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Edge Deletion, Singular Values and *ABC* Energy of Graphs

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Abstract

Let $\mathcal{A}$ be the ABC matrix of graph G. The $\mathcal{A}$-energy $\mathcal{E}_{\mathcal{A}}(G)$ is the sum of absolute values of the eigenvalues of matrix $\mathcal{A}$. In this paper, we are interested how the $\mathcal{A}$-energy of an isolated-free graph changes when a non-leaf edge is deleted. The aim of this paper is to study graph energy change due to edge deletion. Further, we present several new results concerning with the $\mathcal{A}$-energy of a graph. Besides, we compute the energy and energy change due to edge deletion of some classes of well-known graphs.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$, whose adjacency matrix is $A(G)$. A graph is isolate-free if it has no isolated vertex. The complete graph, the cycle graph and the path graph on n vertices are denoted by K_n, C_n and P_n, respectively. For each $x \in V$, let $N_G(x)$ denote the neighborhood of x. Let $e = xy$ be an edge of $E(G)$. Then $N_G(x) \cap N_G(y) = \emptyset$ if and only if e is not on a cycle C_3.

Let M be a real square matrix of order n. The eigenvalues of matrix M are the roots of characteristic polynomial $P_M(\lambda) = \det(\lambda I_n - M)$, where I_n is the identity matrix of order n. If M has exactly s distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ with respectively multiplicities

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$t_1, t_2, \ldots, t_s$, then we use $\mathrm{Spec}(M) = \{[\lambda_1]^{t_1}, [\lambda_2]^{t_2}, \ldots, [\lambda_s]^{t_s}\}$, for showing the spectrum of M.

The adjacency energy or briefly the energy of G is a graph invariant which was introduced by Ivan Gutman [18]. The energy $\mathcal{E}_A(G)$ is defined as the sum of absolute values of the eigenvalues of $A(G)$. For its basic properties and applications, including various lower and upper bounds, see the book [23], a survey [19], the recent papers [7, 9, 10, 17, 20, 21, 25, 26] and the references cited there in. The energy of a vertex, as introduced by Arizmendi et al. [4], is defined as $\mathcal{E}_G(v_i) = |A|_{ii}$, $(1 \leq i \leq n)$, where $|A| = (AA^t)^{\frac{1}{2}}$ and A is the adjacency matrix of G, see [3, 5, 14] for further properties. Given $\mathcal{E}_A(G) = \mathrm{Tr}(|A|)$, we can recover the energy of a graph by adding the energies of the vertices in the graph G,

$$\mathcal{E}_A(G) = \mathcal{E}_G(v_1) + \cdots + \mathcal{E}_G(v_n).$$

The structure of this paper is as follows. In Section 2, we give some auxiliary results concerning with the $\mathcal{A}$-energy of a graph. In Section 3, we provide some preparatory results. Besides, by constructing three examples, we indicate that, in general, by an edge-deletion operation, the $\mathcal{A}$-energy of a graph increases, decreases or remains unchanged. The main results are given in Section 4.

2 The auxiliary results

In this section, we give some equations and two upper bounds for the $\mathcal{A}$-energy of a graph. The matrix $\mathcal{A} = (\mathcal{A}_{ij})$ has been defined as $\mathcal{A}_{ij} = \sqrt{\frac{d_i+d_j-2}{d_i d_j}}$, if two vertices v_i and v_j are adjacent, and $\mathcal{A}_{ii} = 0$ otherwise. The eigenvalues of this matrix are the $\mathcal{A}$-eigenvalues of G denoted by $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$. The $\mathcal{A}$-spectral radius of G is the largest eigenvalue of the $\mathcal{A}$-matrix of G, which is denoted by ν_1. The $\mathcal{A}$-energy $\mathcal{E}_{\mathcal{A}}(G)$ is the sum of absolute values of the eigenvalues of $\mathcal{A}$, see [6, 13]. In [6], Chen conjectured that among all trees of order n, the star graph S_n has the minimum $\mathcal{A}$-energy and Gao et al. in [15] proved this conjecture. As usual, the binomial coefficients are defined by $\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$, where $n \geq r$. Ghorbani et al. in [16], proved that for a connected graph G of order $n \geq 3$, we have the following result about the $\mathcal{A}$-energy of a graph.

Theorem 2.1. [16] *Let G be a connected graph of order $n \geq 3$. Then*

$$\mathcal{E}_{\mathcal{A}}(G) = \nu_1 \ \mathrm{Tr} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \sum_{j=0}^{\infty} \binom{i}{j} (-1)^j \left(\frac{\mathcal{A}}{\nu_1}\right)^{2j}. \tag{1}$$

Proof. Let G be a connected graph. Suppose that the $\mathcal{A}$ matrix of G is a square, symmetric matrix with spectral decomposition $\mathcal{A} = QDQ^T$, where $Q = [\overrightarrow{\psi}_1 \cdots \overrightarrow{\psi}_n]$ is the matrix of orthonormalized eigenvectors $\overrightarrow{\psi}_j$ associated with the eigenvalues ν_j, and $D = \mathrm{diag}(\nu_1, \ldots, \nu_n)$. Since every symmetric positive semidefinite matrix has a unique positive semidefinite square root, we yield that $|\mathcal{A}| = Q|D|Q^T = \sqrt{\mathcal{A}^2}$.

Let $\nu_1 > 0$ be the largest eigenvalue of $\mathcal{A}$. We note in passing that since G is connected, ν_1 is a simple eigenvalue. Then, $\frac{\mathcal{A}}{\nu_1}$ has spectral radius 1, and the matrix $M = (\frac{\mathcal{A}}{\nu_1})^2 - I$ has all its eigenvalues in the interval $[-1, 0]$. Hence, M is negative semidefinite and has spectral radius 1. Let us write

$$|\mathcal{A}| = \sqrt{\mathcal{A}^2} = \nu_1 \sqrt{\left(\frac{\mathcal{A}}{\nu_1}\right)^2} = \nu_1 \sqrt{I + \left(\left(\frac{\mathcal{A}}{\nu_1}\right)^2 - I\right)} = \nu_1 (I + M)^{\frac{1}{2}}. \tag{2}$$

Since, for $-1 \leq x \leq 1$, we have

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots, \quad 0 \neq \alpha \in \mathbb{R},$$

Eq.(2) can be reformulated as follows:

$$|\mathcal{A}| = \nu_1 \left(I + \frac{1}{2}M - \frac{1}{4 \cdot 2} M^2 + \frac{3}{2 \cdot 4 \cdot 6} M^3 + \cdots\right) = \nu_1 \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \left(\left(\frac{\mathcal{A}}{\nu_1}\right)^2 - I\right)^i.$$

Therefore,

$$\begin{aligned} \mathcal{E}_{\mathcal{A}}(G) = \mathrm{Tr}|\mathcal{A}| &= \nu_1 \mathrm{Tr}\left[\sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \left(\left(\frac{\mathcal{A}}{\nu_1}\right)^2 - I\right)^i\right] \\ &= \nu_1 \mathrm{Tr} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \sum_{j=0}^{\infty} \binom{i}{j} (-1)^j \left(\frac{\mathcal{A}}{\nu_1}\right)^{2j}. \end{aligned}$$

∎

Equivalently, the Eq.(1), can be rewritten as follows:

$$\mathcal{E}_{\mathcal{A}}(G) = \nu_1 \sum_{i=0}^{\infty} \binom{2i}{i} \frac{(-1)^{i+1}}{2^{2i}(2i-1)} \ \mathrm{Tr} \left(\left(\frac{\mathcal{A}}{\nu_1}\right)^2 - I\right)^i.$$

The general Randić index or the branching index was defined as $R_{-1}(G) = \sum_{v_i \sim v_j} (1/d_i d_j)$, see [27].

Lemma 2.2. [6]. For any graph G of order $n \geq 3$ with no isolated vertices, we have

1) $\displaystyle\sum_{i=1}^{n} \nu_i = 0,$

2) $\displaystyle\sum_{i=1}^{n} \nu_i^2 = 2(n - 2R_{-1}(G)).$

Theorem 2.3. *Let G be a graph of order $n \geq 3$ with no isolated vertex. Then*

$$\mathcal{E}_{\mathcal{A}}(G) \geq \sqrt{2\Big(n - 2R_{-1}(G) + \binom{n}{2}(\det(\mathcal{A}))^{\frac{2}{n}}\Big)}\ .$$

Proof. Applying Geometric-Arithmetic mean inequality yields that

$$\begin{aligned}
\Big(\sum_{i=1}^{n} |\nu_i|\Big)^2 &= \sum_{i=1}^{n} |\nu_i|^2 + \sum_{i\neq j, 1\leq i,j\leq n} |\nu_i||\nu_j| \\
&\geq 2(n - 2R_{-1}(G)) + n(n-1)\Big(\prod_{i\neq j, 1\leq i,j\leq n} |\nu_i||\nu_j|\Big)^{\frac{1}{n(n-1)}} \\
&= 2(n - 2R_{-1}(G)) + 2\binom{n}{2}\Big(\prod_{i=1}^{n} (\nu_i)^{2(n-1)}\Big)^{\frac{1}{n(n-1)}} \\
&= 2(n - 2R_{-1}(G)) + 2\binom{n}{2}\Big(\prod_{i=1}^{n} \nu_i\ \Big)^{\frac{2}{n}} \\
&= 2(n - 2R_{-1}(G)) + 2\binom{n}{2}(\det(\mathcal{A}))^{\frac{2}{n}}.
\end{aligned}$$

Since $\mathcal{E}_{\mathcal{A}}(G) = \sum_{i=1}^{n} |\nu_i|$, we get

$$\mathcal{E}_{\mathcal{A}}(G) \geq \sqrt{2(n - 2R_{-1}(G)) + 2\binom{n}{2}\sqrt[n]{(\det(\mathcal{A}))^2}}\ .$$

This completes the proof. ■

Theorem 2.4. [8]. (Maclaurin's inequality). Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Then

$$S_1 \geq \sqrt[2]{S_2} \geq \cdots \geq \sqrt[n]{S_n},$$

where

$$S_k = \frac{1}{\binom{n}{k}} \sum_{1\leq i_1 < \cdots < i_k \leq n} a_{i_1} a_{i_2} \cdots a_{i_n}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Theorem 2.5. *Let G be a graph of order $n \geq 3$ with no isolated vertex and zero $\mathcal{A}$ eigenvalues. Then*

$$\mathcal{E}_{\mathcal{A}}(G) > \sqrt{\frac{2n}{n-1}|2R_{-1}(G)-n|}\ .$$

Proof. Consider $a_i = |\nu_i| > 0$ where ν_i's are $\mathcal{A}$ eigenvalues of G, $(1 \leq i \leq n)$. By putting a_i's that in Theorem 2.4, we get $S_1 = \frac{1}{n}\mathcal{E}_{\mathcal{A}}(G)$. Also

$$S_2 = \frac{1}{\binom{n}{2}} \sum_{1\leq i<j\leq n} |\nu_i||\nu_j| \geq \frac{2}{n(n-1)} \left| \sum_{1\leq i<j\leq n} \nu_i\nu_j \right|.$$

By using Lemma 2.2, we have

$$\sum_{1\leq i<j\leq n} \nu_i\nu_j = \frac{1}{2}\left(\sum_{i=1}^{n}\nu_i\right)^2 - \frac{1}{2}\sum_{i=1}^{n}\nu_i^2 = 2R_{-1}(G) - n.$$

Then $S_2 \geq \frac{2}{n(n-1)}|2R_{-1}(G)-n|$. We know that $S_1 \geq \sqrt[2]{S_2}$. Thus

$$\mathcal{E}_{\mathcal{A}}(G) \geq \sqrt{\frac{2n}{n-1}|2R_{-1}(G)-n|}.$$

Equality holds if and only if $|\nu_i| = |\nu_j|$. By [6, Proposition 3.2], we conclude that all ν_i's are zero which is impossible. ■

3 Graph energy change due to edge deletion

Let $G-e$ denote the graph obtained by removing an edge e from G. We introduce three examples to show that the $\mathcal{A}$-energy of $G-e$ increases, decreases or remains unchanged. Indeed, we are interested in how the $\mathcal{A}$-energy of a graph changes when an edge is deleted from a graph. Let us begin with elementary examples.

Example 3.1. *Consider the graph G of order 4 as depicted in Figure 1. The $\mathcal{A}$-matrix is*

$$\mathcal{A}(G) = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \frac{2}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{2}{3} & 0 \end{bmatrix}.$$

We have $\mathrm{Spec}_{\mathcal{A}}(G) = \{[-1.12]^1, [0]^1, [\frac{2}{3}]^1, [1.79]^1\}$ *and* $\mathcal{E}_{\mathcal{A}}(G) \approx 3.57$. *If H_1 is a graph obtained from G by deleting the edge $e = v_3v_4$. Then* $\mathrm{Spec}_{\mathcal{A}}(H_1) = \{[-\sqrt{2}]^1, [0]^2, [\sqrt{2}]^1\}$ *and* $\mathcal{E}_{\mathcal{A}}(H_1) = 2\sqrt{2} < \mathcal{E}_{\mathcal{A}}(G)$. *Also, if H_2 is a graph obtained from G by deleting the edge $e = v_2v_4$. Then* $\mathrm{Spec}_{\mathcal{A}}(H_2) = \{[-1.13]^1, [-\frac{\sqrt{2}}{2}]^1, [0.26]^1, [1.57]^1\}$ *and* $\mathcal{E}_{\mathcal{A}}(H_2) \approx 3.68 > \mathcal{E}_{\mathcal{A}}(G)$.

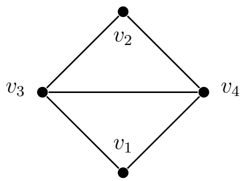

Figure 1. The graph G in Example 3.1.

In what follows, by mG we mean the union of m copies of G, namely $\underbrace{G \cup \cdots \cup G}_{m \text{ times}}$.

Example 3.2. *Suppose $G = \frac{n}{2}K_2$. Then $\mathrm{Spec}_{\mathcal{A}}(G) = \{[0]^4\}$ and $\mathcal{E}_{\mathcal{A}}(G) = 0$. If H is a graph obtained from $G - e$, then $\mathcal{A}$-energy of $G - e$ remains unchanged.*

Question 1. If e is an edge of an isolated-free graph G such that $\mathcal{E}_{\mathcal{A}}(G) = \mathcal{E}_{\mathcal{A}}(G - e)$, then is it true that $G = \frac{n}{2}K_2$?

A vertex-cover of a graph G is a set $S \subseteq V(G)$ such that for each edge $uv \in E(G)$, at least one of u or v is in S.

Theorem 3.3. [6] *If G has a vertex-cover consisting of only the vertices of degree 2, then $\mathcal{E}_{\mathcal{A}}(G) = \frac{\sqrt{2}}{2}\mathcal{E}_{A}(G)$.*

Example 3.4. *Here, we compare the $\mathcal{A}$-energies of C_n with P_n. By* [23, p.26] *and Theorem* 3.3, *we have*

$$\mathcal{E}_{\mathcal{A}}(P_n) = \begin{cases} \sqrt{2}\csc(\frac{\pi}{2(n+1)} - 1) & n \equiv 0 \ (mod\ 2), \\ \sqrt{2}\cot(\frac{\pi}{2(n+1)} - 1) & n \equiv 1 \ (mod\ 2). \end{cases}$$

Then

$$\mathcal{E}_{\mathcal{A}}(C_n) = \begin{cases} 2\sqrt{2}\cot\frac{\pi}{n} & n \equiv 0 \ (mod\ 4), \\ 2\sqrt{2}\csc\frac{\pi}{n} & n \equiv 2 \ (mod\ 4), \\ \sqrt{2}\csc\frac{\pi}{2n} & n \equiv 1 \ (mod\ 2). \end{cases}$$

Therefore

$$\mathcal{E}_{\mathcal{A}}(C_n) - \mathcal{E}_{\mathcal{A}}(P_n) = \begin{cases} \sqrt{2}\left(2\cot\frac{\pi}{n} - \csc(\frac{\pi}{2(n+1)} - 1)\right) & n \equiv 0 \ (mod\ 4), \\ \sqrt{2}\left(2\csc\frac{\pi}{n} - \csc(\frac{\pi}{2(n+1)} - 1)\right) & n \equiv 2 \ (mod\ 4), \\ \sqrt{2}\left(\csc\frac{\pi}{2n} - \cot(\frac{\pi}{2(n+1)} - 1)\right) & n \equiv 1 \ (mod\ 2). \end{cases}$$

In Figure 2, *the difference between $\mathcal{E}_{\mathcal{A}}(C_n)$ and $\mathcal{E}_{\mathcal{A}}(P_n)$ is shown. One can yields that the difference numbers tend to* 0.5, *if n is sufficiently large.*

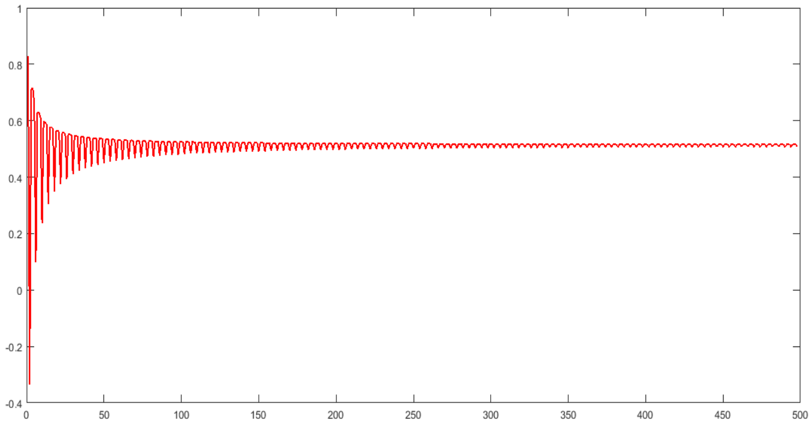

Figure 2. The difference between $\mathcal{E}_{\mathcal{A}}(C_n)$ and $\mathcal{E}_{\mathcal{A}}(P_n)$, where $3 \leq n \leq 500$.

4 Main results

Here, we give some upper bounds for the $\mathcal{A}$-energy of the isolated-free graphs, when a non-leaf edge is deleted. The singular values of a rectangular matrix M with complex entries, are defined to be the square roots of the eigenvalues of the positive semi-definite matrix M^tM, where M^t is the conjugate transpose of M. We denote a singular value by $\sigma_i(M)$, where $1 \leq i \leq n$.

Lemma 4.1. [22] *Let M and N be two square matrices of order n. Then*

$$\sum_{i=1}^{n} \sigma_i(M+N) \leq \sum_{i=1}^{n} \sigma_i(M) + \sum_{i=1}^{n} \sigma_i(N).$$

Lemma 4.2. [22] *The singular values of a real symmetric matrix M are the absolute values of the eigenvalues of M.*

Energy change relating to the adjacency and normalized Laplacian matrices of a graph has been studied in several papers, see [1, 2, 11, 12] for more details. Here, we investigate the conditions that the $\mathcal{A}$-energy of an isolated-free graph changes when a non-leaf edge is deleted.

Lemma 4.3. *Let G_1 and G_2 be two graphs of order n, and $M = \mathcal{A}(G_1) - \mathcal{A}(G_2)$. Then*

$$|\mathcal{E}_{\mathcal{A}}(G_1) - \mathcal{E}_{\mathcal{A}}(G_2)| \leq \sum_{i=1}^{n} \sigma_i(M).$$

Proof. Since $M+\mathcal{A}(G_2)=\mathcal{A}(G_1)$, by Lemma 4.1, we have

$$\sum_{i=1}^{n}\sigma_i\big(M+\mathcal{A}(G_2)\big) \;\leq\; \sum_{i=1}^{n}\sigma_i(M)+\sum_{i=1}^{n}\sigma_i\big(\mathcal{A}(G_2)\big).$$

Lemma 4.2 implies that $\mathcal{E}_{\mathcal{A}}(G_1)-\mathcal{E}_{\mathcal{A}}(G_2)\leq\sum_{i=1}^{n}\sigma_i(M)$. Also, $-M=\mathcal{A}(G_2)-\mathcal{A}(G_1)$ and thus

$$\sum_{i=1}^{n}\sigma_i\big(\mathcal{A}(G_2)\big) \;\leq\; \sum_{i=1}^{n}\sigma_i(-M)+\sum_{i=1}^{n}\sigma_i\big(\mathcal{A}(G_1)\big).$$

Hence, by Lemma 4.1, we get $\mathcal{E}_{\mathcal{A}}(G_2)-\mathcal{E}_{\mathcal{A}}(G_1)\leq\sum_{i=1}^{n}\sigma_i(-M)\leq\sum_{i=1}^{n}\sigma_i(M)$. Therefore $|\mathcal{E}_{\mathcal{A}}(G_1)-\mathcal{E}_{\mathcal{A}}(G_2)|\leq\sum_{i=1}^{n}\sigma_i(M)$. This completes the proof. ■

Lemma 4.3 enables us to prove the following theorems about the variations of $\mathcal{A}$-energy due to edge deletion.

Theorem 4.4. *Let G be an isolated-free graph of order n and $e=xy$ be a non-leaf edge of G. Then*

$$|\mathcal{E}_{\mathcal{A}}(G)-\mathcal{E}_{\mathcal{A}}(G-e)|\leq 2\sqrt{3}.$$

Proof. Suppose $H=\mathcal{A}(G)-\mathcal{A}(G-e)$. It is not difficult to see that $\operatorname{rank}(H)\leq 4$. Suppose that $\operatorname{Spec}(H)=\{[0]^{n-4},[\lambda_1]^1,[\lambda_2]^1,[\lambda_3]^1,[\lambda_4]^1\}$. Thus

$$\sum_{i=1}^{n}\sigma_i(H)=|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4|.$$

The Cauchy-Schwartz inequality and Lemmas 4.2, 4.3 imply that

$$|\mathcal{E}_{\mathcal{A}}(G)-\mathcal{E}_{\mathcal{A}}(G-e)|\leq 2\Big(\sum_{i=1}^{n}(\lambda_i(H))^2\Big)^{\frac{1}{2}}=2\sqrt{\operatorname{Tr}(H^2)}.$$

Since $d_x,d_y\geq 2$ and

$$\begin{aligned}\frac{1}{2}\operatorname{Tr}(H^2)=&\left(\sqrt{\frac{d_x+d_y-2}{d_xd_y}}\right)^2+\sum_{i\neq y,i\in N_G(x)}\left(\sqrt{\frac{d_x+d_i-2}{d_xd_i}}-\sqrt{\frac{d_x+d_i-3}{(d_x-1)d_i}}\right)^2\\&+\sum_{j\neq x,j\in N_G(y)}\left(\sqrt{\frac{d_y+d_j-2}{d_yd_j}}-\sqrt{\frac{d_y+d_j-3}{(d_y-1)d_j}}\right)^2,\end{aligned}$$

we obtain

$$|\mathcal{E}_{\mathcal{A}}(G)-\mathcal{E}_{\mathcal{A}}(G-e)|\leq 2\sqrt{3},$$

as we required. ■

Corollary 4.5. *Let G be an d-regular graph of order n and $e = xy \in E(G)$. Then*

$$|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \le \frac{2\sqrt{2}}{d}\sqrt{(2d-2) + \left(\frac{1}{d^2}\left(\sqrt{\frac{2d-2}{d^2}} - \sqrt{\frac{2d-3}{d(d-1)}}\right)^2\right)}.$$

In Theorem 4.4, it is shown that if e is not incident to a pendant vertex, then $|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \le 2\sqrt{3}$. In following, we show that if x and y also have no common neighbors, then the change in $\mathcal{A}$-energy is less than $\sqrt{10}$.

Suppose $M = \mathcal{A}(G) - \mathcal{A}(G-e)$, where $e = xy$ is a non-pendent edge and $N_G(x) \cap N_G(y) = \phi$. The matrix M is symmetric with zero diagonal entries. Suppose we partition $V - \{x, y\}$ into subsets $N_G(x) = \{u_1, \ldots, u_p\}$ and $N_G(y) = \{u_{p+1}, \ldots, u_{p+q}\}$ such that $N_G(x) - \{y\} \subset N_G(x)$ and $N_G(y) - \{x\} \subset N_G(y)$, where $n = 2 + p + q$, $(p, q \ge 1)$. Then the non-zero entries are the entries of the first two rows or first two columns of M. This means that, the structure of M is

$$\begin{bmatrix} 0 & w & x_1 & \cdots & x_p & 0 & \cdots & 0 \\ w & 0 & 0 & \cdots & 0 & y_{p+1} & \cdots & y_{p+q} \\ x_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_p & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & y_{p+1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & y_{p+q} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where

$$M(x,y) = w = \sqrt{\frac{d_x + d_y - 2}{d_x d_y}}, \tag{3}$$

$$M(x,u_i) = x_i = \begin{cases} 0 & u_i \notin N_G(x) \\ \sqrt{\frac{d_x+d_{u_i}-2}{d_x d_{u_i}}} - \sqrt{\frac{d_x+d_{u_i}-3}{(d_x-1)d_{u_i}}} & u_i \in N_G(x) - \{y\} \end{cases}, \tag{4}$$

$$M(y,u_i) = y_i = \begin{cases} 0 & u_i \notin N_G(y) \\ \sqrt{\frac{d_y+d_{u_i}-2}{d_y d_{u_i}}} - \sqrt{\frac{d_y+d_{u_i}-3}{(d_y-1)d_{u_i}}} & u_i \in N_G(y) - \{x\} \end{cases}, \tag{5}$$

and $d_x, d_y \ge 2$ which implies $w \ne 0$. Consider two vectors $X = [x_1, \ldots, x_p]$ and $Y = [y_{p+1}, \ldots, y_{p+q}]$. The Euclidean norm of the vector $z \in \mathbb{R}^n$, is denoted by $\|z\|$. We have the following theorem.

Theorem 4.6. *Let G be a graph of order n with no isolated vertex. Let $e = xy$ be an edge where $d_x, d_y \ge 2$ and $N_G(x) \cap N_G(y) = \phi$. Then*

1) *If $X \neq 0$,$Y \neq 0$, then $|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq \sqrt{10}$. Moreover, if $d_x, d_y \geq d \geq 2$, Then*

$$|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq \frac{2}{d}\sqrt{8d^2(1-d)\sqrt{\frac{d-2}{d}} + 8d^3 - 16d^2 + 6d - 2}.$$

2) *If $X = 0$, $Y \neq 0$ or $X \neq 0$, $Y = 0$, then $|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq 2$.*

3) *If $X = Y = 0$, then $|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq \sqrt{2}$. In addition, if $d_x, d_y \geq d \geq 2$, Then*

$$|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq \frac{2}{d}\sqrt{2d-2}.$$

Proof. We can distinguish following three cases:

Case 1. Both X and Y have non-zero entries. It is easy to see that in this case $\mathrm{rank}(M) = 4$ and therefore $\mathrm{Spec}(M) = \{[-\lambda_2]^1, [-\lambda_1]^1, [0]^{n-4}, [\lambda_1]^1, [\lambda_2]^1\}$. Let $[\theta, \beta, \rho, \tau]^t$ be an eigenvector corresponding to non-zero eigenvalue λ of M, where $\theta, \beta \in \mathbb{R}$, $\rho \in \mathbb{R}^p$ and $\tau \in \mathbb{R}^q$. Then

$$M \begin{bmatrix} \alpha \\ \beta \\ \rho \\ \tau \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \\ \rho \\ \tau \end{bmatrix}. \tag{6}$$

Hence, Eq.(6) implies that

$$w\beta + X\rho = \lambda\alpha, \tag{7}$$
$$w\alpha + Y\tau = \lambda\beta, \tag{8}$$
$$X^t\alpha = \lambda\rho,$$
$$Y^t\beta = \lambda\tau.$$

Since $\lambda \neq 0$, we have $\rho = \frac{\alpha X^t}{\lambda}$, $\tau = \frac{\beta Y^t}{\lambda}$ and Eq.s (7) and (8), yield to obtain

$$\beta = \tfrac{1}{w}\big(\lambda\alpha - \|X\|^2\tfrac{\alpha}{\lambda}\big), \tag{9}$$
$$\alpha = \tfrac{1}{w}\big(\lambda\beta - \|Y\|^2\tfrac{\beta}{\lambda}\big). \tag{10}$$

Now by Eq.s (9) and (10), we get

$$\lambda^4 - \lambda^2(w^2 + \|X\|^2 + \|Y\|^2) + \|X\|^2\|Y\|^2 = 0.$$

Suppose that $B = w^2 + \|X\|^2 + \|Y\|^2$ and $C = \|X\|^2\|Y\|^2$. Then

$$\lambda_{1,2} = \pm\sqrt{\frac{B \pm \sqrt{B^2 - 4C}}{2}}.$$

Since $d_x, d_y \geq 2$, we yield $w^2 \leq \frac{1}{2}$ with equality if and only if $d_x = d_y = 2$. Also, $\|X\|^2 \leq \frac{1}{2}$ and $\|Y\|^2 \leq \frac{1}{2}$ with equality if and only if $d_{u_i} = 1 (1 \leq i \leq p)$ and $d_{u_j} = 1$ $(p+1 \leq j \leq p+q)$, respectively. Applying Lemma 4.3, we conclude that

$$\begin{aligned}
|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| &\leq 2(\lambda_1 + \lambda_2) \\
&= 2\left[\left(\frac{B + \sqrt{B^2 - 4C}}{2}\right)^{\frac{1}{2}} + \left(\frac{B - \sqrt{B^2 - 4C}}{2}\right)^{\frac{1}{2}}\right] \\
&= 2\sqrt{B + 2\sqrt{C}} \\
&= 2\sqrt{w^2 + \|X\|^2 + \|Y\|^2 + 2\|X\|\|Y\|} \leq \sqrt{10}\ . \qquad (11)
\end{aligned}$$

Also, if $d_x, d_y \geq d \geq 2$, then $w^2 \leq \frac{2d-2}{d^2}$.

Case 2. Suppose either X or Y has a non-zero entry. Let $X = 0$ and Y has a non-zero entry. Then by Eq.(4), we obtain $d_{u_i} = 2$ $(1 \leq i \leq p)$. It is not difficult to see that $\text{null}(M) \geq n-2$ and thus $\text{rank}(M) \leq 2$ which yields that $\text{rank}(M) = 2$ and so M has exactly two non-zero eigenvalues. Let $\lambda \neq 0$ be an eigenvalue of M and $[\alpha, \beta, \mathbf{0}_p, \tau]^t$ be an eigenvector corresponding to λ, where $\alpha, \beta \in \mathbb{R}$ and $\tau \in \mathbb{R}^q$. Then

$$M \begin{bmatrix} \alpha \\ \beta \\ \mathbf{0}_p \\ \tau \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \\ \mathbf{0}_p \\ \tau \end{bmatrix}.$$

Hence,

$$w\beta = \lambda\alpha, \qquad (12)$$

$$w\alpha + Y\tau = \lambda\beta, \qquad (13)$$

$$Y^t\beta = \lambda\tau. \qquad (14)$$

Since $\lambda \neq 0$, by Eq.(14) we have $\tau = \frac{Y^t\beta}{\lambda}$ and by Eq.(13) $w\alpha + Y.\frac{Y^t\beta}{\lambda} = w\alpha + \|Y\|^2\frac{\beta}{\lambda} = \lambda\beta$. Hence, $\alpha = \frac{\beta}{w}(\lambda - \frac{\|Y\|^2}{\lambda})$. Since $\beta \neq 0$, Eq.(12) implies that $w\beta = \lambda\Big(\frac{\beta}{w}(\lambda - \frac{\|Y\|^2}{\lambda})\Big)$. Thus $\lambda^2 - \|Y\|^2 - w^2 = 0$ and so $\lambda = \pm\sqrt{w^2 + \|Y\|^2}$. On the other hand, since $d_x, d_y \geq 2$, we get $w^2 \leq \frac{1}{2}$ and so

$$\|Y\|^2 = \sum_{v_i \in N_G(y) - \{x\}} \left(\sqrt{\frac{d_y + d_{v_i} - 2}{d_y d_{v_i}}} - \sqrt{\frac{d_y + d_{v_i} - 3}{(d_y - 1)d_{v_i}}}\right)^2 \leq \frac{1}{2}.$$

By applying Lemma 4.3, we have

$$|\mathcal{E}_{\mathcal{A}}(G_1) - \mathcal{E}_{\mathcal{A}}(G_2)| \leq 2\sqrt{w^2 + \|Y\|^2} \leq 2\sqrt{\frac{1}{2} + \frac{1}{2}} = 2.$$

Case 3. Both X and Y are zero vectors. Then $x_i = 0$ $(1 \leq i \leq p)$ and $y_j = 0$ $(p+1 \leq j \leq p+q)$. By Eq.s (4) and (5), we have $d_{u_i} = d_{u_j} = 2$. Also, $\mathrm{rank}(M) = 2$ and $\mathrm{Spec}(M) = \{[-w]^1, [0]^{n-2}, [w]^1\}$. Since $d_x, d_y \geq 2$, we get $w \leq \frac{\sqrt{2}}{2}$ with equality if and only if $d_x = d_y = 2$. It is not difficult to see that $\frac{d_x+d_y-2}{d_x d_y} = \frac{1}{2}$ if and only if $2d_x + 2d_y - 4 = d_x d_y$ if and only if $d_x = d_y = 2$. Lemma 4.3 implies that

$$|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq \sum_{i=1}^{n} \sigma_i(M) = \sum_{i=1}^{n} |\lambda_i(M)| = 2w \leq \sqrt{2}\,.$$

■

Here, we determine an upper bound for absolute difference between $\mathcal{E}_{\mathcal{A}}(G)$ and $\mathcal{E}_{\mathcal{A}}(G-e)$, where $e = xy \in E(G)$ is not pendant edge and $|N_G(x) \cap N_G(y)| = k$ $(1 \leq k \leq n-2)$. If we partition $V - \{x, y\}$ into subsets $N_G(x) = \{u_1, \ldots, u_k, u_{k+1}, \ldots, u_p\}$ and $N_G(y) = \{u_1, \ldots, u_k, u_{p+1}, \ldots, u_{p+q}\}$, such that $N_G(x) - \{y\} \subset N_G(x)$ and $N_G(y) - \{x\} \subset N_G(y)$, where $n = 2 + k + p + q$ $(p, q \geq 0)$, then the structure of M is

$$M = \begin{bmatrix} 0 & w & X_1 & X_2 & \mathbf{0}_q \\ w & 0 & Y_1 & \mathbf{0}_p & Y_2 \\ X_1^t & Y_1^t & \mathbf{0}_k^t & \cdots & \mathbf{0}_k^t \\ X_2^t & \mathbf{0}_p^t & \mathbf{0}_p^t & \cdots & \mathbf{0}_p^t \\ \mathbf{0}_q^t & Y_2^t & \mathbf{0}_q^t & \cdots & \mathbf{0}_q^t \end{bmatrix}.$$

Consider now the vectors $\mathbf{0} = [0, 0, \ldots, 0]$, $X_1 = [x_1, \ldots, x_k]$, $X_2 = [x_{k+1}, \ldots, x_p]$, $Y_1 = [y_1, \ldots, y_k]$, $Y_2 = [y_{p+1}, \ldots, y_{p+q}]$, where x_i $(1 \leq i \leq p)$ and y_j $(1 \leq j \leq k)$, $(p+1 \leq j \leq p+q)$ are defined in the Eq.(4) and Eq.(5), respectively.

Theorem 4.7. *If $X_1, X_2, Y_1, Y_2 = 0$, then $|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq \sqrt{2}$.*

Proof. Suppose $X_1, X_2, Y_1, Y_2 = 0$. Then $\mathrm{rank}(M) = 2$, $\mathrm{Spec}(M) = \{[-w]^1, [0]^{n-2}, [w]^1\}$ and similar to the proof of Theorem 4.6 (1), we have

$$|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq 2w = \sqrt{2}\,,$$

which completes the proof. ■

Theorem 4.8. *If $X_1, X_2, Y_1 = 0$ and $Y_2 \neq 0$, or $X_1, X_2, Y_2 = 0$ and $Y_1 \neq 0$, or $X_1, X_2, Y_1 = 0$ and $X_2 \neq 0$, or $X_2, Y_1, Y_2 = 0$ and $X_1 \neq 0$, then $|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq 2$.*

Proof. Suppose $X_1, X_2, Y_1 = 0$ but $Y_2 \neq 0$, then $\mathrm{rank}(M) = 2$. Let $\lambda \neq 0$ be an eigenvalue of M corresponded to eigenvector $\mathbf{v} = [\alpha, \beta, \gamma, \mathbf{0}_{k+p}, \tau]^t$, where $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R}^k$ and

$\tau \in \mathbb{R}^q$. Then

$$w\beta = \lambda\alpha, \tag{15}$$

$$w\alpha + Y_2\tau = \lambda\beta, \tag{16}$$

$$Y_2^t\beta = \lambda\tau. \tag{17}$$

By using Eq.s (15) and (17), we get $\alpha = \frac{w\beta}{\lambda}$ and $\tau = \frac{Y_2^t\beta}{\lambda}$. Since $\lambda \neq 0$, Eq.(16) implies that $\lambda^2 = w^2 + \|Y_2\|^2$ and thus $\mathrm{Spec}(M) = \{[0]^{n-2}, [\pm\sqrt{w^2 + \|Y_2\|^2}]^1\}$. Knowing that $d_x, d_y \geq 2$, we conclude that $w^2, \|Y_2\|^2 \leq \frac{1}{2}$. Thus $|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)|$ is bounded above by $2\sqrt{w^2 + \|Y_2\|^2} \leq 2$. By a similar argument, we obtain a similar result. This completes the proof. ■

Theorem 4.9. *If either $X_1, Y_1 = 0$ and $X_2, Y_2 \neq 0$, or $X_1, Y_2 = 0$ and $X_2, Y_1 \neq 0$, or $X_2, Y_1 = 0$ and $X_1, Y_2 \neq 0$, then $|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| \leq \sqrt{10}$.*

Proof. Suppose that $X_1, Y_1 = 0$ and $X_2, Y_2 \neq 0$. Then $\mathrm{rank}(M) = 4$ and

$$\mathrm{Spec}(M) = \{[-\lambda_2]^1, [-\lambda_1]^1, [0]^{n-4}, [\lambda_1]^1, [\lambda_2]^1\}.$$

Let $\lambda \neq 0$ be an eigenvalue of M with eigenvector $\mathbf{v} = [\alpha, \beta, \mathbf{0}_k, \rho, \tau]^t$, where $\alpha, \beta \in \mathbb{R}$, $\rho \in \mathbb{R}^p$ and $\tau \in \mathbb{R}^q$. Then we have

$$w\beta + X_2\rho = \lambda\alpha, \tag{18}$$

$$w\alpha + Y_2\tau = \lambda\beta, \tag{19}$$

$$X_2^t\alpha = \lambda\rho, \tag{20}$$

$$Y_2^t\beta = \lambda\tau. \tag{21}$$

Eq.s (18), (19), (20) and (21) imply that

$$\lambda^4 - (w^2 + \|X_2\|^2 + \|Y_2\|^2)\lambda^2 + \|X_2\|^2\|Y_2\|^2 = 0. \tag{22}$$

If $B = w^2 + \|X_2\|^2 + \|Y_2\|^2$ and $C = \|X_2\|^2\|Y_2\|^2$, then $B^2 - 4C \geq 0$ and thus the roots of Eq.(22) are

$$\lambda_{1,2} = \pm\sqrt{\frac{1}{2}(B \pm \sqrt{B^2 - 4C})}.$$

This yields that

$$\begin{aligned}|\mathcal{E}_{\mathcal{A}}(G) - \mathcal{E}_{\mathcal{A}}(G-e)| &\leq 2\sqrt{B + 2\sqrt{C}} \\ &\leq 2\sqrt{w^2 + \|X_2\|^2 + \|Y_2\|^2 + 2\|X_2\|\|Y_2\|} \leq \sqrt{10}\ .\end{aligned}$$

■

Let $p, q \geq 0$. The tree Su_p of order $n = 2p + 1$, containing with p pendent vertices, each attached to a vertex of degree 2, and a vertex of degree p, will be called the p-sun. The tree $Su_{p,q}$ of order $n = 2(p+q+1)$, obtained from a p-sun and a q-sun, by connecting their central vertices, will be called a (p, q)-double sun, see Figure 3.

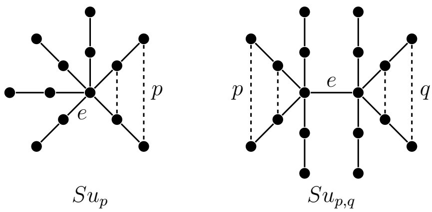

Figure 3. Two graphs Su_p and $Su_{p,q}$.

Example 4.10. *Concider the (p, q)-double sun. It has an edge $e = xy$ for which $d_x = p$ and $d_y = q$. Also, $Su_{p,q} - e$ is disjoint suns Su_p and Su_q. According to Theorem* 4.6 (3), *we have*

$$|\mathcal{E}_{\mathcal{A}}(Su_{p,q}) - \mathcal{E}_{\mathcal{A}}(Su_{p,q} - e)| = |\mathcal{E}_{\mathcal{A}}(Su_{p,q}) - \mathcal{E}_{\mathcal{A}}(Su_p) - \mathcal{E}_{\mathcal{A}}(Su_q)| \leq \sqrt{2}.$$

On the other hand, it is not difficult to see that the $\mathcal{A}$-spectrum of the sun with $p \geq 1$ is

$$\mathrm{Spec}_{\mathcal{A}}(Su_p) = \{[-\sqrt{(n+1)/4}]^1, [-\sqrt{2}/2]^{\frac{n-3}{2}}, [0]^1, [\sqrt{2}/2]^{\frac{n-3}{2}}, [\sqrt{(n+1)/4}]^1\},$$

where $n = 2p + 1$. Then

$$\mathcal{E}_{\mathcal{A}}(Su_p) = \sqrt{2}(p-1) + \sqrt{2p+2}. \tag{23}$$

Therefore

$$\begin{aligned}\mathcal{E}_{\mathcal{A}}(Su_{p,q}) &\leq \sqrt{2} + \sqrt{2}(p-1) + \sqrt{2p+2} + \sqrt{2}(q-1) + \sqrt{2q+2} \\ &= \sqrt{2}\left(1 + (p-1)(q-1) + \sqrt{p+1} + \sqrt{q+1}\right).\end{aligned}$$

Example 4.11. *Concider the p-sun, where $p \geq 2$. Since $Su_p - e$ is disjoint suns Su_{p-1} and K_2, see Figure* 3. *By Eq.* (23) *and Theorem* 4.6 (2), *we have*

$$\begin{aligned}|\mathcal{E}_{\mathcal{A}}(Su_p) - \mathcal{E}_{\mathcal{A}}(Su_p - e)| &= |\mathcal{E}_{\mathcal{A}}(Su_p) - \mathcal{E}_{\mathcal{A}}(Su_{p-1}) - \mathcal{E}_{\mathcal{A}}(K_2)| \\ &= \left(\sqrt{2}(p-1) + \sqrt{2p+2}\right) - \left(\sqrt{2}(p-2) + \sqrt{2(p-1)+2}\right) \\ &\leq \sqrt{2}\left(1 - \sqrt{p} + \sqrt{p+1}\right) < 2.\end{aligned}$$

In Figure 4, the difference between $\mathcal{E}_{\mathcal{A}}(Su_p)$ and $\mathcal{E}_{\mathcal{A}}(Su_p) - e$ is shown. One can yields that the difference numbers tend to $\sqrt{2}$, if p is sufficiently large.

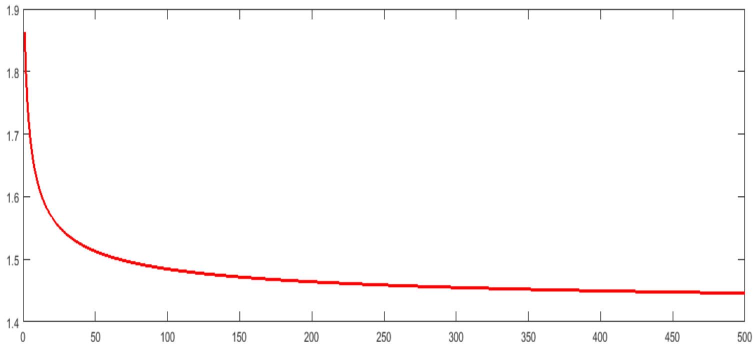

Figure 4. The difference between $\mathcal{E}_{\mathcal{A}}(Su_p)$ and $\mathcal{E}_{\mathcal{A}}(Su_p) - e$, where $2 \leq p \leq 500$.

A complete bipartite graph of order n with a bipartition of sizes n_1 and n_2 is denoted by K_{n_1,n_2}, where $n_1 + n_2 = n$. The double star $S(p,q)$, where $p \geq q \geq 0$, is the graph consisting of the union of two stars $K_{1,p}$ and $K_{1,q}$ together with a line joining their centers.

Example 4.12. *Consider the graph G of order n in Figure 5.*

1) *Supppse $k = 0$ and $n \geq 4$. If $n = 4$, then*

$$M = \mathcal{A}(P_4) - \mathcal{A}(P_4 - e) = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 \end{bmatrix},$$

the path graph P_4 is satisfied in conditions of Theorem 4.6(1). We know

$$Spec_{\mathcal{A}}(P_4) = \{[-1.1441]^1, [-0.4370]^1, [0.4370]^1, [1.1441]^1\},$$

$$Spec_{\mathcal{A}}(P_4 - e) = \{[0]^4\}.$$

Therefore $|\mathcal{E}_{\mathcal{A}}(P_4) - \mathcal{E}_{\mathcal{A}}(P_4 - e)| = \sqrt{10}$ and the bound in Eq.(11) is sharp. Now, suppose $n \geq 5$. Then $G = S(p,q)$ is satisfied in conditions of Theorem 4.6(1). Thus we have

$$|\mathcal{E}_{\mathcal{A}}(S(p,q)) - \mathcal{E}_{\mathcal{A}}(K_{1,p}) - \mathcal{E}_{\mathcal{A}}(K_{1,q})| \leq \sqrt{10}.$$

Since $\mathrm{Spec}_{\mathcal{A}}(K_{1,p}) = \{[-\sqrt{p-1}]^1, [0]^{p-1}, [\sqrt{p-1}]^1\}$. *Therefore*

$$\mathcal{E}_{\mathcal{A}}(S(p,q)) \leq \sqrt{10} + 2(\sqrt{p-1} + \sqrt{q-1}).$$

2) *Suppose* $k \neq 0$. *If graph* H_1 *is obtained from the graph* G *with* $p, q = 0$, *then* $H_1 = K_{2,n-2}+e$ *is satisfied in conditions of Theorem* 4.7. *Thus we have* $|\mathcal{E}_{\mathcal{A}}(K_{2,n-2}+e) - \mathcal{E}_{\mathcal{A}}(K_{2,n-2})| \leq \sqrt{2}$. *Since by* [6, Proposition 4.2], $\mathcal{E}_{\mathcal{A}}(K_{2,n-2}) = 2\sqrt{n-2}$, *we may obtain*

$$\mathcal{E}_{\mathcal{A}}(K_{2,n-2}+e) \leq \sqrt{2} + 2\sqrt{n-2}.$$

Also, if graph H_2 *is obtained from the graph* G *with* $p = 0$ *and* $q \neq 0$, *then this graph is satisfied in conditions of Theorem* 4.8. *Thus we have* $|\mathcal{E}_{\mathcal{A}}(H_2) - \mathcal{E}_{\mathcal{A}}(H_2 - e)| \leq 2$.

Moreover, if graph H_3 *is obtained from the graph* G *with* $p, q \neq 0$, *then this graph is satisfied in conditions of Theorem* 4.9. *Thus we have* $|\mathcal{E}_{\mathcal{A}}(H_3) - \mathcal{E}_{\mathcal{A}}(H_3 - e)| \leq \sqrt{10}$.

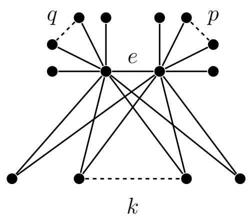

Figure 5. The graph G in Example 4.12.

Here, we determine the $\mathcal{A}$-eigenvalues and $\mathcal{E}_{\mathcal{A}}$ of probabilistic neural networks. In general, a probabilistic neural network or briefly a PNN is a neural network which is widely used in classification and pattern recognition. In graph approach, some problems such as $G = \mathrm{PNN}(n, k, m)$ can be constructed as follows: There are three types of vertices of degrees respectively km, $n+1$ and m. Thus, we have

$$V_1 = \{v \in V(G) \mid d_v = km\},$$

$$V_2 = \{v \in V(G) \mid d_v = n+1\},$$

$$V_3 = \{v \in V(G) \mid d_v = m\},$$

where $|V_1| = n$, $|V_2| = km$ and $|V_3| = k$ and $V(G) = V_1 \cup V_2 \cup V_3$. Consequently, $|V(G)| = |V_1| + |V_2| + |V_3| = n + k(m+1)$. The set of edges can be divided as following subsets:

$$E_1 = E_{km,n+1} = \{uv \in E(G) \mid d_u = km, d_v = n+1\},$$

$$E_2 = E_{n+1,m} = \{uv \in E(G) \mid d_u = n+1, d_v = m\},$$

where $|E_{km,n+1}| = kmn$ and $|E_{n+1,m}| = km$. Consequently, $|E(G)| = |E_1| + |E_2| = km(n+1)$. The probabilistic neural network G for $n = 4$, $k = 2$ and $m = 3$ is depicted in Figure 6.

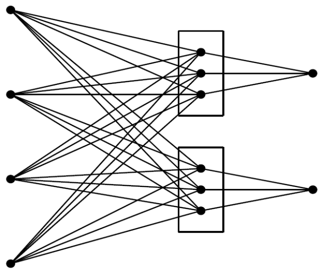

Figure 6. Probabilistic neural network $\mathrm{PNN}(4,2,3)$.

Example 4.13. *Take $G = \mathrm{PNN}(n,k,m)$ graph on $n + k(m+1)$ vertices. One can easily prove that the $\mathcal{A}$-matrix is as follows:*

$$\mathcal{A}(G) = \begin{bmatrix} 0_{n\times n} & M_{n\times mk} & 0_{n\times k} \\ M_{mk\times k} & 0_{mk\times mk} & N^T_{mk\times k} \\ 0_{k\times n} & N_{k\times mk} & 0_{k\times k} \end{bmatrix},$$

where

$$N = \begin{bmatrix} \overbrace{\alpha\ldots\alpha}^{k} & 0\ldots 0 & \cdots & 0\ldots 0 \\ 0\ldots 0 & \overbrace{\alpha\ldots\alpha}^{k} & \cdots & 0\ldots 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0\ldots 0 & 0\ldots 0 & \cdots & \overbrace{\alpha\ldots\alpha}^{k} \end{bmatrix},$$

and $M = \beta J$, $\alpha = \sqrt{\frac{m+n-1}{m(n+1)}}$, $\beta = \sqrt{\frac{km(n+1)-2}{km+n+1}}$. Suppose that $\det(\nu I - \mathcal{A}(G)) = 0$. Then we have

$$P(G,\nu) = \nu^{km+n-k}\left(\nu^2 - \frac{m+n-1}{n+1}\right)^{k-1}\left(\nu^2 - \frac{m+n-1}{n+1} + n\left(\frac{mk+n-1}{n+1}\right)\right).$$

This yields that

$$\mathrm{Spec}_{\mathcal{A}}(G) = \left\{\left[\pm\sqrt{\frac{m+n-1}{n+1}}\right]^{k-1}, [0]^{km+n-k}, \left[\pm\sqrt{\frac{m+n-1}{n+1} + n\left(\frac{mk+n-1}{n+1}\right)}\right]^{1}\right\}.$$

Thus

$$\mathcal{E}_{\mathcal{A}}(G) = (2k-2)\sqrt{\frac{m+n-1}{n+1}} + 2\sqrt{\frac{m+n-1}{n+1} + n\left(\frac{mk+n-1}{n+1}\right)}.$$

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