MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **87** (2022) 109-114
ISSN: 0340-6253
doi: 10.46793/match.87-1.109T

Graphs with Given Cyclomatic Number Extremal Relatively to Vertex Degree Function Index for Convex Functions

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(Received April 21, 2021)

Abstract

In this paper it is shown that the unique graph obtained from the star S_n by adding γ edges between a fixed pendant vertex v and γ other pendant vertices, has the maximum (minimum) vertex degree function index $H_f(G)$ in the set of all n-vertex connected graphs having cyclomatic number γ when $1 \leq \gamma \leq n-2$ if $f(x)$ is strictly convex (concave) and satisfies an additional property. This property holds for example if $f(x)$ is differentiable and its derivative is also strictly convex (concave). The general zeroth-order Randić index ${}^0R_\alpha(G)$ is strictly convex and verifies this property for $\alpha > 2$.

1 Introduction and notation

In this paper we shall use the notation and terminology from [16]. We recall some of them. A universal vertex in a graph of order n is a vertex v having $d(v) = n-1$. All extremal graphs considered in this paper will contain universal vertices. An (n,m)-graph is a graph having n vertices and m edges. The set of (n,m)-graphs will be denoted by $G(n,m)$.

The disjoint union of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph whose vertex and edge sets are $V(G_1) \cup V(G_2)$ and $E(G_1) \cup E(G_2)$, respectively. The union of k copies of a graph G will be denoted by kG.

For two vertex-disjoint graphs G and H, the join $G \vee H$ is obtained by joining by edges each vertex of G to all vertices of H. The n-vertex star graph is denoted by S_n.

The first Zagreb index $M_1(G)$ [6] is defined as $M_1(G) = \sum_{v \in V(G)} d(v)^2$. The general first Zagreb index (sometimes referred as "zeroth-order general Randić index"), denoted by ${}^0R_\alpha(G)$ was defined [8, 10] as ${}^0R_\alpha(G) = \sum_{v \in V(G)} d(v)^\alpha$, where α is a real number, $\alpha \notin \{0, 1\}$.

Some extremal results concerning the first Zagreb index or the general zeroth-order Randić index were deduced in [1, 5, 7–14, 18]; see also the surveys [2, 4].

The cyclomatic number or the circuit rank of a graph G, denoted $\gamma(G)$ is the minimum number of edges whose deletion transforms G into an acyclic graph. For a connected graph $G \in G(n, m)$ it holds that $\gamma(G) = m - n + 1$.

Let $\mathbb{G}_{n,\gamma}$ be the set of all connected n-vertex graphs with cyclomatic number γ.
Let v be a fixed pendant vertex of the n-vertex star S_n, where $n \geq 3$. As in [3], for $0 \leq \gamma \leq n - 2$ denote by $H_{n,\gamma}$ the graph obtained from S_n by joining by edges v with γ other pendant vertices. We have $H_{n,\gamma} \in \mathbb{G}_{n,\gamma}$ and $H_{n,0} = S_n$.

The vertex-degree function index $H_f(G)$ was introduced in [17] as

$$H_f(G) = \sum_{v \in V(G)} f(d(v)),$$

for a function $f(x)$ defined on non-negative real numbers. In this paper we will impose to function $f(x)$ to be strictly convex (concave) and satisfy an additional property. ${}^0R_\alpha(G)$ corresponds to $f(x) = x^\alpha$, which is strictly convex for $\alpha < 0$ or $\alpha > 1$ and strictly concave for $0 < \alpha < 1$.

2 Graphs with given cyclomatic number and maximum vertex degree function index

We need the following property, which was proved in [16]:

Lemma 2.1. In the set of connected (n, m)-graphs G having $m \geq n$, the graph which maximizes (minimizes) $H_f(G)$ where $f(x)$ is strictly convex (concave) possesses the following properties:

(1) G has a universal vertex v;

(2) The subgraph $G - v$ consists of some isolated vertices and a nontrivial connected component C which is maximum (minimum) relatively to H_g, where $g(x) = f(x + 1)$ and C also contains a universal vertex.

For unicyclic, bicyclic and tricyclic graphs we obtain the following corollary:

Corollary 2.2. [16] Let $G \in G(n,m)$ be a connected graph such that $H_f(G)$ is maximum and $f(x)$ is strictly convex. Then for:

a) $m = n$: $G = K_1 \vee (K_2 \cup (n-3)K_1)$;

b) $m = n+1$: $G = K_1 \vee (K_{1,2} \cup (n-4)K_1)$;

c) $m = n+2$: $G = K_1 \vee (K_{1,3} \cup (n-5)K_1)$ or $G = K_1 \vee (K_3 \cup (n-4)K_1)$. The first case occurs when $f(4)+3f(2) > 3f(3)+f(1)$; when the inequality is reversed then the second graph is maximum. In case of equality both graphs are maximum. A similar result holds when convex is replaced by concave and maximum by minimum.

Note that $K_1 \vee (K_{1,3} \cup (n-5)K_1) = H_{n,3}$ and the inequality occuring in case c) of this Corollary can be written as

$$f(4) + f(2) - 2f(3) > f(3) + f(1) - 2f(2).$$

This inequality is a particular case of the following property:

We say that function $f(x)$ has property $(P_{\nearrow}; P_{\searrow})$ if $\varphi(i+1) > \varphi(i); \varphi(i+1) < \varphi(i)$, respectively for every integer $i \geq 0$, where $\varphi(x) = f(x+2) + f(x) - 2f(x+1)$. Note that if $f(x)$ is differentiable and its derivative is strictly convex, then $f(x)$ has property $P_{\nearrow}$, since $\varphi'(x) = f'(x+2) + f'(x) - 2f'(x+1) > 0$ by Jensen inequality. This implies that $\varphi(x)$ is a strictly increasing function, hence $f(x)$ has property $P_{\nearrow}$. A similar situation occurs when $f'(x)$ is strictly concave. Also if $\varphi(x)$ is strictly convex, then $\varphi(1) > \varphi(0)$ implies property $P_{\nearrow}$. Indeed, if $\varphi(2) \leq \varphi(1)$ holds we would have $\varphi(0) + \varphi(2) < 2\varphi(1)$, which contradicts Jensen inequality and by induction we deduce that $\varphi(i+1) > \varphi(i)$ for every $i \geq 0$.

We shall state a preliminary result which will be used in the proof of the main result of this paper.

Theorem 2.3. Let $n \geq 2$ and G be an n-vertex graph with m edges such that $1 \leq m \leq n-1$. If $f(x)$ is a strictly convex function having property $P_{\nearrow}$, then it holds

$$H_f(G) \leq f(m) + mf(1) + (n-m-1)f(0),$$

with equality if and only if $G = S_{m+1} \cup (n-m-1)K_1$.

Proof. The proof is by induction on m. For $m = 1$ the result holds. Suppose that the theorem is true for $m = k-1$, where $2 \leq k \leq n-1$, and let G be an n-vertex graph with k edges. Consider an edge $uv \in E(G)$. We get $d(u) + d(v) \leq k+1$. If $d(u) \leq d(v)$ it follows that $d(u) \leq (k+1)/2$. We can write $H_f(G) - H_f(G - uv) =$

$f(d(u)) - f(d(u)-1) + f(d(v)) - f(d(v)-1)$. $f(x)$ being strictly convex, it follows that $f(x+1) - f(x)$ is strictly increasing in x. Since $d(v) \leq k+1-d(u)$ we obtain that

$$H_f(G) - H_f(G-uv) \leq f(d(u)) - f(d(u)-1) + f(k+1-d(u)) - f(k-d(u)).$$

We will show that the maximum of this alternating sum is reached only for $d(u) = 1$ if $d(u) \leq (k+1)/2$. By letting $d(u) = x$ and $g(x) = f(x) - f(x-1) + f(k+1-x) - f(k-x)$ we get that $g(x+1) - g(x) = \varphi(x-1) - \varphi(k-x-1)$. Since $f(x)$ has property $P_{\nearrow}$ we have $\varphi(x-1) < \varphi(k-x-1)$ for every $1 \leq x < k/2$, which implies that $g(x+1) < g(x)$ for every $1 \leq x < k/2$. We shall consider two cases: A1. k is odd; A2. k is even.

A1. In this case it is necessary to prove that $g(1) > g(2) > \ldots > g((k+1)/2)$, or $g(x) > g(x+1)$ for every $x = 1, \ldots, (k-1)/2$. This property holds since $(k-1)/2 < k/2$.

A2. Now we must prove that $g(1) > g(2) > \ldots > g(k/2)$, or $g(x) > g(x+1)$ for every $x = 1, \ldots, k/2 - 1$. This inequality is true since $k/2 - 1 < k/2$.

We deduce that

$$H_f(G) - H_f(G-uv) \leq f(1) - f(0) + f(k) - f(k-1)$$

and equality holds only for $d(u) = 1$ and $d(v) = k$. $G - uv$ has $k-1$ edges and by the induction hypothesis we have $H_f(G-uv) \leq f(k-1) + (k-1)f(1) + (n-k)f(0)$ with equality if and only if $G = S_k \cup (n-k-2)K_1$. Consequently, $H_f(G) \leq f(k) + kf(1) + (n-k-1)f(0)$ and equality is reached if and only if $G = S_{k+1} \cup (n-k-1)K_1$. ■

The following theorem gives the maximum value of the vertex degree function index $H_f(G)$ in the set $\mathbb{G}_{n,\gamma}$ for every $1 \leq \gamma \leq n-2$.

Theorem 2.4. If $n \geq 3$, $1 \leq \gamma \leq n-2$, $f(x)$ is strictly convex which has property $P_{\nearrow}$ and G is a connected n-vertex graph with cyclomatic number γ, then

$$H_f(G) \leq f(n-1) + f(\gamma+1) + \gamma f(2) + (n-\gamma-2)f(1),$$

with equality if and only if $G = H_{n,\gamma} = K_1 \vee (K_{1,\gamma} \cup (n-\gamma-2)K_1)$.

Proof. Let $G \in \mathbb{G}_{n,\gamma}$ be such that $H_f(G)$ is maximum. By Lemma 2.1 there exists a universal vertex $v \in V(G)$. We can write:

$$H_f(G) = f(n-1) + H_g(G-v),$$

where $g(x) = f(x+1)$. Since $f(x)$ is strictly convex and satisfies $P_{\nearrow}$, it follows that $g(x)$ has the same properties. $G - v$ has $n' = n-1$ vertices and $m' = m-n+1 = \gamma(G)$ edges.

Since by hypothesis we have $1 \leq \gamma(G) \leq n-2$ it follows that $1 \leq m' \leq n'-1$. We can apply Theorem 2.3 for $G-v$, since $H_g(G-v)$ must be maximum also, which concludes the proof. ■

Analogous results for general sum-connectivity index $\chi_\alpha(G) = \sum_{uv\in E(G)} (d(u)+d(v))^\alpha$ were deduced in [3] for $\alpha \geq 2$ and in [15] for $1 < \alpha < 2$.

A similar result holds for strictly concave functions $f(x)$ which have property $P_\searrow$: the minimum of $H_f(G)$ is reached in $\mathbb{G}_{n,\gamma}$ if and only if $G = H_{n,\gamma} = K_1 \vee (K_{1,\gamma} \cup (n-\gamma-2)K_1)$.

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