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On Sombor Index of Graphs

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Abstract

Recently, Gutman defined a new vertex-degree-based graph invariant, named the Sombor index SO of a graph G, and is defined by

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2},$$

where $d_G(v)$ is the degree of the vertex v of G. In this paper, we obtain the sharp lower and upper bounds on $SO(G)$ of a connected graph, and characterize graphs for which these bounds are attained.

1 Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is denoted by n. The degree of the vertex v is denoted by $d_G(v)$. For $v \in V(G)$, $N_G(v)$ denotes the set of all neighbors of v. An edge uv of a graph G is called a cut edge if the graph $G - uv$ is disconnected. For $uv \in E(G)$, denote by $G - uv$ the subgraph of G obtained from G by deleting the edge uv. For two nonadjacent vertices u and v of G, denote by $G + uv$ the graph obtained from G by adding the edge uv. The girth of a graph G is the length of the shortest cycle which is contained in G. The maximum degree of G

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is denoted by Δ. The complete graph and the cycle of order n are denoted by K_n and C_n, respectively. The clique number of a graph G is the maximal order of a complete subgraph of G.

Gutman [2] defined a new vertex-degree-based graph invariant, named "Sombor index" of a graph G, denoted by $SO(G)$ and is defined by

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

Mathematical properties and applications of SO index were established in [2].

In this paper, we obtain the sharp lower bounds on $SO(G)$ of a graph of order n with the maximum degree Δ and of a graph of order n with girth g. Also, we give the sharp upper bound on $SO(G)$ of a unicyclic graph of order n with girth g. Very recently, for the graphs of order n with k pendent vertices, the graphs were characterized that have the extremal classical Zagreb indices [1], multiplicative sum Zagreb index [3], and reduced second Zagreb index [4]. Hence, furthermore, we obtain the sharp upper bound on $SO(G)$ of a graph of order n with k pendent vertices (r cut edges). Moreover, the corresponding extremal graphs are characterized for which all the above bounds are attained.

2 Graphs with minimum Sombor index

In this section, we study the graphs with minimum Sombor index. Let $P = uu_1u_2 \cdots u_k$ be a path of length k in G such that $d_G(u) \geq 3$, $d_G(u_k) = 1$ and $d_G(u_i) = 2$ for $i = 1, 2, \ldots, k-1$. Then it is called a pendent path in G, u and k are called the origin and the length of P. Let us consider a function $\theta(t) = \sqrt{t^2+4} - \sqrt{t^2+1}$ and one can easily see that $\theta(t)$ is decreasing on $[0, +\infty)$.

Lemma 2.1. *Let P and Q be two pendent paths with origins u and v in graph G, respectively. Let x be a neighbor vertex of u who lies on P and y be the pendent vertex on Q. Denote $G' = G - ux + xy$. Then $SO(G) > SO(G')$.*

Proof. Let z be the neighbor vertex of y in G. Suppose first that $u \neq v$. Then

$$\begin{aligned} & SO(G) - SO(G') \\ = & \sum_{w \in N_G(u) \setminus x} \sqrt{d_G(u)^2 + d_G(w)^2} + \sqrt{d_G(u)^2 + d_G(x)^2} + \sqrt{1 + d_G(z)^2} \\ & - \sum_{w \in N_G(u) \setminus x} \sqrt{(d_G(u)-1)^2 + d_G(w)^2} - \sqrt{2^2 + d_G(x)^2} - \sqrt{2^2 + d_G(z)^2} \\ > & \sqrt{d_G(u)^2 + d_G(x)^2} + \sqrt{1 + d_G(z)^2} - \sqrt{2^2 + d_G(x)^2} - \sqrt{2^2 + d_G(z)^2}. \end{aligned} \tag{1}$$

Suppose now that $u = v$. If the length of Q is equal to one, then $u = z$ and

$$\begin{aligned} & SO(G) - SO(G') \\ = & \sum_{w \in N_G(u) \setminus \{x,y\}} \sqrt{d_G(u)^2 + d_G(w)^2} + \sqrt{d_G(u)^2 + 1} + \sqrt{d_G(u)^2 + d_G(x)^2} \\ & - \sum_{w \in N_G(u) \setminus \{x,y\}} \sqrt{(d_G(u)-1)^2 + d_G(w)^2} - \sqrt{(d_G(u)-1)^2 + 2^2} - \sqrt{2^2 + d_G(x)^2} \\ > & \sqrt{d_G(u)^2 + d_G(x)^2} - \sqrt{2^2 + d_G(x)^2} \geq 0 \end{aligned}$$

since $d_G(u) \geq 2$. If the length of Q is greater than one then let x' be the neighbor of u on path Q. Hence

$$\begin{aligned} & SO(G) - SO(G') \\ = & \sum_{w \in N_G(u) \setminus \{x,x'\}} \sqrt{d_G(u)^2 + d_G(w)^2} + \sqrt{d_G(u)^2 + d_G(x')^2} \\ & + \sqrt{d_G(u)^2 + d_G(x)^2} + \sqrt{1 + d_G(z)^2} \\ & - \sum_{w \in N_G(u) \setminus \{x,x'\}} \sqrt{(d_G(u)-1)^2 + d_G(w)^2} - \sqrt{(d_G(u)-1)^2 + d_G(x')^2} \\ & - \sqrt{2^2 + d_G(x)^2} - \sqrt{2^2 + d_G(z)^2} \\ > & \sqrt{d_G(u)^2 + d_G(x)^2} + \sqrt{1 + d_G(z)^2} - \sqrt{2^2 + d_G(x)^2} - \sqrt{2^2 + d_G(z)^2}. \end{aligned} \tag{2}$$

Therefore from the inequalities (1) or (2), it follows that

$$SO(G) - SO(G') > \sqrt{9 + d_G(x)^2} - \sqrt{4 + d_G(x)^2} - \theta(2) \tag{3}$$

since $d_G(u) \geq 3$, $d_G(z) \geq 2$ and $\theta(t)$ is decreasing. Clearly $d_G(x) \leq 2$. If $d_G(x) = 1$ then we have $SO(G) > SO(G')$ from (3). If $d_G(x) = 2$ then we also get $SO(G) - SO(G') > \sqrt{13} - \sqrt{8} - \theta(2) > 0$ from (3). ■

A tree is said to be star-like if it has exactly one vertex of degree greater than two. Connected graphs of order n with the maximum degree at most two are only P_n and C_n.

In [2], it has been proved that $SO(G) > SO(P_n)$ for any connected graph G of order n. Therefore we consider a graph G which is different from P_n and C_n.

Theorem 2.2. *Let G be a connected graph of order n with maximum degree $\Delta \geq 3$.*
(i) If $2\Delta \leq n-1$ then

$$SO(G) \geq \Delta(\sqrt{\Delta^2+4}+\sqrt{5}) + 2(n-2\Delta-1)\sqrt{2} \tag{4}$$

with equality if and only if G is isomorphic to a star-like tree of order n with maximum degree Δ in which all neighbors of the maximum degree vertex have degree two.
(ii) If $2\Delta > n-1$ then

$$SO(G) \geq (n-1-\Delta)(\sqrt{\Delta^2+4}+\sqrt{5}) + (2\Delta-n+1)\sqrt{\Delta^2+1} \tag{5}$$

with equality if and only if G is isomorphic to a star-like tree of order n with maximum degree Δ in which the maximum degree vertex has exactly $2\Delta-n+1$ pendent neighbors.

Proof. Let $SO(G)$ be minimum in the class of graphs of order n with maximum degree Δ and w be the maximum degree vertex of G. If there is a non-cut edge xy in G such that $x \neq w$ and $y \neq w$, then $SO(G) > SO(G-xy)$ and it follows that G is a tree. Now, we prove that G is isomorphic to a star-like tree of order n with maximum degree Δ. If not there is a pendent path $uu_1 \cdots u_k$ such that $u \neq w$. Clearly there is a pendent vertex z $(\neq u_k)$ in G. Then $SO(G) > SO(G-uu_1+u_1z)$ by Lemma 2.1 and it contradicts the fact that $SO(G)$ is minimum.

Hence G is a star-like tree of order n with maximum degree Δ. Let k be the number of pendent neighbors of w. Then

$$SO(G) = k(\theta(2)-\theta(\Delta)) + \Delta(\sqrt{\Delta^2+4}+\sqrt{5}) + 2(n-1-2\Delta)\sqrt{2}. \tag{6}$$

Since θ is a decreasing function and $\Delta \geq 3$, we have $\theta(2) > \theta(\Delta)$. Therefore we distinguish the following two cases.

(i) If $2\Delta \leq n-1$ then there are star-like trees of order n with maximum degree Δ such that $k=0$. Hence from (6), we obtain the required result.

(ii) If $2\Delta > n-1$ then $k \geq 2\Delta-n+1$. Hence from (6), we easily get the inequality (5) and with equality if and only if G is isomorphic to a star-like tree of order n with maximum degree Δ in which the maximum degree vertex has exactly $2\Delta-n+1$ pendent neighbors. ■

Denote by $C_{n,1}$ the graph obtained by attaching one pendent edge to a vertex of C_{n-1}.

Theorem 2.3. *Let $SO(G)$ be minimum in the class of graphs of order n with girth g. If G is different from C_n, then G is isomorphic to the unicyclic graph with girth g that has exactly one pendent path.*

Proof. Let C be a cycle of length g and $xy \notin C$ be a non-cut edge of G. Then $SO(G) > SO(G - xy)$ and it follows that G is a unicyclic graph. Therefore, if $g = n - 1$ then G is isomorphic to $C_{n,1}$ and hence the theorem in this case. Let now $g \leq n - 2$ and G is not isomorphic to the unicyclic graph that has exactly one pendent path of length at least two. Then repeatedly using the transformation in Lemma 2.1, we get the required result. ■

The following result easily follows from Theorem 2.3.

Theorem 2.4. *Let G be a unicyclic graph order n which is different from C_n. Then $SO(C_n) < SO(G)$.*

Proof. Let g be the girth of G. Suppose that $SO(G)$ is minimum in the class of graphs of order n with girth g. Since G is different from C_n, G is isomorphic to the unicyclic graph with girth g that has exactly one pendent path by Theorem 2.3. If $g = n - 1$ then G is isomorphic to $C_{n,1}$ and it follows that $SO(C_{n,1}) = 2\sqrt{2}(n-3) + 2\sqrt{13} + \sqrt{10} > 2n\sqrt{2} = SO(C_n)$. If $g \leq n - 2$ then G is isomorphic to the unicyclic graph that has exactly one pendent path of length at least two and it follows that $SO(G) = \sqrt{5} + 3\sqrt{13} + 2\sqrt{2}(n-4) > 2n\sqrt{2} = SO(C_n)$. ■

3 Graphs with maximum Sombor index

In this section, we study the graphs with maximum Sombor index. Namely, we obtain the sharp upper bounds on SO index of a unicyclic graph of order n with girth g and of a graph of order n with k pendent vertices (r cut edges).

Lemma 3.1. *Let G be a connected graph and uv be a non-pendent cut edge in G. Denote by G' the graph obtained by the contraction of uv onto the vertex u and adding a pendent vertex v to u. Then $SO(G) < SO(G')$.*

Proof. Let $N_G(u)\backslash\{v\} = \{u_1, u_2, \ldots, u_s\}$ and $N_G(v)\backslash\{u\} = \{v_1, v_2, \ldots, v_t\}$, then $d_G(u) = s+1$ and $d_G(v) = t+1$. Since uv is a non-pendent cut edge of G, we have $st > 0$. Hence, by the definition of SO, we obtain

$$\begin{aligned} SO(G') - SO(G) &= \sum_{i=1}^{s} \sqrt{(s+t+1)^2 + d_G(u_i)^2} - \sum_{i=1}^{s} \sqrt{(s+1)^2 + d_G(u_i)^2} \\ &\quad + \sum_{j=1}^{t} \sqrt{(s+t+1)^2 + d_G(v_j)^2} - \sum_{j=1}^{t} \sqrt{(t+1)^2 + d_G(v_j)^2} \\ &\quad + \sqrt{(s+t+1)^2 + 1} - \sqrt{(s+1)^2 + (t+1)^2} \\ &> \sqrt{(s+t+1)^2 + 1} - \sqrt{(s+1)^2 + (t+1)^2} \end{aligned}$$

and it follows that $SO(G) < SO(G')$ since $[(s+t+1)^2+1] - [(s+1)^2+(t+1)^2] = 2st > 0$. ∎

Proposition 3.2. *Let G be a connected graph of order n with k cut edges. If $SO(G)$ is maximum in the class of graphs of order n with k cut edges, then all k cut edges of G are pendent.*

Proof. Suppose, on the contrary, that G contains a non-pendent cut edge uv. Let G' be the graph obtained by the contraction of uv onto the vertex u and adding a pendent vertex v to u. Then $SO(G) < SO(G')$ by Lemma 3.1. Therefore, we have a contradiction to the assumption that $SO(G)$ is maximum in the class of graphs of order n with k cut edges. ∎

Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be non-increasing two sequences on an interval I of real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. If

$$a_1 + a_2 + \cdots + a_i \geq b_1 + b_2 + \cdots + b_i \text{ for all } 1 \leq i \leq n-1$$

then we say that A majorizes B.

Now we introduce Karamata's inequality, named after Jovan Karamata [5], also known as the majorization inequality.

Lemma 3.3. [5] *Let $f\colon I \to \mathbb{R}$ be a strictly convex function. Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be non-increasing sequences on I. If A majorizes B then*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n)$$

with equality if and only if $a_i = b_i$ for all $1 \leq i \leq n$.

Theorem 3.4. *Let G be a unicyclic graph of order n with girth g. Then*

$$SO(G) \leq 2\sqrt{(n-g+2)^2+4}+(n-g)\sqrt{(n-g+2)^2+1}+2\sqrt{2}(g-2) \tag{7}$$

with equality if and only if G is isomorphic to the graph obtained by attaching $n-g$ pendent edges to a vertex of C_g.

Proof. Denote by $U_{n,g}$ the graph obtained by attaching $n-g$ pendent edges to a vertex of C_g. If G is isomorphic to $U_{n,g}$ then the equality holds in (7). Suppose that G is not isomorphic to this graph and $SO(G)$ is maximum among all unicyclic graphs of order n with girth g. Then by Proposition 3.2, G is isomorphic to a graph such that each pendent edge is attached to the unique cycle.

Denote (in clockwise order) by $u_1, u_2, \ldots, u_g$ the vertices on the cycle. Let k be the number of pendent edges in G. For simplicity's sake we denote $d_G(u_i) = d_i$, $i = 1, 2, \ldots, g$. Then, we have

$$2 \leq d_i \leq k+2 \quad \text{and} \quad d_1+d_2+\cdots+d_g = k+2g = n+g. \tag{8}$$

Consider a non-increasing sequence $A = \{a_i\}$ with length $n+g$ as follows:

$$\underbrace{\frac{k+2}{2}, \frac{k+2}{2}}_{2}, \ \underbrace{1, 1, \ldots, 1}_{2g-4}, \ \underbrace{\frac{2}{k+2}, \frac{2}{k+2}, \ldots, \frac{2}{k+2}}_{k+2}.$$

Let $c_1, c_2, \ldots, c_g$ be a permutation of the sequence $d_1, d_2, \ldots, d_g$. Then, we consider a non-increasing sequence $B = \{b_i\}$ with length $n+g$ as follows:

$$\underbrace{\frac{c_2}{c_1}, \ldots, \frac{c_2}{c_1}}_{c_1}, \ \underbrace{\frac{c_3}{c_2}, \ldots, \frac{c_3}{c_2}}_{c_2}, \ \ldots, \ \underbrace{\frac{c_g}{c_{g-1}}, \ldots, \frac{c_g}{c_{g-1}}}_{c_{g-1}}, \ \underbrace{\frac{c_1}{c_g}, \ldots, \frac{c_1}{c_g}}_{c_g},$$

where for all $1 \leq i \leq g$ there exists j such that $c_i/c_{i-1} = d_j/d_{j-1}$ with $c_0 = c_g$ and $d_0 = d_g$.

Now we prove that A majorizes B. Denote $A_i = a_1 + a_2 + \cdots + a_i$ and $B_i = b_1 + b_2 + \cdots + b_i$ for $1 \leq i \leq n+g$. Then, one can easily see that $A_{n+g} = B_{n+g} = n+g$, $A_1 \geq B_1$ and $A_2 \geq B_2$ from (8) because $c_1, c_2, \ldots, c_g$ is a permutation of $d_1, d_2, \ldots, d_g$.

Suppose first that $3 \leq i \leq 2g-2$. Then, we have $A_i = k+2+i-2 = k+i$ and

$$B_i = c_2 + c_3 + \cdots + c_{s-1} + \frac{pc_s}{c_{s-1}} \quad \text{for some positive integers } s \text{ and } p,$$

such that $c_1 + \cdots + c_{s-2} + p = i$ and $p \leq c_{s-1}$. Therefore, we get

$$A_i - B_i = k + c_1 + p - c_{s-1} - \frac{pc_s}{c_{s-1}}. \tag{9}$$

On the other hand, for $1 \leq i < j \leq g$ we have $d_i + d_j \leq k + 4$ and it follows that $c_i + c_j \leq k + 4$. If $p \geq 2$, then $c_{s-1} + pc_s/c_{s-1} \leq c_{s-1} + c_s \leq k + 4 \leq k + c_1 + p$ since $p \leq c_{s-1}$ and $c_1 \geq 2$. Therefore, we have $A_i \geq B_i$ from (9). If $p = 1$, then from (9), we get

$$A_i - B_i \geq k + 3 - c_{s-1} - \frac{c_s}{2} = k + 3 + \frac{c_s}{2} - (c_{s-1} + c_s) \geq 0 \tag{10}$$

since $c_1, c_{s-1}, c_s \geq 2$ and $c_{s-1} + c_s \leq k + 4$.

Suppose now that $2g - 2 < i \leq n + g$. Then since $A_{n+g} = n + g$,

$$A_i = n + g - (n + g - i) \cdot \frac{2}{k+2}. \tag{11}$$

Moreover, since $B_{n+g} = n + g$ and the sequence B is non-increasing, we get

$$B_i \leq n + g - (n + g - i)\frac{c_1}{c_g}. \tag{12}$$

Therefore, from (11) and (12) we get $A_i \geq B_i$ using $2 \leq c_1, c_g \leq k+2$. Hence we conclude that A majorizes B.

Now, we prove that $SO(G) < SO(U_{n,g})$ by using Karamata's inequality. For this purpose, let us consider a function $f(x) = \sqrt{1 + x^2}$ and it is easy to see that this function is strictly convex for $x \in [0, +\infty)$. By the definition of $SO(G)$ and G is not isomorphic to $U_{n,g}$, we obtain

$$\begin{aligned} SO(G) &= \sqrt{d_1^2 + d_2^2} + \cdots + \sqrt{d_{g-1}^2 + d_g^2} + \sqrt{d_g^2 + d_1^2} \\ &\quad + (d_1 - 2)\sqrt{d_1^2 + 1} + \cdots + (d_g - 2)\sqrt{d_g^2 + 1} \\ &< d_1\sqrt{1 + \left(\frac{d_2}{d_1}\right)^2} + \cdots + d_{g-1}\sqrt{1 + \left(\frac{d_g}{d_{g-1}}\right)^2} + d_g\sqrt{1 + \left(\frac{d_1}{d_g}\right)^2} \\ &\quad + k\sqrt{(k+2)^2 + 1} \\ &= d_1 f\left(\frac{d_2}{d_1}\right) + \cdots + d_{g-1} f\left(\frac{d_g}{d_{g-1}}\right) + d_g f\left(\frac{d_1}{d_g}\right) \\ &\quad + k\sqrt{(k+2)^2 + 1} \end{aligned} \tag{13}$$

by (8). Without loss of generality we may assume that

$$\frac{d_2}{d_1} \geq \frac{d_3}{d_2} \geq \cdots \geq \frac{d_g}{d_{g-1}} \geq \frac{d_1}{d_g}.$$

Then we have proved that A majorizes the sequence

$$\underbrace{\frac{d_2}{d_1},\ldots,\frac{d_2}{d_1}}_{d_1},\ \underbrace{\frac{d_3}{d_2},\ldots,\frac{d_3}{d_2}}_{d_2},\ \ldots,\ \underbrace{\frac{d_g}{d_{g-1}},\ldots,\frac{d_g}{d_{g-1}}}_{d_{g-1}},\ \underbrace{\frac{d_1}{d_g},\ldots,\frac{d_1}{d_g}}_{d_g}.$$

Therefore from (13), we get the required strict inequality in (7) by using Karamata's inequality. ■

Lemma 3.5. *If $x \geq y \geq 0$ and $a \geq 1$ then*
$(x+1)\sqrt{(x+a)^2+1} + y\sqrt{(y+a-1)^2+1} \geq x\sqrt{(x+a-1)^2+1} + (y+1)\sqrt{(y+a)^2+1}.$

Proof. Let us consider a function

$$\phi(x) = (x+1)\sqrt{(x+a)^2+1} - x\sqrt{(x+a-1)^2+1}, \quad x \in [0, +\infty).$$

Then, we have

$$\begin{aligned}\phi'(x) &= \sqrt{(x+a)^2+1} + \frac{(x+1)(x+a)}{\sqrt{(x+a)^2+1}} - \sqrt{(x+a-1)^2+1} - \frac{x(x+a-1)}{\sqrt{(x+a-1)^2+1}} \\ &> \frac{(x+1)(x+a)}{\sqrt{(x+a)^2+1}} - \frac{x(x+a-1)}{\sqrt{(x+a-1)^2+1}} \\ &> \frac{a+2x}{\sqrt{(x+a-1)^2+1}} > 0\end{aligned}$$

and it follows that $\phi(x)$ is an increasing function. Therefore we get the required inequality since $x \geq y$. ■

Theorem 3.6. *Let G be a connected graph of order n with k pendent vertices. Then*
$SO(G) \leq \frac{(n-k-2)(n-k-1)^2}{\sqrt{2}} + k\sqrt{(n-1)^2+1} + (n-k-1)\sqrt{(n-1)^2+(n-k-1)^2}$
with equality if and only if G is isomorphic to a graph obtained by attaching k pendent edges to a vertex of K_{n-k}.

Proof. If G is isomorphic to a graph obtained by attaching k pendent edges to a vertex of K_{n-k}, then equality holds in the inequality of the statement of the theorem. Suppose that G is not isomorphic to this graph and $SO(G)$ is maximum among all graphs of order n with k pendent vertices. Then by Proposition 3.2, G is isomorphic to a graph such that each pendent edge is attached to the clique with $n-k$ vertices.

Denote by $u_1, u_2, \ldots, u_{n-k}$ the vertices of the clique. Denote $d_G(u_i) = d_i$, $i = 1, 2, \ldots, n-k$. Without loss of generality we may assume that $d_1 \geq d_2 \geq \cdots \geq d_{n-k}$. Then, we have

$$d_1 + d_2 + \cdots + d_{n-k} = k + (n-k)(n-k-1) \quad \text{and} \quad n-k-1 \leq d_i < n-1. \tag{14}$$

Assume that $d_t = \min\{d_i \mid n-k-1 < d_i < n-1\}$. Then there is a pendent edge $u_t x$ in G and consider the graph $G' = G - u_t x + u_1 x$. If we set $x = d_1 - n + k + 1$, $a = n - k$ and $y = d_t - n + k$ in the inequality of the statement of Lemma 3.5, then

$$\begin{aligned}(d_1 - n + k + 2)\sqrt{(d_1+1)^2+1} + (d_t - n + k)\sqrt{(d_t-1)^2+1} \\ \geq (d_1 - n + k + 1)\sqrt{d_1^2+1} + (d_t - n + k + 1)\sqrt{d_t^2+1}.\end{aligned} \tag{15}$$

Therefore, we have

$$\begin{aligned}SO(G') - SO(G) = & \sum_{i\neq 1,t}\sqrt{(d_1+1)^2+d_i^2} + \sum_{i\neq 1,t}\sqrt{(d_t-1)^2+d_i^2} + \sqrt{(d_1+1)^2+(d_t-1)^2} \\ & +(d_1-n+k+2)\sqrt{(d_1+1)^2+1} + (d_t-n+k)\sqrt{(d_t-1)^2+1} \\ & - \sum_{i\neq 1,t}\sqrt{d_1^2+d_i^2} - \sum_{i\neq 1,t}\sqrt{d_t^2+d_i^2} - \sqrt{d_1^2+d_t^2} \\ & -(d_1-n+k+1)\sqrt{d_1^2+1} - (d_t-n+k+1)\sqrt{d_t^2+1} \\ > & \sum_{i\neq 1,t} d_i\sqrt{1+\left(\frac{d_1+1}{d_i}\right)^2} + \sum_{i\neq 1,t} d_i\sqrt{1+\left(\frac{d_t-1}{d_i}\right)^2} \\ & - \sum_{i\neq 1,t} d_i\sqrt{1+\left(\frac{d_1}{d_i}\right)^2} - \sum_{i\neq 1,t} d_i\sqrt{1+\left(\frac{d_t}{d_i}\right)^2}\end{aligned} \tag{16}$$

by (15) and $\sqrt{(d_1+1)^2+(d_t-1)^2} \geq \sqrt{d_1^2+d_t^2}$.

Consider non-increasing two sequences $A = \{a_i\}$ and $B = \{b_i\}$ as follows:

$$A: \underbrace{\frac{d_1+1}{d_{n-k}},\cdots,\frac{d_1+1}{d_{n-k}}}_{d_{n-k}},\cdots,\underbrace{\frac{d_1+1}{d_{t+1}},\cdots,\frac{d_1+1}{d_{t+1}}}_{d_{t+1}},\underbrace{\frac{d_1+1}{d_{t-1}},\cdots,\frac{d_1+1}{d_{t-1}}}_{d_{t-1}},\cdots,\underbrace{\frac{d_1+1}{d_2},\cdots,\frac{d_1+1}{d_2}}_{d_2},$$

$$\underbrace{\frac{d_t-1}{d_{n-k}},\cdots,\frac{d_t-1}{d_{n-k}}}_{d_{n-k}},\cdots,\underbrace{\frac{d_t-1}{d_{t+1}},\cdots,\frac{d_t-1}{d_{t+1}}}_{d_{t+1}},\underbrace{\frac{d_t-1}{d_{t-1}},\cdots,\frac{d_t-1}{d_{t-1}}}_{d_{t-1}},\cdots,\underbrace{\frac{d_t-1}{d_2},\cdots,\frac{d_t-1}{d_2}}_{d_2};$$

$$B: \underbrace{\frac{d_1}{d_{n-k}},\cdots,\frac{d_1}{d_{n-k}}}_{d_{n-k}},\cdots,\underbrace{\frac{d_1}{d_{t+1}},\cdots,\frac{d_1}{d_{t+1}}}_{d_{t+1}},\underbrace{\frac{d_1}{d_{t-1}},\cdots,\frac{d_1}{d_{t-1}}}_{d_{t-1}},\cdots,\underbrace{\frac{d_1}{d_2},\cdots,\frac{d_1}{d_2}}_{d_2},$$

$$\underbrace{\frac{d_t}{d_{n-k}},\cdots,\frac{d_t}{d_{n-k}}}_{d_{n-k}},\cdots,\underbrace{\frac{d_t}{d_{t+1}},\cdots,\frac{d_t}{d_{t+1}}}_{d_{t+1}},\underbrace{\frac{d_t}{d_{t-1}},\cdots,\frac{d_t}{d_{t-1}}}_{d_{t-1}},\cdots,\underbrace{\frac{d_t}{d_2},\cdots,\frac{d_t}{d_2}}_{d_2}.$$

Denote $A_i = a_1 + a_2 + \cdots + a_i$ and $B_i = b_1 + b_2 + \cdots + b_i$ for $1 \leq i \leq 2\sum_{i\neq 1,t} d_i$. From the above, it is easy to see that both the summations of all elements of A and B are equal to $(n-k-2)(d_1+d_t)$, and $A_i \geq B_i$ for all $1 \leq i \leq 2\sum_{i\neq 1,t} d_i$. Hence A majorizes B.

On the other hand, $f(x) = \sqrt{1+x^2}$ is a strictly convex function on $[0, +\infty)$. Therefore, using Karamata's inequality in (16), we get $SO(G') > SO(G)$ and it contradicts the fact that $SO(G)$ is maximum among all graphs of order n with k pendent vertices. ■

The same argument as in the proof of Theorem 3.6 yields the following result.

Theorem 3.7. *If $SO(G)$ is maximum in the class of connected graphs of order n with r cut edges, then G is isomorphic to the graph obtained by attaching r pendent edges to a vertex of K_{n-r}.*

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