

On the Variable Sum Exdeg Index/Coindex of Graphs

Akbar Ali^{a,1}, Emina Milovanović^b, Marjan Matejić^b,
Igor Milovanović^b

^aDepartment of Mathematics, Faculty of Science,
University of Ha'il, Ha'il, Saudi Arabia
akbarali.maths@gmail.com

^bFaculty of Electronic Engineering,
University of Niš, Niš, Serbia
ema@elfak.ni.ac.rs, marjan.matejic@elfak.ni.ac.rs,
igor.milovanovic@elfak.ni.ac.rs

(Received November 16, 2020)

Abstract

Let G be a connected graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$, where $n \geq 2$. Denote by d_i the degree of the vertex v_i for $i = 1, 2, \dots, n$. If v_i and v_j are adjacent in G , we write $i \sim j$, otherwise we write $i \not\sim j$. The variable sum exdeg index and coindex of G are defined as $SEI_a(G) = \sum_{i \sim j} (a^{d_i} + a^{d_j}) = \sum_{i=1}^n d_i a^{d_i}$ and $\overline{SEI}_a(G) = \sum_{i \not\sim j} (a^{d_i} + a^{d_j}) = \sum_{i=1}^n (n-1-d_i) a^{d_i}$, respectively, where 'a' is a positive real number different from 1. Some inequalities involving $SEI_a(G)$ or/and $\overline{SEI}_a(G)$ are derived. Special cases of the obtained inequalities are also discussed for unicyclic graphs.

1 Introduction

All the graphs discussed in the present paper are simple and connected. Let G be a graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$, edge set E and with the vertex-degree sequence (d_1, d_2, \dots, d_n) satisfying $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta > 0$, where $n \geq 2$, $|E| = m$ and d_i is the degree of the vertex v_i for $i = 1, 2, \dots, n$. If v_i and v_j are adjacent in G , we write $i \sim j$, otherwise we write $i \not\sim j$. Denote by \overline{G} the complement of G and let

¹Corresponding author

$D = \text{diag}(d_1, d_2, \dots, d_n)$ be the diagonal degree matrix of G . The cyclomatic number of G is the minimum number of edges whose removal from G gives a graph without containing any cycle. Graphs with the cyclomatic number 1, 2, 3 and 4 are usually called unicyclic, bicyclic, tricyclic and tetracyclic graphs, respectively.

The variable sum exdeg index of G is denoted by $SEI_a(G)$ and is defined [19] as

$$SEI_a(G) = \sum_{i=1}^n d_i a^{d_i} = \sum_{i \sim j} (a^{d_i} + a^{d_j}),$$

where ‘ a ’ is an arbitrary positive real number different from 1. The main motivation of introducing and studying the variable sum exdeg index SEI_a was its very good chemical applicability – see [19] for detail. Because of its chemical applications, mathematical properties of SEI_a have been studied in a considerable number of papers. Vukičević [20] initiated the mathematical study of SEI_a and he reported several extremal results concerning SEI_a . For $a > 1$, Ghalavand and Ashrafi [8] determined the graphs having extremum values of SEI_a from the classes of all trees and unicyclic graphs of a fixed order. The graphs with the maximum SEI_a value (for $a > 1$) in the classes of all bicyclic and tricyclic graphs of a fixed order were also characterized in [8]. Ali and Dimitrov [1] reported alternative proofs of some of the main results given in [8], and also they proved an extremal result regarding the variable sum exdeg index of tetracyclic graphs. For $a > 1$, the results of [1,8] were generalized by Dimitrov and Ali [5] by considering the graphs of a fixed order and cyclomatic number; in [5], the case $0 < a < 1$ was also discussed but only the partial solutions to the considered problems for this certain case were obtained. The problem of finding graphs having the extremum values of the variable sum exdeg index of the trees of a fixed order and with the vertices having prescribed degrees was attacked in [11]. Additional recent results about the variable sum exdeg index can be found in the papers [3, 7, 9, 13, 18].

Motivated from the concept of the Zagreb coindices [6], we define the variable sum exdeg coindex [13] as

$$\overline{SEI}_a(G) = \sum_{i \not\sim j} (a^{d_i} + a^{d_j}),$$

which can be rewritten as

$$\overline{SEI}_a(G) = \sum_{i=1}^n (n-1-d_i) a^{d_i}.$$

The main purpose of this paper is to derive some inequalities for $SEI_a(G)$ or/and $\overline{SEI}_a(G)$. Special cases of the obtained inequalities are also considered when G is a unicyclic graph.

2 Preliminaries

In this section, we recall some analytical inequalities for real number sequences that will be used later in the paper.

Let $a = (a_i)$, $i = 1, 2, \dots, n$, be a sequence of positive real numbers. In [12], the following inequality was proven

$$\left(\sum_{i=1}^n \sqrt{a_i} \right)^2 \leq (n-1) \sum_{i=1}^n a_i + n \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}}. \quad (1)$$

Equality in (1) holds if and only if $a_1 = a_2 = \dots = a_n$.

If $a = (a_i)$, $i = 1, 2, \dots, n$, is a sequence of positive real numbers with the property $a_1 \geq a_2 \geq \dots \geq a_n$ then it holds [4] that

$$\sum_{i=1}^n a_i \geq n \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}} + (\sqrt{a_1} - \sqrt{a_n})^2, \quad (2)$$

with equality if and only if $a_2 = \dots = a_{n-1} = \sqrt{a_1 a_n}$.

Let $p = (p_i)$, $i = 1, 2, \dots, n$, be a sequence of non-negative real numbers, and let $a = (a_i)$, $i = 1, 2, \dots, n$, be a sequence of positive real numbers. Then, for any real r with $r \leq 0$ or $r \geq 1$, it holds [10] (see also [15]) that

$$\left(\sum_{i=1}^n p_i \right)^{r-1} \sum_{i=1}^n p_i a_i^r \geq \left(\sum_{i=1}^n p_i a_i \right)^r. \quad (3)$$

When $0 \leq r \leq 1$, the opposite inequality in (3) is valid. Equality sign in (3) holds if and only if $r = 0$, or $r = 1$, or $a_1 = a_2 = \dots = a_n$, or $p_1 = p_2 = \dots = p_t = 0$ and $a_{t+1} = \dots = a_n$ for some t satisfying $1 \leq t \leq n-1$.

Let $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \dots, n$, be two real number sequences with the properties $0 < r_1 \leq a_i \leq R_1$ and $0 < r_2 \leq b_i \leq R_2$. In [2] (see also [16]), the following inequality was proven

$$\left| n \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i \sum_{i=1}^n b_i \right| \leq (R_1 - r_1)(R_2 - r_2)n \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right). \quad (4)$$

Equality in (4) holds if and only if $a_1 = a_2 = \dots = a_n$ or $b_1 = b_2 = \dots = b_n$. Since

$$\frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) = \begin{cases} \frac{1}{4} & \text{if } n \text{ is even,} \\ \frac{(n-1)(n+1)}{4n^2} & \text{if } n \text{ is odd,} \end{cases}$$

we have

$$\alpha(n) = \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) = \frac{1}{4} \left(1 - \frac{(-1)^{n+1} + 1}{2n^2} \right). \quad (5)$$

Therefore, (4) can be rewritten as

$$\left| n \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i \sum_{i=1}^n b_i \right| \leq n^2 \alpha(n) (R_1 - r_1)(R_2 - r_2). \quad (6)$$

Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, \dots, n$, be positive real number sequences such that $a = (a_i)$ is monotone and $0 < r \leq a_i \leq R < +\infty$. Let S be a subset of $I_n = \{1, 2, \dots, n\}$ which minimizes the expression

$$\left| \sum_{i \in S} p_i - \frac{1}{2} \sum_{i=1}^n p_i \right|. \quad (7)$$

In [14], it was proven that

$$\sum_{i=1}^n p_i a_i \sum_{i=1}^n \frac{p_i}{a_i} \leq \left(1 + \gamma(S) \frac{(R-r)^2}{rR} \right) \left(\sum_{i=1}^n p_i \right)^2, \quad (8)$$

where

$$\gamma(S) = \frac{\sum_{i \in S} p_i}{\sum_{i=1}^n p_i} \left(1 - \frac{\sum_{i \in S} p_i}{\sum_{i=1}^n p_i} \right). \quad (9)$$

Equality in (8) is attained if $R = a_1 = \dots = a_n = r$.

3 Main results

In this section, by making use of the inequalities specified in Section 2 we derive inequalities involving $SEI_a(G)$ or/and $\overline{SEI}_a(G)$. Firstly, we establish a lower bound and an upper bound on $SEI_a(G)$ in terms of the parameters n , m , Δ , δ , $\det D$, and a for $a > 1$.

Theorem 1. *If G is a connected graph of order $n \geq 2$, size m , minimum degree δ , maximum degree Δ , and if the determinant of its diagonal degree matrix is $\det D$ then for any $a > 1$ it holds that*

$$\left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta} \right)^2 \leq SEI_a(G) - na^{\frac{2m}{n}} (\det D)^{\frac{1}{n}} \leq n^2 \alpha(n) \left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta} \right)^2, \quad (10)$$

with equalities if and only if G is a regular graph, where $\alpha(n)$ is defined via (5).

Proof. Let (d_1, d_2, \dots, d_n) be the vertex-degree sequence of G such that $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta > 0$. For $a_i = b_i = \sqrt{d_i a^{d_i}}$, $i = 1, 2, \dots, n$, $R_1 = R_2 = \sqrt{\Delta a^\Delta}$, $r_1 = r_2 = \sqrt{\delta a^\delta}$, the inequality (6) transforms into

$$0 \leq n \sum_{i=1}^n d_i a^{d_i} - \left(\sum_{i=1}^n \sqrt{d_i a^{d_i}} \right)^2 \leq n^2 \alpha(n) \left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta} \right)^2,$$

that is

$$nSEI_a(G) \leq \left(\sum_{i=1}^n \sqrt{d_i a^{d_i}} \right)^2 + n^2 \alpha(n) \left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta} \right)^2. \quad (11)$$

On the other hand, for $a_i = d_i a^{d_i}$, $i = 1, 2, \dots, n$, the inequality (1) becomes

$$\left(\sum_{i=1}^n \sqrt{d_i a^{d_i}} \right)^2 \leq (n-1) \sum_{i=1}^n d_i a^{d_i} + n \left(\prod_{i=1}^n d_i a^{d_i} \right)^{\frac{1}{n}},$$

i.e.,

$$\left(\sum_{i=1}^n \sqrt{d_i a^{d_i}} \right)^2 \leq (n-1)SEI_a(G) + na^{\frac{2m}{n}} (\det D)^{\frac{1}{n}}. \quad (12)$$

From (11) and (12) we arrive at the right-hand side of (10).

Equalities in (11) and (12) hold if and only if $d_1 a^{d_1} = \dots = d_n a^{d_n}$, which implies that equality in the right-hand side of (10) holds if and only if G is regular.

Now, setting $a_i = d_i a^{d_i}$, $i = 1, 2, \dots, n$, in (2), we get

$$\sum_{i=1}^n d_i a^{d_i} \geq n \left(\prod_{i=1}^n d_i a^{d_i} \right)^{\frac{1}{n}} + \left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta} \right)^2,$$

i.e.

$$SEI_a(G) \geq na^{\frac{2m}{n}} (\det D)^{\frac{1}{n}} + \left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta} \right)^2, \quad (13)$$

from which we obtain the left-hand side of (10).

Equality in (13) holds if and only if $d_2 a^{d_2} = \dots = d_{n-1} a^{d_{n-1}} = \sqrt{\Delta \delta a^{\Delta+\delta}}$, that is, if and only if $\sqrt{\Delta \delta} = \frac{\Delta+\delta}{2}$, $\Delta = \delta$. This implies that equality in the left-hand side of (10) holds if and only if G is regular. \blacksquare

As $\left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta} \right)^2 \geq 0$ and $\alpha(n) \leq \frac{1}{4}$, the next two results follow from Theorem 1.

Corollary 1. *If G is a connected graph having the parameters mentioned in the statement of Theorem 1 then for any $a > 1$ it holds that*

$$na^{\frac{2m}{n}} (\det D)^{\frac{1}{n}} \leq SEI_a(G) \leq na^{\frac{2m}{n}} (\det D)^{\frac{1}{n}} + \frac{n^2}{4} \left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta} \right)^2,$$

with equality if and only if G is regular.

Corollary 2. *If U is a connected unicyclic graph of order $n \geq 3$, minimum degree δ , maximum degree Δ , and if the determinant of its diagonal degree matrix is $\det D$ then for any $a > 1$ it holds that*

$$\left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta}\right)^2 \leq SEI_a(U) - na^2 (\det D)^{\frac{1}{n}} \leq n^2 \alpha(n) \left(\sqrt{\Delta a^\Delta} - \sqrt{\delta a^\delta}\right)^2.$$

with equalities if and only if $U \cong C_n$, where $\alpha(n)$ is defined via (5).

Next, we prove an inequality involving $SEI_a(G)$ and $SEI_a(G)^{-1}$.

Theorem 2. *If G is a connected graph of size $m \geq 1$, minimum degree δ , maximum degree Δ , and if $a > 0$ with $a \neq 1$, then it holds that*

$$SEI_a(G) + 4m^2 a^{\Delta+\delta} SEI_a(G)^{-1} \leq 2m (a^\Delta + a^\delta), \quad (14)$$

with equality if and only if G is regular.

Proof. Let (d_1, d_2, \dots, d_n) be the vertex-degree sequence of G such that $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta > 0$. Clearly, for every $i \in \{1, 2, \dots, n\}$, it holds that $(a^\Delta - a^{d_i})(a^\delta - a^{d_i}) \leq 0$, which gives $a^{2d_i} + a^{\Delta+\delta} \leq (a^\Delta + a^\delta) a^{d_i}$, and hence we have

$$a^{d_i} + \frac{a^{\Delta+\delta}}{a^{d_i}} \leq a^\Delta + a^\delta. \quad (15)$$

After multiplying (15) with d_i and then summing over i for $i = 1, 2, \dots, n$, we get

$$SEI_a(G) + a^{\Delta+\delta} \sum_{i=1}^n d_i a^{-d_i} \leq 2m(a^\Delta + a^\delta). \quad (16)$$

On the other hand, for $r = -1$, $p_i = d_i$, $a_i = a^{d_i}$, with $i = 1, 2, \dots, n$, (3) gives

$$\left(\sum_{i=1}^n d_i\right)^{-2} \sum_{i=1}^n d_i a^{-d_i} \geq \left(\sum_{i=1}^n d_i a^{d_i}\right)^{-1},$$

i.e.,

$$\sum_{i=1}^n d_i a^{-d_i} \geq \frac{4m^2}{SEI_a(G)}. \quad (17)$$

From (16) and (17), we arrive at (14).

Equality in (15) is attained if and only if $a^{d_i} \in \{a^\delta, a^\Delta\}$ for every $i = 1, 2, \dots, n$. Equality in (17) holds if and only if $a^\Delta = a^{d_1} = \dots = a^{d_n} = a^\delta$, i.e. $\Delta = d_1 = \dots = d_n = \delta$.

Therefore we conclude that equality in (14) holds if and only if G is regular. ■

Corollary 3. *If U is a connected unicyclic graph of order $n \geq 3$, minimum degree δ , maximum degree Δ , and if $a > 0$ with $a \neq 1$, then it holds that*

$$SEI_a(U) + 4n^2 a^{\Delta+\delta} SEI_a(U)^{-1} \leq 2n (a^\Delta + a^\delta),$$

with equality if and only if $U \cong C_n$.

Let χ and $\bar{\chi}$ be the chromatic numbers of G and \bar{G} , respectively. Nordhaus and Gaddum [17] reported bounds on $\chi \cdot \bar{\chi}$ and $\chi + \bar{\chi}$. Motivated by these results of Nordhaus and Gaddum [17], next we prove an inequality involving $SEI_a(G)$ and $SEI_a(\bar{G})$.

Theorem 3. *If G is a connected graph of order $n \geq 2$, size m , minimum degree δ , maximum degree Δ , and if $a > 0$ with $a \neq 1$, then it holds that*

$$a^{\Delta+\delta-n+1} SEI_a(\bar{G}) - SEI_a(G) \leq (a^\Delta + a^\delta) (n(n-1) - 2m) - n(n-1)a^{\frac{2m}{n}}, \quad (18)$$

with equality if and only if G is regular, where \bar{G} is the complement of G .

Proof. Let (d_1, d_2, \dots, d_n) be the vertex-degree sequence of G such that $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta > 0$. After multiplying the inequality (15) with $n-1-d_i$ and summing over i for $i = 1, 2, \dots, n$, we get

$$\sum_{i=1}^n (n-1-d_i)a^{d_i} + a^{\Delta+\delta} \sum_{i=1}^n (n-1-d_i)a^{-d_i} \leq (a^\Delta + a^\delta) \sum_{i=1}^n (n-1-d_i),$$

which gives

$$(n-1) \sum_{i=1}^n a^{d_i} - SEI_a(G) + a^{\Delta+\delta-n+1} SEI_a(\bar{G}) \leq (a^\Delta + a^\delta)(n(n-1) - 2m). \quad (19)$$

Now, by arithmetic-geometric mean inequality, AM-GM (see e.g. [15]), we have

$$\sum_{i=1}^n a^{d_i} \geq n \left(\prod_{i=1}^n a^{d_i} \right)^{\frac{1}{n}} = na^{\frac{2m}{n}}. \quad (20)$$

From (19) and (20), we arrive at (18).

Equality in (15) is attained if and only if $a^{d_i} \in \{a^\delta, a^\Delta\}$ for every i , $i = 1, 2, \dots, n$. Equality in (20) holds if and only if $a^{d_1} = \dots = a^{d_n}$. Therefore, we conclude that equality in (18) holds if and only if G is a regular graph. ■

Corollary 4. *Let G be a connected graph of order $n \geq 3$, size m , minimum degree δ and maximum degree Δ . If \bar{G} (complement of G) is connected and if $a > 0$ with $a \neq 1$, then it holds that*

$$\overline{SEI}_a(G) + a^{\Delta+\delta-n+1} SEI_a(\bar{G}) \leq (a^\Delta + a^\delta) (n(n-1) - 2m),$$

with equality if and only if every vertex of G has degree either Δ or δ .

Theorem 4. Let G be a connected graph of order $n \geq 2$ and size m . Let (d_1, d_2, \dots, d_n) be the vertex-degree sequence of G such that $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta > 0$. For any $a > 1$, it holds that

$$SEI_a(G) \leq \frac{4m^2 a^{\frac{2m}{n}}}{n(\det D)^{\frac{1}{n}}} \left(1 + \gamma(S) \frac{(a^\Delta - a^\delta)^2}{a^{\Delta+\delta}} \right), \quad (21)$$

where $\det D$ is the determinant of the diagonal degree matrix of G ,

$$\gamma(S) = \frac{\sum_{i \in S} d_i}{2m} \left(1 - \frac{\sum_{i \in S} d_i}{2m} \right), \quad (22)$$

and S is a subset of $I_n = \{1, 2, \dots, n\}$ which minimizes the expression

$$\left| \sum_{i \in S} d_i - m \right|. \quad (23)$$

Equality sign in (21) holds if G is regular.

Proof. For $p_i = d_i$ with $i = 1, 2, \dots, n$, (7) becomes (23). Let S be a subset of $I_n = \{1, 2, \dots, n\}$ which minimizes the expression (23). Then, from (9) we get $\gamma(S)$ defined by (22). Also, for $p_i = d_i$ and $a_i = a^{d_i}$ with $i = 1, 2, \dots, n$, and for $a > 1$, $R = a^\Delta$, $r = a^\delta$, the inequality (8) transforms into

$$\sum_{i=1}^n d_i a^{d_i} \sum_{i=1}^n d_i a^{-d_i} \leq \left(1 + \gamma(S) \frac{(a^\Delta - a^\delta)^2}{a^{\Delta+\delta}} \right) 4m^2. \quad (24)$$

On the other hand, according to the arithmetic-geometric mean inequality, we have

$$\sum_{i=1}^n d_i a^{-d_i} \geq n \left(\prod_{i=1}^n d_i a^{-d_i} \right)^{\frac{1}{n}},$$

that is,

$$\sum_{i=1}^n d_i a^{-d_i} \geq \frac{n(\det D)^{\frac{1}{n}}}{a^{\frac{2m}{n}}}. \quad (25)$$

Now, from (24) and (25) we obtain (21).

Equality in (24) holds if G is a regular graph. In that case the equality in (15) is attained also. Therefore, we conclude that equality in (21) holds if G is regular. ■

Since for any $S \subset I_n = \{1, 2, \dots, n\}$, it holds $\gamma(S) \leq \frac{1}{4}$, we have the following corollaries of Theorem 4.

Corollary 5. If G is a connected graph of order $n \geq 2$, size m , minimum degree δ , maximum degree Δ , and if the determinant of its diagonal degree matrix is $\det D$ then for any $a > 1$ it holds that

$$SEI_a(G) \leq \frac{m^2 a^{\frac{2m}{n}} (a^\Delta + a^\delta)^2}{na^{\Delta+\delta} (\det D)^{\frac{1}{n}}},$$

with equality if G is regular.

Corollary 6. If U is a connected unicyclic graph of order $n \geq 3$, minimum degree δ , maximum degree Δ , and if the determinant of its diagonal degree matrix is $\det D$ then for any $a > 1$ it holds that

$$SEI_a(U) \leq \frac{na^2 (a^\Delta + a^\delta)^2}{a^{\Delta+\delta} (\det D)^{\frac{1}{n}}},$$

with equality if $U \cong C_n$.

Theorem 5. Let G be a connected graph of order $n \geq 2$, size m , minimum degree δ and maximum degree Δ . If $a > 1$ then

$$\begin{aligned} (n-1) \left(na^{\frac{2m}{n}} + \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2 \right) &\leq SEI_a(G) + \overline{SEI}_a(G) \\ &\leq n(n-1) \left(a^{\frac{2m}{n}} + n\alpha(n) \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2 \right), \end{aligned} \quad (26)$$

and if $0 < a < 1$ then

$$\begin{aligned} (n-1) \left(na^{\frac{2m}{n}} + \frac{\left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2}{a^{\Delta+\delta}} \right) &\leq SEI_a(G) + \overline{SEI}_a(G) \\ &\leq n(n-1) \left(a^{\frac{2m}{n}} + \frac{n\alpha(n) \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2}{a^{\Delta+\delta}} \right), \end{aligned} \quad (27)$$

where $\alpha(n)$ is defined via (5). Left equality signs in (26) and (27) hold if $d_2 = \dots = d_{n-1} = \frac{\Delta+\delta}{2}$. Right equality sign in (26) and (27) hold if and only if G is regular.

Proof. Let (d_1, d_2, \dots, d_n) be the vertex-degree sequence of G such that $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta > 0$. For any $a > 0$ with $a \neq 1$, the following identity is valid

$$SEI_a(G) + \overline{SEI}_a(G) = (n-1) \sum_{i=1}^n a^{d_i}. \quad (28)$$

First, we assume that $a > 1$. For $a_i = b_i = \sqrt{a^{d_i}}$ with $i = 1, 2, \dots, n$, and $R_1 = R_2 = \sqrt{a^\Delta}$, $r_1 = r_2 = \sqrt{a^\delta}$, the inequality (6) transforms into

$$0 \leq n \sum_{i=1}^n a^{d_i} - \left(\sum_{i=1}^n \sqrt{a^{d_i}} \right)^2 \leq n^2 \alpha(n) \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2,$$

that is

$$n \sum_{i=1}^n a^{d_i} \leq \left(\sum_{i=1}^n \sqrt{a^{d_i}} \right)^2 + n^2 \alpha(n) \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2. \quad (29)$$

Now, setting $a_i = a^{d_i}$ with $i = 1, 2, \dots, n$, in (1), we get

$$\left(\sum_{i=1}^n \sqrt{a^{d_i}} \right)^2 \leq (n-1) \sum_{i=1}^n a^{d_i} + na^{\frac{2m}{n}}. \quad (30)$$

From (29) and (30), it follows that

$$n \sum_{i=1}^n a^{d_i} \leq (n-1) \sum_{i=1}^n a^{d_i} + na^{\frac{2m}{n}} + n^2 \alpha(n) \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2,$$

that is

$$\sum_{i=1}^n a^{d_i} \leq na^{\frac{2m}{n}} + n^2 \alpha(n) \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2. \quad (31)$$

From (28) and (31), we obtain the right-hand side of (26).

Equalities in (29) and (30) hold if and only if $a^{d_1} = a^{d_2} = \dots = a^{d_n}$, therefore equality in the right-hand side of (26) holds if and only if G is regular.

Now, for $a_i = a^{d_i}$ with $i = 1, 2, \dots, n$, the inequality (2) becomes

$$\sum_{i=1}^n a^{d_i} \geq na^{\frac{2m}{n}} + \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2. \quad (32)$$

From (28) and (32), we get the left-hand side of (26).

Equality in (32) holds if $a^{d_2} = \dots = a^{d_{n-1}} = a^{\frac{\Delta+\delta}{2}}$, therefore equality in the left-hand side of (26) holds if $d_2 = \dots = d_{n-1} = \frac{\Delta+\delta}{2}$.

In a similar way, one can prove that (27) holds for $0 < a < 1$. ■

Since $\alpha(n) \leq \frac{1}{4}$, the next result is a direct consequence of Theorem 5.

Corollary 7. *Let G be a connected graph of order $n \geq 2$, size m , minimum degree δ and maximum degree Δ . If $a > 1$ then*

$$SEI_a(G) + \overline{SEI}_a(G) \leq n(n-1) \left(a^{\frac{2m}{n}} + \frac{n \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2}{4} \right),$$

and if $0 < a < 1$ then

$$SEI_a(G) + \overline{SEI}_a(G) \leq n(n-1) \left(a^{\frac{2m}{n}} + \frac{n \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2}{4a^{\Delta+\delta}} \right),$$

where the equalities hold if and only if G is regular.

For unicyclic graphs, we have the following two corollaries of Theorem 5.

Corollary 8. *Let U be a connected unicyclic graph of order $n \geq 3$, minimum degree δ and maximum degree Δ . If $a > 1$ then*

$$\begin{aligned} (n-1) \left(na^2 + \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2 \right) &\leq SEI_a(U) + \overline{SEI}_a(U) \\ &\leq n(n-1) \left(a^2 + n\alpha(n) \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2 \right), \end{aligned}$$

and if $0 < a < 1$ then

$$\begin{aligned} (n-1) \left(na^2 + \frac{\left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2}{a^{\Delta+\delta}} \right) &\leq SEI_a(U) + \overline{SEI}_a(U) \\ &\leq n(n-1) \left(a^2 + \frac{n\alpha(n) \left(\sqrt{a^\Delta} - \sqrt{a^\delta} \right)^2}{a^{\Delta+\delta}} \right), \end{aligned}$$

where the equality signs hold if $U \cong C_n$.

Corollary 9. *If U is a connected unicyclic graph with $n \geq 3$ vertices then for any $a > 0$ with $a \neq 1$, it holds that*

$$SEI_a(U) + \overline{SEI}_a(U) \geq n(n-1)a^2,$$

with equality if and only if $U \cong C_n$.

Theorem 6. *If G is a connected graph with $n \geq 2$ vertices and m edges then for any $a > 0$ with $a \neq 1$, it holds that*

$$SEI_a(\overline{G}) \cdot \overline{SEI}_a(G) \geq a^{n-1}(n(n-1) - 2m)^2. \quad (33)$$

Equality in (33) holds if and only if either G is regular or $n-1 = d_1 = \dots = d_t > d_{t+1} = \dots = d_n = \delta$, for some t with $1 \leq t \leq n-1$.

Proof. If $G \cong K_n$, then obviously the equality holds in (33). In what follows, we assume that $G \not\cong K_n$. Let (d_1, d_2, \dots, d_n) be the vertex-degree sequence of G such that $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta > 0$. For any $a > 0$ with $a \neq 1$, it holds that

$$SEI_a(\overline{G}) = \sum_{i=1}^n (n-1-d_i)a^{n-1-d_i} = a^{n-1} \sum_{i=1}^n (n-1-d_i)a^{-d_i}. \quad (34)$$

For $r = -1$, $p_i = n - 1 - d_i$ and $a_i = a^{d_i}$ with $i = 1, 2, \dots, n$, the inequality (3) gives

$$\left(\sum_{i=1}^n (n-1-d_i) \right)^{-2} \sum_{i=1}^n (n-1-d_i) a^{-d_i} \geq \left(\sum_{i=1}^n (n-1-d_i) a^{d_i} \right)^{-1},$$

that is

$$(n(n-1) - 2m)^{-2} \sum_{i=1}^n (n-1-d_i) a^{-d_i} \geq (\overline{SEI}_a(G))^{-1}. \quad (35)$$

From (34) and (35), we get (33).

Equality in (35), and consequently in (33), is attained if and only if either G is a regular graph or $n-1 = d_1 = \dots = d_t > d_{t+1} = \dots = d_n = \delta$, for some t with $1 \leq t \leq n-1$. ■

Corollary 10. *If U is a connected unicyclic graph with $n \geq 3$ vertices then for any $a > 0$ with $a \neq 1$, it holds that*

$$SEI_a(\overline{U}) \cdot \overline{SEI}_a(U) \geq a^{n-1} n^2 (n-3)^2,$$

with equality if and only if $U \cong C_n$.

Theorem 7. *If G is a connected graph with $n \geq 2$ vertices and m edges then for any $a > 1$, it holds that*

$$\overline{SEI}_a(G) + SEI_a(\overline{G}) \geq 2(n(n-1) - 2m) a^{\frac{n-1}{2}}. \quad (36)$$

If $0 < a < 1$, the opposite inequality sign in (36) is valid. The equality sign in (36) holds if and only if n is odd and G is an $\left(\frac{n-1}{2}\right)$ -regular graph.

Proof. Let (d_1, d_2, \dots, d_n) be the vertex-degree sequence of G such that $d_1 \geq d_2 \geq \dots \geq d_n$. Here, one has

$$\overline{SEI}_a(G) + SEI_a(\overline{G}) = \sum_{i=1}^n (n-1-d_i) (a^{d_i} + a^{n-1-d_i}). \quad (37)$$

First, we assume that $a > 1$. Since the function $f(x) = a^x$, with $x \in [1, +\infty)$, is convex, so for any $x, y \in [1, +\infty)$ we have $\frac{1}{2}(a^x + a^y) \geq a^{\frac{x+y}{2}}$ (see e.g. [16]), and thereby

$$\frac{1}{2}(a^{d_i} + a^{n-1-d_i}) \geq a^{\frac{n-1}{2}}. \quad (38)$$

From (37) and (38), we get

$$\overline{SEI}_a(G) + SEI_a(\overline{G}) \geq 2 \sum_{i=1}^n (n-1-d_i) a^{\frac{n-1}{2}} = 2(n(n-1) - 2m) a^{\frac{n-1}{2}},$$

which completes the proof of (36).

Equality in (38) holds if and only if $d_i = \frac{n-1}{2}$ for $i = 1, 2, \dots, n$, which implies that equality in (36) holds if and only if n is odd and G is an $\left(\frac{n-1}{2}\right)$ -regular graph.

Similarly, we prove that the opposite inequality in (36) is valid when $0 < a < 1$. ■

Corollary 11. *If G be a connected graph with $n \geq 2$ vertices and m edges then for any $a > 1$, it holds that*

$$SEI_a(G) + \overline{SEI}_a(\overline{G}) \geq 4ma^{\frac{n-1}{2}}. \quad (39)$$

If $0 < a < 1$ then the opposite inequality sign in (39) holds. The equality sign in (39) holds if and only if n is odd and G is an $\left(\frac{n-1}{2}\right)$ -regular graph.

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