left\{5, \sqrt{2}, \sqrt{5}, \sqrt{10}, \sqrt{13}, \sqrt{17}\right\}$$

is a linearly independent set over $\mathbb{Z}$, it follows by (7) that

$$p = m'_{1,2} + 2m'_{2,4} \tag{12}$$

$$q = 2m'_{2,2} + 3m'_{3,3} + 4m'_{4,4} \tag{13}$$

and

$$\begin{array}{ll} m_{1,3} = m'_{1,3}, & m_{1,4} = m'_{1,4} \\ m_{2,3} = m'_{2,3}, & m_{3,4} = m'_{3,4}. \end{array} \tag{14}$$

Again by (8), (12), (13), and (14),

$$\begin{aligned} 12n &= 12\left(n'_1 + n'_2 + n'_3 + n'_4\right) \\ &= 18p + 6q - 10m'_{3,3} - 18m'_{4,4} - 27m'_{2,4} \\ &\quad +16m_{1,3} + 15m_{1,4} + 10m_{2,3} + 7m_{3,4}. \end{aligned} \tag{15}$$

Now from relations (9) and (15) we conclude that

$$10\left(m_{3,3} - m'_{3,3}\right) + 18\left(m_{4,4} - m'_{4,4}\right) + 27\left(m_{2,4} - m'_{2,4}\right) = 0. \tag{16}$$

The general solution of the Diophantine equation

$$10x + 18y + 27z = 0$$

is

$$\left.\begin{array}{c} x = -54 + 54k \\ y = 27 - 27k \\ z = 2 - 2k \end{array}\right\}, \tag{17}$$

where $k \in \mathbb{Z}$. Hence, from (16) we deduce

$$m_{3,3} - m'_{3,3} = -54 + 54k, \tag{18}$$

$$m_{4,4} - m'_{4,4} = 27 - 27k, \tag{19}$$

$$m_{2,4} - m'_{2,4} = 2 - 2k, \tag{20}$$

for some $k \in \mathbb{Z}$. On the other hand, from (11), (13), and (17),

$$\begin{aligned} 2\left(m_{22} - m'_{22}\right) &= 3\left(m'_{3,3} - m_{3,3}\right) + 4\left(m'_{4,4} - m_{4,4}\right) \\ &= 3\left(54 - 54k\right) + 4\left(-27 + 27k\right). \end{aligned}$$

Hence,

$$m_{2,2} - m'_{2,2} = 27 - 27k. \tag{21}$$

Now by (10), (12), and (17),

$$m_{1,2} - m'_{1,2} = 2\left(m'_{2,4} - m_{2,4}\right) = 2\left(2k - 2\right).$$

Consequently,

$$m_{1,2} - m'_{1,2} = 4k - 4. \tag{22}$$

From (14),

$$m_{1,3} - m'_{1,3} = 0 \tag{23}$$

$$m_{1,4} - m'_{1,4} = 0 \tag{24}$$

$$m_{2,3} - m'_{2,3} = 0 \tag{25}$$

$$m_{3,4} - m'_{3,4} = 0. \tag{26}$$

Adding the left sides and right sides of relations (18)-(26), we obtain

$$\begin{aligned} \sum_{1 \le i \le j \le 4} m_{i,j} - \sum_{1 \le i \le j \le 4} m'_{i,j} &= (-54 + 54k) + (27 - 27k) + (2 - 2k) \\ + (27 - 27k) + (4k - 4) &= 2k - 2. \end{aligned} \tag{27}$$

Since

$$n - 1 = \sum_{1 \le i \le j \le 4} m_{i,j} = \sum_{1 \le i \le j \le 4} m'_{i,j},$$

it follows from (27) that $k = 1$. Thus, from relations (18)-(22),

$$\begin{aligned} &m_{3,3} = m'_{3,3} \quad m_{4,4} = m'_{4,4} \quad m_{2,4} = m'_{2,4} \\ &m_{2,2} = m'_{2,2} \quad m_{1,2} = m'_{1,2}. \end{aligned} \tag{28}$$

Finally, from (14) and (28),

$$m\left(T\right) = m\left(T'\right).$$

■

3 Conditions which assure the weakly discrimination property on an arbitrary class

In our next result we give conditions on a VDB topological index which assures the weakly discrimination property over an arbitrary class $\mathcal{H} \subseteq \mathcal{G}_n$.

Theorem 3.1 *Let $\varphi \in \mathbb{R}^h$ be a VDB topological index defined over $\mathcal{H} \subseteq \mathcal{G}_n$ such that $\{\varphi_{i,j}\}_{(i,j)\in K}$ is a linearly independent set over $\mathbb{Z}$, the set of integers. Then φ is weakly discriminating over $\mathcal{H}$.*

Proof. Let $H, H' \in \mathcal{H}$ and assume that $\varphi(H) = \varphi(H')$. Then

$$0 = (m(H) - m(H')) \cdot \varphi = \sum_{(i,j)\in K} (m_{i,j}(H) - m_{i,j}(H'))\, \varphi_{i,j}.$$

Since $m_{i,j}(H) - m_{i,j}(H') \in \mathbb{Z}$ for all $(i,j) \in K$ and $\{\varphi_{i,j}\}_{(i,j)\in K}$ is a linearly independent set over $\mathbb{Z}$, $m_{i,j}(H) - m_{i,j}(H') = 0$ for all $(i,j) \in K$. Consequently, $m(H) = m(H')$. ■

Remark 3.2 *The converse of Theorem 3.1 does not hold. For instance, the Sombor index is weakly discriminating over $\mathcal{CT}_n$. However, the components $SO_{2,2}, SO_{3,3}$, and $SO_{4,4}$ are $2\sqrt{2}, 3\sqrt{2}$, and $4\sqrt{2}$, respectively. Note that*

$$3SO_{2,2} + 2SO_{3,3} + (-3)\, SO_{4,4} = 0,$$

so the components of SO are not linearly independent over $\mathbb{Z}$.

Besicovitch's Theorem can be used to construct many examples of VDB topological indices that are weakly discriminating over $\mathcal{H} \subseteq \mathcal{G}_n$.

Theorem 3.3 *[4] (Besicovitch) Let $n > 1$ be any integer and $p_1, \ldots, p_k$ distinct positive prime numbers. The set of n^k radicals,*

$$\left\{ \sqrt[n]{p_1^{m(1)} p_2^{m(2)} \cdots p_k^{m(k)}} : 0 \le m(i) < n,\ 1 \le i \le k \right\},$$

is linearly independent over $\mathbb{Z}$.

Example 3.4 *Consider the sum-connectivity Gourava index $\varphi \in \mathbb{R}^9$ induced by the symmetric function $\varphi(x,y) = \frac{1}{\sqrt{x+y+xy}}$ [14]. Then φ is weakly discriminating over $\mathcal{CT}_n$. In fact,*

$$\varphi = \left(\frac{\sqrt{5}}{5}, \frac{\sqrt{7}}{7}, \frac{1}{3}, \frac{\sqrt{2}}{4}, \frac{\sqrt{11}}{11}, \frac{\sqrt{2\times 7}}{14}, \frac{\sqrt{3\times 5}}{15}, \frac{\sqrt{19}}{19}, \frac{\sqrt{2\times 3}}{12} \right).$$

It follows easily from Besicovitch's Theorem that the set of components of φ is a linearly independent set over $\mathbb{Z}$, and by Theorem 3.1, the sum-connectivity Gourava index is weakly discriminating over $\mathcal{CT}_n$.

Based on Besicovitch's Theorem we can introduce weakly discriminating topological indices over arbitrary classes $\mathcal{H} \subseteq \mathcal{G}_n$.

Definition 3.5 *The **prime topological index** is defined as follows: let $\mathcal{H} \subseteq \mathcal{G}_n$, $\Delta = \max\{\Delta(H) : H \in \mathcal{H}\}$ and $h = \frac{\Delta(\Delta+1)}{2}$. Consider the sequence of the first h prime numbers $\{p_{i,j}\}_{(i,j)\in K}$. Then we define the VDB topological index $\Pi = \left\{\sqrt{p_{i,j}}\right\}_{(i,j)\in K} \in \mathbb{R}^h$.*

Clearly, by Besicovitch's Theorem and Theorem 3.1, Π is weakly discriminating over $\mathcal{H}$.

Example 3.6 *If $\mathcal{H}$ is the set of chemical trees, then $\Pi \in \mathbb{R}^9$ is the vector*

$$\Pi = \left(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}, \sqrt{17}, \sqrt{19}, \sqrt{23}\right).$$

and for a chemical tree U,

$$\begin{aligned}\Pi(U) &= \sqrt{2}m_{1,2} + \sqrt{3}m_{1,3} + \sqrt{5}m_{1,4} + \sqrt{7}m_{2,2} + \\ &\quad \sqrt{11}m_{2,3} + \sqrt{13}m_{2,4} + \sqrt{17}m_{3,3} + \sqrt{19}m_{3,4} + \sqrt{23}m_{4,4}.\end{aligned}$$

We next show how to construct weakly discriminating VDB topological indices φ over $\mathcal{H} \subseteq \mathcal{G}_n$, when all the components of the vector φ are different. First we recall a classical result from algebra. Let $\mathcal{A}$ be the field of algebraic numbers.

Theorem 3.7 *[2] (Lindemann-Weierstrass's Theorem) If $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\mathcal{A}$.*

In [19], the exponential of a VDB topological index $\varphi \in \mathbb{R}^h$ over $\mathcal{H} \subseteq \mathcal{G}_n$ was defined for $H \in \mathcal{H}$ as

$$e^{\varphi}(H) = m(H) \cdot e^{\varphi},$$

where $e^{\varphi} \in \mathbb{R}^h$ has components $\{e^{\varphi_{i,j}}\}_{(i,j)\in K}$.

Theorem 3.8 *Let $\varphi \in \mathbb{R}^h$ be a VDB topological index defined over $\mathcal{H} \subseteq \mathcal{G}_n$ such that all components of φ are distinct algebraic numbers. Then e^{φ} is weakly discriminating over $\mathcal{H} \subseteq \mathcal{G}_n$.*

Proof. Since $\{\varphi_{i,j}\}_{(i,j)\in K}$ are distinct algebraic numbers, it follows from Theorem 3.7 that $\{e^{\varphi_{i,j}}\}_{(i,j)\in K}$ are linearly independent over $\mathcal{A}$. But $\mathbb{Z} \subseteq \mathcal{A}$. Hence $\{e^{\varphi_{i,j}}\}_{(i,j)\in K}$ are linearly independent over $\mathbb{Z}$, and the result follows from Theorem 3.1. ■

Example 3.9 *The components of the Sombor index over $\mathcal{CT}_n$ are distinct algebraic numbers:*

$$SO = \left(\sqrt{5}, \sqrt{10}, \sqrt{17}, 2\sqrt{2}, \sqrt{13}, 2\sqrt{5}, 3\sqrt{2}, 5, 4\sqrt{2}\right).$$

It follows from Theorem 3.8 that e^{SO} is weakly discriminating over $\mathcal{CT}_n$.

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