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Garden of Laplacian Borderenergetic Graphs

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Abstract

Let G be a graph of order n. G is said to be L-borderenergetic if its Laplacian energy is the same as the energy of the complete graph K_n, i.e. $LE(G) = 2(n-1)$. In this paper, we construct 36 infinite classes of L-borderenergetic graphs. The L-borderenergetic graphs we construct are composition of the complete graphs and the cycle graphs under the operators join, union and complements. They are non-complete and distinct from the previously known L-borderenergetic graphs.

1 Introduction

A graph G of order n consists of the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges E. If there exists an edge between v_i and v_j for some $i, j = 1, 2, \ldots, n$, it is said that v_i is *adjacent to* v_j. The number of edges connected to v_i is called as the degree of v_i, and it is shown as d_i for $i = 1, 2, \ldots, n$. In this paper, we consider only simple and undirected graphs. A graph with at most one edge between two distinct vertices v_i and v_j, and no edge from v_i to v_i for any $i = 1, 2, \ldots, n$ is called as *simple graph*. A graph with no direction associated with its edge is called as *undirected graph*.

Let G be a graph of order n. The *adjacency matrix* $A(G)$ of G has the entry $a_{ij} = 1$ if v_i is adjacent to v_j, and 0 otherwise for $i, j = 1, 2, \ldots, n$. The diagonal matrix $D(G)$ associated with G is defined as $D(G) = diag(d_1, d_2, \ldots, d_n)$, where d_i is the degree of the vertex v_i of G for $i = 1, 2, \ldots, n$. In this paper, we study the Laplacian matrix associated

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with a graph G. The *Laplacian matrix* $L(G)$ of G is defined as $L(G) = D(G) - A(G)$. The Laplacian matrix has been studied by many researchers, see [22] and references therein.

Let M be a real symmetric matrix associated with a graph G of order n. Let $Spec(M) = \{\lambda_i(M), i = 1, 2, \ldots, n\}$ be the set of eigenvalues of M, that is called as the spectrum of M. Then the M-energy of G is defined as

$$E_M(G) = \sum_{i=1}^{n} \left| \lambda_i(M) - \frac{tr(M)}{n} \right|. \tag{1}$$

For further details on the theory of graph energy, see [13, 19, 23], and for its applications in chemistry, see [18, 19].

Similarly, the Laplacian energy of G, introduced by Gutman and Zhou [14], is given by

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \bar{d} \right|, \tag{2}$$

where μ_i are the Laplacian eigenvalues of G and $\bar{d}$ is the average degree of G. The Laplacian energy is also studied by [1–3, 8–10, 24, 30, 31].

Furthermore, a graph G of order n having energy equal to the energy of a complete graph of order n is called as *borderenergetic graph*, introduced by [12] and its properties are studied by [5, 11, 16, 17, 20, 25, 29]. In particular, Laplacian borderenergetic graphs are first studied by Tura [27] and then in [4, 6, 7, 15, 21, 26, 28]. In other words, a graph is called L-borderenergetic if it satisfies $LE(G) = LE(K_n)$, i.e. $LE(G) = 2n - 2$.

In this paper, we continue in this direction and study L-borderenergetic graphs so that new L-borderenergetic graphs are constructed and proven. Our graphs are non-cospectral to complete graphs and distinct than any previously known L-borderenergetic graphs. We first consider join operator on complete graphs, then construct 8 infinite classes of L-borderenergetic graphs. Secondly, we consider join operator on cycle graphs and construct 2 infinite classes of L-borderenergetic graphs. Similarly, 18 infinite classes of L-borderenergetic graphs are constructed by using union operator. Then, we consider join, union and complement operators together (i.e. mixed operators) to get 8 new classes of L-borderenergetic graphs.

The outline of this paper is follows. In Section 2, some previous results on L-borderenergetic graphs are presented. Then, we present our results on construction of new infinite classes of L-borderenergetic graphs by using join operator, union operator and mixed operators in Sections 3.1, 3.2 and 3.3, respectively. Finally, we give our conclusion in Section 4.

2 Previous results

In this section, we give the known results on Laplacian energy and L-borderenergetic graphs. We begin with two theorems that we use in the proof of our main results in Section 3.

We denote the complete graph of order n with $K_n = (V_{K_n}, E_{K_n})$, which is the graph of n vertices having an edge between all distinct vertices. Let $G = (V, E)$ be a graph of order n with the set of vertices V and E. Then, the complement of G is defined as $\overline{G} = (V, E_{K_n} \backslash E)$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs on n_1 and n_2 vertices, respectively. Then, the union $G = G_1 \cup G_2$ of G_1 and G_2 is defined as $G = (V_1 \cup V_2, E_1 \cup E_2)$. Similarly, the join $G = G_1 \nabla G_2$ of G_1 and G_2 is defined as $G = \overline{\overline{G_1} \cup \overline{G_2}}$. We note that G^n represents the join of n-copies of G, i.e. $G^n = \underbrace{G \nabla G \nabla \cdots \nabla G}_{\text{n-copies}}$. Similarly, the union of n-copies of G is shown as $nG = \underbrace{G \cup G \cup \cdots \cup G}_{\text{n-copies}}$.

The following result gives the Laplacian spectrum of a join of two graphs.

Theorem 1. *[27] Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. Let L_1 and L_2 be the Laplacian matrices for G_1 and G_2, respectively, and let L be the Laplacian matrix for $G_1 \nabla G_2$. If $0 = \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_{n_1}$ and $0 = \beta_1 \leq \beta_2 \leq ... \leq \beta_{n_2}$ are the eigenvalues of L_1 and L_2, respectively. Then the eigenvalues of L are*

$$\{0, n_2 + \alpha_2, n_2 + \alpha_3, ..., n_2 + \alpha_{n_1}, n_1 + \beta_2, n_1 + \beta_3, ..., n_1 + \beta_{n_2}, n_1 + n_2\}.$$

Similarly, the Laplacian spectrum of the complement of a graph can be given as in the following result.

Lemma 2. *[27] Let G be a graph on n vertices with Laplacian matrix L. Let $0 = \mu_1 \leq \mu_2 \leq ... \leq \mu_n$ be the eigenvalues of L. Then the eigenvalues of Laplacian matrix of $\overline{G}$ are*

$$0 \leq n - \mu_n \leq n - \mu_{n-1} \leq n - \mu_{n-2} \leq ... \leq n - \mu_2$$

with the same corresponding eigenvectors.

Tura [27] mainly uses the results given above and presents four classes of disconnected non-complete L-borderenergetic graphs, which are the compositions of complete graphs

K_1 and K_2 under the join and union operators. In the next section, we will improve the results of Tura [27], and give many new classes of L-borderenergetic graphs by using arbitrary complete and cycle graphs under join, union and complement operators.

3 Results

In this part of the paper, our new classes of L-borderenergetic graphs will be presented. In this paper, we consider only complete and cycle graphs to get new classes of L-borderenergetic graphs. We divide our new classes into three subsections. The classes we obtain only by using join operator are given in Section 3.1. Then, the classes obtained by using only union operator are given in Section 3.2. Finally, the new classes consisting of join, union and complement operators are given in Section 3.3.

3.1 Join operator

In this section we construct and prove infinite classes of L-borderenergetic graphs by using join operator on complete graphs and cycle graphs. We separate into two cases: complete graphs and cycle graphs. We begin with a lemma which is needed in the main theorems. We use the notation μ^m to denote the Laplacian eigenvalue μ with the multiplicity equals to m.

Lemma 3. *Let K_1 be the complete graph of order 1 and $t, n \in \mathbb{Z}^+$. Then $(tK_1)^n$ has Laplacian spectrum*

$$\{0, [(n-1)t]^{(t-1)n}, [nt]^{(n-1)}\}.$$

Proof. We will prove by induction on n. We start with the case $n = 1$ and consider the graph (tK_1), which has t many eigenvalues of 0: 0^t. On the other hand, the spectrum $\{0, [(n-1)t]^{(t-1)n}, [nt]^{(n-1)}\}$ for $n = 1$ simplifies to $\{0, 0^{(t-1)}\}$, that is, 0^t. Therefore, it holds for $n = 1$. We assume that the lemma holds for $n = k$, and consider the case $n = k + 1$. The graph $(tK_1)^{k+1} = (tK_1)^k \nabla (tK_1)$ has the spectrum

$$\{0, [kt]^{(t-1)k}, [(k+1)t]^{(k-1)}, [kt]^{(t-1)}, t[k+1]\}$$

by Theorem 1. Then by rearranging the terms we get the specturm $\{0, [kt]^{(t-1)(k+1)}, [(k+1)t]^{k}\}$, as desired. ■

Theorem 4. *Let K_t be the complete graph of order t. Then the graphs of order n_i in the infinite classes Ω_i are Laplacian borderenergetic and L-noncospectral graph with K_{n_i} for $i = 1, 2, \ldots, 8$.*

1. $\Omega_1 = \{G_r = [rK_1 \nabla (r+1)K_1]^2, r = 1, 2, \ldots\}$ *of order* $n_1 = 4r + 2$.
2. $\Omega_2 = \{G_r = K_1 \nabla rK_1 \nabla (r+1)K_1, r = 1, 2, \ldots\}$ *of order* $n_2 = 2r + 2$.
3. $\Omega_3 = \{G_r = K_1^2 \nabla (2K_1)^r, r = 1, 2, \ldots\}$ *of order* $n_3 = 2r + 2$.
4. $\Omega_4 = \{G_r = (r+1)K_1 \nabla (rK_1)^{(r+1)}, r = 1, 2, \ldots\}$ *of order* $n_4 = (r+1)^2$.
5. $\Omega_5 = \{G_r = [K_1 \nabla (2K_1)^r]^2, r = 1, 2, \ldots\}$ *of order* $n_5 = 4r + 2$.
6. $\Omega_6 = \{G_{r,s} = [(r+1)K_2 \nabla [(2r+1)K_1]^s]^2, r, s = 1, 2, \ldots\}$ *of order* $n_6 = [4(r+1) + 2s(2r+1)]$.
7. $\Omega_7 = \{G_{r,s} = [(r+1)K_2]^2 \nabla [(2r+1)K_1]^s, r, s = 1, 2, \ldots\}$ *of order* $n_7 = 4(r+1) + s(2r+1)$.
8. $\Omega_8 = \{G_r = 3K_{r+1} \nabla [2(r+1)K_1]^3, r = 0, 1, 2, \ldots\}$ *of order* $n_8 = 9(r+1)$.

Proof. We will prove (1) and (6), the others follows similarly. Let $G_r \in \Omega_1$ be a graph of order $n_1 = 4r + 2$. Consider $H = [rK_1 \nabla (r+1)K_1]$ such that $G_r = H \nabla H$. We see by using Theorem 1 that the Laplacian spectrum of H is $\{0, (r+1)^{r-1}, r^r, 2r+1\}$. Since H has order $2r+1$, the Laplacian spectrum of G_r is obtained by using Theorem 1 as follows

$$\{0, (3r+2)^{2r-2}, (3r+1)^{2r}, (4r+2)^3\}. \tag{3}$$

Clearly G_r and K_{4r+2} are L-noncospectral. Let $\bar{d}$ be the average degree of G_r. We can find $\bar{d}$ as in the following equation array

$$\bar{d} = \frac{(2r-2)(3r+2) + 2r(3r+1) + 3(4r+2)}{4r+2} = \frac{6r^2 + 6r + 1}{2r+1}.$$

By (3), we have $LE(G_r) = 3[(4r+2) - (\frac{6r^2+6r+1}{2r+1})] + (2r-2)[(3r+2) - (\frac{6r^2+6r+1}{2r+1})] - 2r[(3r+1) - (\frac{6r^2+6r+1}{2r+1})] + \frac{6r^2+6r+1}{2r+1} = \frac{16r^2+12r+2}{2r+1} = \frac{(2r+1)(8r+2)}{2r+1} = 8r + 2 = LE(K_{4r+2})$.

Next, we give the proof of (6). Let $G_r \in \Omega_6$ be a graph of order $n_6 = [4(r+1) + 2s(2r+1)]$. By Lemma 3.1, the graph $[(2r+1)K_1]^s$ has the spectrum

$$\{0, [(s-1)(2r+1)]^{(2r)s}, [s(2r+1)]^{s-1}\}.$$

As the complete graph K_2 has the Laplacian eigenvalues 0 and 2, the spectrum of the graph $H = (r+1)K_2\nabla[(2r+1)K_1]^s$ is obtained by Theorem 1 as follows

$$\{0, [s(2r+1)]^r, [s(2r+1)+2]^{r+1}, [s(2r+1)+1]^{(2r)s}, [2(r+1)+s(2r+1)]^s\}.$$

Note that $G_{r,s} = H\nabla H$. Hence, by applying Theorem 1 to $H\nabla H$, we get the Laplacian spectrum of $G_{r,s}$ as given below:

$$\begin{array}{l} \{0, [2s(2r+1)+2(r+1)]^{2r}, [2s(2r+1)+2r+4]^{2r+2}, \\ \quad [2s(2r+1)+2r+3]^{2s(2r)}, [2s(2r+1)+4(r+1)]^{2s+1}\}. \end{array} \tag{4}$$

Clearly, the spectrum of $G_{r,s}$ shows that it is non-cospectral with a complete graph of order $n_6 = [4(r+1)+2s(2r+1)]$. On the other hand, in order to find the Laplacian energy of $G_{r,s}$ we need to find its average degree $\bar{d}$, that is

$$\begin{aligned} \bar{d} &= \frac{(2r)[2s(2r+1)+2(r+1)]+2s(2r)[2s(2r+1)+2r+3]}{[2s(2r+1)+4(r+1)]} \\ &\quad + \frac{2r[2s(2r+1)+2r+4]+(2s+1)[2s(2r+1)+4(r+1)]}{[2s(2r+1)+4(r+1)]} \\ &= \frac{[2s(2r+1)+4(r+1)][4(r+1)-2s+4s(r+1)-1-2r]}{[2s(2r+1)+4(r+1)]} \\ &= 2s(2r+1)+2r+3. \end{aligned}$$

Hence, by using (4), the Laplacian energy of $G_{r,s}$ is

$$\begin{aligned} LE(G_{r,s}) &= (2s+1)[2s(2r+1)+4(r+1)-2s(2r+1)-2r-3] \\ &\quad +2(r+1)[2s(2r+1)+2r+4-2s(2r+1)-2r-3] \\ &\quad +2s(2r)[2s(2r+1)+2r+3-2s(2r+1)-2r-3] \\ &\quad +(2r)[2s(2r+1)+2r+3-2s(2r+1)-2(r+1)] \\ &\quad +[2s(2r+1)+2r+3-0] \\ &= (2s+1)(2r+1)+2(r+1)+0+2r+2s(2r+1)+2r+3 \\ &= 4s(2r+1)+8r+6 \\ &= LE(K_{[4(r+1)+2s(2r+1)]}). \end{aligned}$$

Therefore, it is proven that any graph in Ω_6 is L-borderenergetic. ■

Remark 5. *In the following we give two examples of certain L-borderenergetic graphs consisting of complete graphs:*

1. $G_1 = 3K_2\nabla 3K_1$ *of order* $n_1 = 9$.

2. $G_2 = 2K_2\nabla 2K_1\nabla 3K_1\nabla 3K_1$ *of order* $n_2 = 12$.

We note that these 2 graphs are not included into any of the classes given in Theorem 4.

In the remaining part of this section we study borderenergetic graphs consisting of the cycle graph C_4.

Lemma 6. *Let $G_r = C_4^r$ be a graph of order $n = 4r$. $C_4^r = C_4 \nabla C_4 \nabla \cdots \nabla C_4$. Then the Laplacian spectrum of C_4^r is given by*

$$0, [4r]^{(2r-1)}, [4r-2]^{2r}.$$

Proof. We will prove by induction. For $r = 1$ that is $G = C_4$, we have $\mathrm{spec}(G) = \{4^1, 2^2, 0\}$. We see that the lemma holds for $r = 1$. Assume that the lemma holds for $r = k$. We assume that

$$\mathrm{spec}(C_4^k) = \{0, [4k]^{(2k-1)}, [4k-2]^{2k}\}$$

holds. Now consider $r = k+1$. We get the spectrum of $C_4^k \nabla C_4$ by [27, Theorem 1] and $n_1 = 4k$, $n_2 = 4$ as follows:

$$\{0, [4k+4]^{(2k-1)}, [4k+2]^{2k}, [4k+4], [4k+2], [4k+2], [4k+4]\}$$

$$= \{0, [4k+4]^{(2k+1)}, [4k+2]^{(2k+2)}\}.$$

We see that the lemma holds for $r = k+1$. Therefore, we are done. ■

We now give some classes of L-borderenergetic graphs by using cycle graph C_4.

Theorem 7. *Let C_t be the cycle graph of order t. Then the graphs of order n_i in the infinite classes Γ_i are Laplacian borderenergetic and L-noncospectral graph with K_{n_i} for $i = 1, 2$.*

1. $\Gamma_1 = \{G_{r,s} = C_4^r \nabla K_2 \nabla (2K_1)^s, r, s = 0, 1, 2, \ldots\}$ *of order* $n_1 = 4r + 2s + 2$.
2. $\Gamma_2 = \{G_{r,s} = [C_4^r \nabla K_1 \nabla (2K_1)^s]^2, r, s = 0, 1, 2, \ldots\}$ *of order* $n_2 = 8r + 4s + 2$.

Proof. We will prove only (1). Proof of (2) can be done similarly. The Laplacian spectrum of C_4^r is given by Lemma 6 as

$$0, [4r]^{(2r-1)}, [4r-2]^{2r}.$$

Then, by [27, Theorem 1] we get

$$\mathrm{spec}(C_4^r \nabla K_2) = \{0, [4r+2]^{(2r-1)}, [4r]^{2r}, [4r+2], [4r+2]\}.$$

Using Lemma 3.1, we know that the Laplacian spectrum of $(2K_1)^s$ is given by

$$0, [2(s-1)]^s, [2s]^{(s-1)}.$$

Then, by [27, Theorem 1] we similarly get

$$\begin{aligned}\operatorname{spec}(C_4^r \nabla K_2 \nabla (2K_1)^s) &= \{0, [4r+2+2s]^{(2r-1)}, [4r+2s]^{2r}, [4r+2+2s]^2, [2(s-1)+4r+2]^s, \\ &\qquad [2s+4r+2]^{(s-1)}, [4r+2+2s]\} \\ &= \{0, [4r+2s+2]^{(2r+s+1)}, [4r+2s]^{(2r+s)}\}.\end{aligned}$$

Since that $\bar{d}$ is equal to average of Laplacian eigenvalues of $G_{r,s}$ then

$$\begin{aligned}\bar{d} &= \frac{(2r+s+1)(4r+2s+2)+(2r+s)(4r+2s)}{4r+2s+2} \\ &= \frac{(2r+s+1)^2+(2r+s)^2}{2r+s+1}.\end{aligned}$$

Using the spectrum of $G_{r,s}$, we get

$$\begin{aligned}LE(G_{r,s}) &= \frac{(2r+s+1)^2+(2r+s)^2}{2r+s+1} \\ &\quad +(2r+s+1)[4r+2s+2-\frac{(2r+s+1)^2+(2r+s)^2}{2r+s+1}] \\ &\quad +(2r+s)[\frac{(2r+s+1)^2+(2r+s)^2}{2r+s+1}-(4r+2s)] \\ &= \frac{(2r+s+1)^2+(2r+s)^2}{2r+s+1} \\ &\quad +2(2r+s+1)^2-(2r+s+1)^2-(2r+s)^2 \\ &\quad +(2r+s)[\frac{(2r+s+1)^2+(2r+s)^2}{2r+s+1}-2(2r+s)^2] \\ &= \frac{(2r+s+1)^2+(2r+s)^2}{2r+s+1}(2r+s+1) \\ &\quad +(2r+s+1)^2-3(2r+s)^2 \\ &= (2r+s+1)^2+(2r+s)^2+(2r+s+1)^2-3(2r+s)^2 \\ &= 2(2r+s+1)^2-2(2r+s)^2 \\ &= 2(4r+2s+1) \\ &= 8r+4s+2 \\ &= LE(K_{4r+2s+2}).\end{aligned}$$

Therefore we prove that Γ_1 is an infinite class of L-borderenergetic graphs. ■

Remark 8. *In the following we give two examples of cycles that are L-borderenergetic graph.*

1. $G_1 = C_4 \nabla 3K_1 \nabla 3K_1$ *of order* $n_1 = 10$.
2. $G_2 = C_4 \nabla 3K_1 \nabla 2K_1$ *of order* $n_2 = 9$.

We note that these 2 graphs are not included into any of the classes given in Theorem 7.

3.2 Union operator

In this section, we give new infinite classes of L-borderenergetic graphs consisting of complete and cycle graphs under union operator. We only consider the graphs in the form $r_1G_1 \cup r_2G_2 \cup r_3G_3$ for some graphs G_1, G_2, G_3 and non-negative integers r_1, r_2, r_3. In other words, the union of 4 or more distinct graphs is not considered in this paper.

We note that the Laplacian spectrum of the complete graph K_t of order t is $\{0, t^{t-1}\}$.

Theorem 9. *Let K_t be the complete graph of order t. Then the graphs of order n_i in the infinite classes Φ_i are Laplacian borderenergetic and L-noncospectral graph with K_{n_i} for $i = 1, 2, \ldots, 18$.*

1. $\Phi_1 = \{G_r = \frac{r(3r-1)}{2}K_4 \cup r(3r+2)K_1, r = 1, 2, \ldots\}$ *of order* $n_1 = 9r^2$.
2. $\Phi_2 = \{G_r = \frac{r(3r+1)}{2}K_4 \cup (r+1)(3r+1)K_1, r = 1, 2, \ldots\}$ *of order* $n_2 = (3r+1)^2$.
3. $\Phi_3 = \{G_r = r(3r-2)K_1 \cup \frac{r(3r+1)}{2}K_4, r = 1, 2, \ldots\}$ *of order* $n_3 = 9r^2$.
4. $\Phi_4 = \{G_r = r(3r+2)K_1 \cup \frac{(r+1)(3r+2)}{2}K_4, r = 1, 2, \ldots\}$ *of order* $n_4 = (3r+2)^2$.
5. $\Phi_5 = \{G_r = (2r-1)K_{r+1} \cup K_{2r+2}, r = 1, 2, \ldots\}$ *of order* $n_5 = (r+1)(2r+1)$.
6. $\Phi_6 = \{G_r = K_r \cup K_{r+2}, r = 1, 2, \ldots\}$ *of order* $n_6 = 2r+2$.
7. $\Phi_7 = \{G_r = (2r+1)K_{r-1} \cup K_{2r}, r = 1, 2, \ldots\}$ *of order* $n_7 = r(2r+1) - 1$.
8. $\Phi_8 = \{G_r = rK_{2(r+1)} \cup (r+1)K_{2(r+2)}, r = 1, 2, \ldots\}$ *of order* $n_8 = 4(r+1)^2$.
9. $\Phi_9 = \{G_{r,s} = [(s+2)r - 1]K_1 \cup [\frac{s(s+3)}{2}r + 1]K_2 \cup rK_{s+4}, r, s = 1, 2, \ldots\}$ *of order* $n_9 = r(s+2)(s+3) + 1$.
10. $\Phi_{10} = \{G_{r,s} = [(s+2)r]K_1 \cup [\frac{s(s+3)}{2}r + 1]K_2 \cup rK_{s+4}, r, s = 1, 2, \ldots\}$ *of order* $n_{10} = r(s+2)(s+3) + 2$.
11. $\Phi_{11} = \{G_{r,s} = [\frac{(s+1)(s+4)}{2}r]K_1 \cup [\frac{s(s+3)}{2}r + 1]K_3 \cup rK_{s+4}, r, s = 1, 2, \ldots\}$ *of order* $n_{11} = 2r(s+1)(s+3) + 3$.
12. $\Phi_{12} = \{G_r = 4rK_1 \cup (5r+1)K_4 \cup rK_6, r = 1, 2, \ldots\}$ *of order* $n_{12} = 30r + 4$.
13. $\Phi_{13} = \{G_r = (4r+1)K_1 \cup (5r+2)K_4 \cup rK_1, r = 1, 2, \ldots\}$ *of order* $n_{13} = 25r + 9$.
14. $\Phi_{14} = \{G_r = 5rK_1 \cup (4r+1)K_4 \cup 3rK_5, r = 1, 2, \ldots\}$ *of order* $n_{14} = 36r + 4$.

15. $\Phi_{15} = \{G_r = (5r+1)K_1 \cup (4r+2)K_4 \cup 3rK_5, r = 1, 2, \ldots\}$ *of order* $n_{15} = 36r + 9$.

16. $\Phi_{16} = \{G_r = 2rK_1 \cup (5r+1)K_4 \cup 2rK_7, r = 1, 2, \ldots\}$ *of order* $n_{16} = 36r + 4$.

17. $\Phi_{17} = \{G_r = 7rK_1 \cup (10r+1)K_4 \cup rK_7, r = 1, 2, \ldots\}$ *of order* $n_{17} = 54r + 4$.

18. $\Phi_{18} = \{G_r = (7r+1)K_1 \cup (10r+2)K_4 \cup rK_7, r = 1, 2, \ldots\}$ *of order* $n_{18} = 54r + 9$.

Proof. We will prove only (1). Proof of others can be done similarly. Let $G_r \in \Phi_1$ be a graph of order $n_1 = 9r^2$. Then the Laplacian spectrum of G_r is given by

$$4^{\frac{3r(3r-1)}{2}}, 0^{\frac{r(3r-1)+2r(3r+2)}{2}}.$$

Let $\bar{d}$ be the average degree of G_r. Since $\bar{d}$ is equal to the average of Laplacian eigenvalues of G_r, we have

$$\bar{d} = \frac{\frac{3r(3r-1)4}{2} + \frac{0[r(3r-1)+2r(3r+2)]}{2}}{9r^2} = \frac{6r-2}{3r}.$$

Using the spectrum of G_r, we get

$$\begin{aligned} LE(G_r) &= \tfrac{3r(3r-1)}{2}(4 - \tfrac{6r-2}{3r}) + \tfrac{r(3r-1)+2r(3r+2)}{2}\tfrac{6r-2}{3r} \\ &= \tfrac{(3r-1)(6r+2)}{2} + \tfrac{(9r+3)(3r-1)}{3} \\ &= 2(3r+1)(3r-1) \\ &= 18r^2 - 2 \\ &= LE(K_{9r^2}). \end{aligned}$$

Therefore we prove that Φ_1 is an infinite class of L-borderenergetic graphs. ■

3.3 Mixed operators

In this section, we consider join, union and complement operators to get new infinite classes of L-borderenergetic graphs consisting of complete and cycle graph.

Theorem 10. *Let K_t be the complete graph of order t. Then the graphs of order n_i in the infinite classes Λ_i are Laplacian borderenergetic and L-noncospectral graph with K_{n_i} for $i = 1, 2, 3, 4$.*

1. $\Lambda_1 = \{G_r = [rK_1 \cup (rK_1 \nabla K_1)]^2, r = 1, 2, \ldots\}$ *of order* $n_1 = 4r + 2$.

2. $\Lambda_2 = \{G_r = [rK_2 \cup (K_1 \nabla 2K_1)]^2, r = 1, 2, \ldots\}$ *of order* $n_2 = 4r + 6$.

3. $\Lambda_3 = \{G_r = [K_{r+1} \cup (rK_1 \nabla rK_1)]^2, r = 1, 2, \ldots\}$ *of order* $n_3 = 6r + 2$.

4. $\Lambda_4 = \{G_r = [K_r \cup (rK_1 \nabla (r+1)K_1)]^2, r = 1, 2, \ldots\}$ *of order* $n_4 = 6r + 2$.

Proof. We will prove only (1), others can be done similarly. Let $G_r \in \Lambda_1$ be a graph of order $n_1 = 4r+2$. Consider $H = [rK_1 \cup (rK_1 \nabla K_1)]$ such that $G_r = H \nabla H$. We see by using Theorem 1 that the Laplacian spectrum of H is $\{0^{r+1}, 1^{r-1}, r+1\}$. Since H has order $2r+1$, the Laplacian spectrum of G_r is obtained by using Theorem 1 as follows

$$\{0, (2r+1)^{2r}, (2r+2)^{2r-2}, (3r+2)^2, 4r+2\}. \tag{5}$$

Clearly G_r and K_{4r+2} are L-noncospectral. Let $\bar{d}$ be the average degree of G_r. We can find $\bar{d}$ as in the following equation array

$$\bar{d} = \frac{2r(2r+1) + (2r-2)(2r+2) + 2(3r+2) + 4r+2}{4r+2} = \frac{4r^2+6r+1}{2r+1}.$$

By (5), we have $LE(G_r) = \frac{4r^2+6r+1}{2r+1} + 2r[\frac{4r^2+6r+1}{2r+1} - (2r+1)] + (2r-2)[2r+2-(\frac{4r^2+6r+1}{2r+1})] + 2[3r+2-(\frac{4r^2+6r+1}{2r+1})] + [4r+2-(\frac{4r^2+6r+1}{2r+1})] = \frac{16r^2+12r+2}{2r+1} = \frac{(2r+1)(8r+2)}{2r+1} = 8r+2 = LE(K_{4r+2})$. ■

Next theorem presents new Laplacian borderenergetic graphs based on cycle graph C_4 and complete graphs K_1 and K_2.

Theorem 11. *Let C_4 be the cycle graph of order 4. Then the graphs of order n_i in the infinite classes Υ_i are Laplacian borderenergetic and L-noncospectral graph with K_{n_i} for $i = 1, 2, 3, 4$.*

1. $\Upsilon_1 = \{G_r = (r+1)C_4 \nabla (2K_2 \cup rC_4), r = 0, 1, 2, \ldots\}$ *of order* $n_1 = 8r+8$.
2. $\Upsilon_2 = \{G_r = C_4 \nabla K_1 \nabla (\overline{K_1 \cup rK_2}), r = 0, 1, 2, \ldots\}$ *of order* $n_2 = 2r+6$.
3. $\Upsilon_3 = \{G_r = C_4 \nabla K_1 \nabla (\overline{2K_1 \cup rK_2}), r = 0, 1, 2, \ldots\}$ *of order* $n_3 = 2r+7$.
4. $\Upsilon_4 = \{G_r = C_4 \nabla C_4 \nabla (\overline{2K_1 \cup rK_2}), r = 0, 1, 2, \ldots\}$ *of order* $n_4 = 2r+10$.

Proof. We will prove only (1), others follow similarly. Let $G_r = [(r+1)C_4 \nabla (2K_2 \cup rC_4)] \in \Upsilon_1$ be a graph of order $n_1 = 8r+8$. Let H_1 and H_2 be graphs such that $H_1 = (r+1)C_4$ and $H_2 = (2K_2 \cup rC_4)$. Then we have $G_r = H_1 \nabla H_2$. Note that $\{0^{r+1}, 2^{2r+2}, 4^{r+1}\}$ and $\{0^{r+2}, 2^{2r+2}, 4^r\}$ are the Laplacian spectrums of H_1 and H_2, respectively. Since G_r has order $8r+8$, the Laplacian spectrum of G_r is obtained by using Theorem 1 as follows

$$\{0, (4r+4)^{2r+1}, (4r+6)^{4r+4}, (4r+8)^{2r+1}, 8r+8\}. \tag{6}$$

Clearly, G_r and K_{8r+8} are L-noncospectral. Let $\bar{d}$ be the average degree of G_r. We can find $\bar{d}$ as in the following equation array

$$\begin{aligned}\bar{d} &= \frac{(2r+1)(4r+8)+(4r+4)(4r+6)+(2r+1)(4r+4)+8r+8}{8r+8}\\ &= \frac{8r^2+20r+11}{2r+2}.\end{aligned}$$

By (6), we have $LE(G_r) = \frac{8r^2+20r+11}{2r+2} + (2r+1)[4r+8-(\frac{8r^2+20r+11}{2r+2})] + (4r+4)[4r+6-(\frac{8r^2+20r+11}{2r+2})] + (2r+1)[\frac{8r^2+20r+11}{2r+2}-(4r+4)] + [8r+8-(\frac{8r^2+20r+11}{2r+2})] = \frac{16r^2+30r+14}{r+1} = \frac{(16r+14)(r+1)}{r+1} = 16r+14 = LE(K_{8r+8})$. ■

4 Conclusion

In this paper, we presented new classes of L-borderenergetic graphs. Our classes are noncospectral with a complete graph and distinct from any known graphs in the literature. We constructed totally 36 classes, each of which consist of infinitely many graphs. Our classes are composition of complete graphs and cycle graphs under the operators union, join and complement. It would be a good future work to find new infinite classes by using different operators on different graph families.

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