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Laplacian Borderenergetic Graphs and Their Complements *

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Abstract

A graph G of order n is *(Laplacian) borderenergetic* if it has the same *(Laplacian) energy* as the complete graph K_n. Recently, Deng and Li showed that for any graph G, except for three graphs, at most one of G and its complement $\overline{G}$ can be a borderenergetic graph. In this paper, we will show that for any graph G, except for four graphs, at most one of G and its complement $\overline{G}$ can be a Laplacian borderenergetic graph. In addition, several bounds on the Laplacian energy of the complement of a Laplacian borderenergetic graph are obtained by using Nordhaus-Gaddum-type results.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let G be a graph of order n and size m. The complement of G is denoted by $\overline{G}$. The complete graph of order n is denoted by K_n. The degree of a vertex v_i in G is denoted by d_i. The maximum

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degree of G is denoted by $\Delta(G)$. The auxiliary quantity $M(G)$ of G was defined in [18] as follows:

$$M(G) = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2.$$

Let $A(G)$ be the adjacency matrix of G and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix $A(G)$. Let $D(G)$ denote the diagonal matrix of the vertex-degrees of G. Then $L(G) = D(G) - A(G)$ is defined as the Laplacian matrix of G. The spectrum of $L(G)$ is composed of $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$, which are the eigenvalues of $L(G)$.

The *energy of a graph* G, denoted by $E(G)$, is defined in [12, 13] as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For additional information on graph energy and its applications in chemistry, we refer to [13, 16, 17, 24].

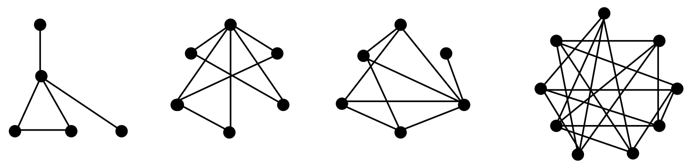
In 2015, Gong et al. [11] proposed the concept of *borderenergetic graphs*, namely graphs of order n satisfying $E(G) = 2(n-1)$. Related results on borderenergetic graphs can be found in [4, 7–9, 20, 25–27]. Actually, similar topics on the energy of graphs have been studied in [1, 14, 15, 21–23, 28, 29].

Analogously, for the Laplacian energy of a graph G [18], the concept of *Laplacian borderenergetic graphs* was proposed by Tura in [29], that is, a graph G of order n is Laplacian borderenergetic, or L-borderenergetic for short, if $LE(G) = LE(K_n)$, where $LE(G)$ is the Laplacian energy of G defined as $LE(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|$ and $\overline{d}$ is the average degree of G. Some classes of L-borderenergetic graphs of order $n = 4r + 4$ $(r \geq 1)$ and a kind of threshold L-borderenergetic graphs were obtained in [29] and [5], respectively. So far, there are few results on the structures of L-borderenergetic graphs.

Recently, Deng and Li in [6] showed that for any graph G, except for three graphs (one of order 9 and two of order 11), at most one of G and its complement $\overline{G}$ can be a borderenergetic graph. Interestingly, in this paper, we can show that for any graph G, except for four graphs, at most one of G and its complement $\overline{G}$ can be an L-borderenergetic graph. The four graphs (one of order 5, two of order 6 and one of order 9) are depicted in Figure 1. One can check that each of the four graphs possesses the property that both itself and its complement are L-borderenergetic. The corresponding complements are presented in Figure 2. The Laplacian spectra of the four graphs and their complements are given, respectively, as follows.

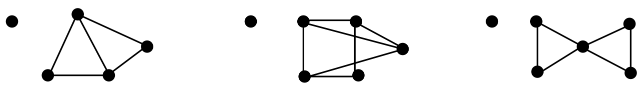
$$\begin{aligned}
LS_p(G_5^1) &= \{5,3,1,1,0\};\\
LS_p(G_6^2) &= \{6,4,3,2,1,0\};\\
LS_p(G_6^3) &= \{6,5,3,3,1,0\};\\
LS_p(G_9^4) &= \{6,6,6,5,5,3,3,2,0\};\\
LS_p(\overline{G}_5^1) &= \{4,4,2,0,0\};\\
LS_p(\overline{G}_6^2) &= \{5,4,3,2,0,0\};\\
LS_p(\overline{G}_6^3) &= \{5,3,3,1,0,0\};\\
LS_p(\overline{G}_9^4) &= \{6,6,6,5,5,3,3,2,0\}.
\end{aligned}$$

Especially, we can easily check that the graph G_9^4 is self-complementary, that is, $G_9^4 \cong \overline{G}_9^4$. In addition, several bounds on the Laplacian energy of the complement of an L-borderenergetic graph are obtained by using Nordhaus-Gaddum-type results.



G_5^1 $\quad$ G_6^2 $\quad$ G_6^3 $\quad$ G_9^4

Figure 1. The L-borderenergetic graphs: G_5^1, G_6^2, G_6^3 and G_9^4.



$\overline{G}_5^1$ $\quad$ $\overline{G}_6^2$ $\quad$ $\overline{G}_6^3$

Figure 2. The complements of graphs G_5^1, G_6^2 and G_6^3.

2 L-borderenergetic graphs and their complements

Our main result of this section is presented below.

Theorem 2.1. *For any connected noncomplete graph G, except for the four graphs G_5^1, G_6^2, G_6^3 and G_9^4, at most one of G and its complement $\overline{G}$ can be L-borderenergetic.*

Proof. For the cases of $4 \leq n \leq 11$, all the L-borderenergetic graphs have been found in [5,28]. Let G be an L-borderenergetic graph with order $n \geq 12$. By contradiction, suppose $\overline{G}$ is also an L-borderenergetic graph, i.e., $LE(\overline{G}) = 2(n-1)$. By $LE(\overline{G}) \geq 2\sqrt{M(\overline{G})}$, we have $2(n-1) \geq 2\sqrt{M(\overline{G})}$ and

$$(n-1)^2 \geq M(\overline{G}). \tag{1}$$

Let $\overline{m}$ be the size of $\overline{G}$. The degree of a vertex v_i in $\overline{G}$ is denoted by $\overline{d_i}$. By investigating the quality $M(\overline{G})$, we obtain

$$\begin{aligned}
M(\overline{G}) &= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}\left(\overline{d_i} - \frac{2\overline{m}}{n}\right)^2 \\
&= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}\left(\overline{d_i}^2 - \frac{4\overline{m}}{n}\overline{d_i} + \frac{4\overline{m}^2}{n^2}\right) \\
&= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}\overline{d_i}^2 - \frac{2\overline{m}}{n}\sum_{i=1}^{n}\overline{d_i} + \frac{2\overline{m}^2}{n} \\
&= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}(n-1-d_i)^2 - \frac{2\overline{m}}{n}\sum_{i=1}^{n}(n-1-d_i) + \frac{2\overline{m}^2}{n} \\
&= \overline{m} + \frac{1}{2}\sum_{i=1}^{n}\left[(n-1)^2 - 2(n-1)d_i + d_i^2\right] - 2\overline{m}(n-1) + \frac{2\overline{m}}{n}\sum_{i=1}^{n}d_i + \frac{2\overline{m}^2}{n} \\
&= \frac{1}{2}n(n-1) - m + \frac{1}{2}n(n-1)^2 - 2m(n-1) + \frac{1}{2}\sum_{i=1}^{n}d_i^2 - 2(n-1)\left[\frac{1}{2}n(n-1) - m\right] \\
&\quad + \frac{4m}{n}\left[\frac{1}{2}n(n-1) - m\right] + \frac{2}{n}\left[\frac{1}{2}n(n-1) - m\right]^2 \\
&= -\frac{2}{n}m^2 - m + \frac{1}{2}\sum_{i=1}^{n}d_i^2 + \frac{1}{2}n(n-1).
\end{aligned}$$

Assume $\sum_{i=1}^{n} d_i^2 = x$. Then

$$M(\overline{G}) = -\frac{2}{n}m^2 - m + \frac{1}{2}x + \frac{1}{2}n(n-1). \tag{2}$$

Combining (1) with (2), it arrives at

$$\begin{aligned}
(n-1)^2 - M(\overline{G}) &= n^2 - 2n + 1 + \frac{2}{n}m^2 + m - \frac{1}{2}x - \frac{1}{2}n^2 + \frac{1}{2}n \\
&= \frac{2}{n}m^2 + m - \frac{1}{2}x + \frac{1}{2}n^2 - \frac{3}{2}n + 1 \geq 0.
\end{aligned} \tag{3}$$

Since $\sum_{i=1}^{n} d_i = 2m$, we get

$$\left(\sum_{i=1}^{n} d_i\right)^2 = 4m^2 = \sum_{i=1}^{n} d_i^2 + 2\sum_{i \neq j} d_i d_j \leq x + n(n-1)^3 = x + n^4 - 3n^3 + 3n^2 - n.$$

Thus,

$$m \leq \sqrt{\frac{1}{4}x + \frac{n^4 - 3n^3 + 3n^2 - n}{4}}.$$

By (3), we obtain

$$\frac{2}{n}\left(\frac{1}{4}x+\frac{n^4-3n^3+3n^2-n}{4}\right)+\sqrt{\frac{1}{4}x+\frac{n^4-3n^3+3n^2-n}{4}}-\frac{1}{2}x+\frac{1}{2}n^2-\frac{3}{2}n+1\geq 0.$$

So,

$$\begin{aligned}&\frac{(n-1)^2}{4n^2}x^2-\frac{2n^4-6n^3+4n^2+3n-2}{4n}x\\&+\frac{n^6-4n^5+3n^4+5n^3-7n^2+n+1}{4}\leq 0.\end{aligned}$$

The left expression of the above inequality can be seen as a function with variable x, i.e., $f(x)$. Then the above inequality can be written as

$$f(x)\leq 0.$$

Obviously, we can see that the discriminant Δ of the quadratic equation $f(x)=0$ satisfies

$$\Delta=\frac{4n^6-16n^5+28n^4-32n^3+25n^2-8n}{16n^2}>0,$$

which implies that there are solutions for the inequality $f(x)\leq 0$. Let $x_1<x_2$ be two roots of the equation $f(x)=0$, possessing that $x_1\leq x\leq x_2$. It is not hard to find that

$$x_1=\frac{2n^5-6n^4+4n^3+3n^2-2n}{2(n-1)^2}-\frac{2n^2}{(n-1)^2}\sqrt{\Delta},$$

$$x_2=\frac{2n^5-6n^4+4n^3+3n^2-2n}{2(n-1)^2}+\frac{2n^2}{(n-1)^2}\sqrt{\Delta}.$$

If $\Delta(G)\leq n-2$, then $x\leq n(n-2)^2$. As $n\geq 12$, we have

$$\begin{aligned}x_1-n(n-2)^2&=\frac{2n^5-6n^4+4n^3+3n^2-2n}{2(n-1)^2}-\frac{2n^2}{(n-1)^2}\sqrt{\Delta}-n(n-2)^2\\&=\frac{n\left(-10+27n-22n^2+6n^3-4n\sqrt{\Delta}\right)}{2(n-1)^2}>0,\end{aligned}$$

which is a contradiction with $x_1\leq x\leq n(n-2)^2$.

Next, we consider the case of $\Delta(G)=n-1$. Note that the number of vertices with degree equal to $n-1$ is at most $n-2$. Thus, we have $x\leq(n-2)(n-1)^2+2(n-2)^2$. Combining $(n-2)(n-1)^2+2(n-2)^2<x_2$, we get

$$x\leq(n-2)(n-1)^2+2(n-2)^2<x_2,$$

and

$$f((n-2)(n-1)^2+2(n-2)^2)\leq 0. \tag{4}$$

But for $n \geq 12$, it holds that

$$f((n-2)(n-1)^2 + 2(n-2)^2) = \frac{2n^5 - 8n^4 + 35n^2 - 48n + 18}{2n^2} > 0,$$

which creates a contradiction with (4). The proof is thus complete. ■

3 Bounds on the Laplacian energy of the complement of an L-borderenergetic graph

In this section, several upper bounds on the Laplacian energy of the complement of an L-borderenergetic graph are given.

Theorem 3.1. *[30] Let G be a graph with n vertices. Then*

$$LE(G) + LE(\overline{G}) < n\sqrt{n^2 - 1}.$$

Immediately, we get

Corollary 3.2. *If G is an L-borderenergetic graph with n vertices. Then*

$$LE(\overline{G}) < n(\sqrt{n^2 - 1} - 2) + 2.$$

Proof. If G is an L-borderenergetic graph with n vertices, then $LE(G) = 2(n-1)$. By Theorem 3.1, the result follows directly. ■

Lemma 3.3. *[3] Let G be a bipartite graph of order n and size m. Then*

$$LE(G) \leq \frac{4m}{n} + \sqrt{(n-2)\left(2M(G) - \frac{8m^2}{n^2}\right)}\ .$$

Suppose $\Delta_0 = max\{\Delta(G), \Delta(\overline{G})$. By Lemma 3.3, a Nordhaus-Gaddum-Type bound for the Laplacian energy is obtained.

Theorem 3.4. *Let G be a bipartite graph of order n and size m. Then*

$$LE(G) + LE(\overline{G}) < 2(n-1) + \sqrt{2}\sqrt{n-2}\sqrt{n\left(n - 1 + 2\Delta_0^2\right) - 2n - 2 + \frac{4}{n}}.$$

Proof. By calculating $M(G)+M(\overline{G})$, we obtain

$$\begin{aligned}
M(G)+M(\overline{G}) &= m+\frac{1}{2}\sum_{i=1}^{n} d_i^2-\frac{2m^2}{n}+\overline{m}+\frac{1}{2}\sum_{i=1}^{n}\overline{d_i}^2-\frac{2\overline{m}^2}{n} \\
&= m+\overline{m}+\frac{1}{2}\sum_{i=1}^{n} d_i^2+\frac{1}{2}\sum_{i=1}^{n}\overline{d_i}^2-\frac{2}{n}(m^2+\overline{m}^2) \\
&\leq \frac{1}{2}n(n-1)+\frac{1}{2}n(\Delta^2+\overline{\Delta}^2)-\frac{2}{n}(m^2+\overline{m}^2) \\
&< \frac{1}{2}n(n-1)+\frac{1}{2}n(\Delta^2+\overline{\Delta}^2)-\frac{2}{n}(m+\overline{m}) \\
&< \frac{1}{2}n(n-1+2\Delta_0^2)-n+1.
\end{aligned}$$

Then by Lemma 3.3, we have

$$\begin{aligned}
LE(G)+LE(\overline{G}) &\leq \frac{4m}{n}+\sqrt{(n-2)\left(2M(G)-\frac{8m^2}{n^2}\right)}+\frac{4\overline{m}}{n}+\sqrt{(n-2)\left(2M(\overline{G})-\frac{8\overline{m}^2}{n^2}\right)} \\
&= 2(n-1)+\sqrt{n-2}\left(\sqrt{2M(G)-\frac{8m^2}{n^2}}+\sqrt{2M(\overline{G})-\frac{8\overline{m}^2}{n^2}}\right) \\
&\leq 2(n-1)+\sqrt{2}\sqrt{n-2}\sqrt{2M(G)+2M(\overline{G})-\frac{8(m^2+\overline{m}^2)}{n^2}} \\
&\leq 2(n-1)+\sqrt{2}\sqrt{n-2}\sqrt{2M(G)+2M(\overline{G})-\frac{8(m+\overline{m})}{n^2}} \\
&= 2(n-1)+\sqrt{2}\sqrt{n-2}\sqrt{2M(G)+2M(\overline{G})-\frac{4(n-1)}{n}} \\
&< 2(n-1)+\sqrt{2}\sqrt{n-2}\sqrt{n(n-1+2\Delta_0^2)-2n-2+\frac{4}{n}}.
\end{aligned}$$

■

From Theorem 3.4, we can directly get

Corollary 3.5. *If G is an L-borderenergetic bipartite graph of order n and size m, then*

$$LE(\overline{G}) < \sqrt{2}\sqrt{n-2}\sqrt{n(n-1+2\Delta_0^2)-2n-2+\frac{4}{n}}.$$

In the case of regular graphs, the upper bounds in Theory 3.1 and Corollary 3.2 can be improved.

Theorem 3.6. *Let G be an r-regular graph with n vertices. Then*

$$LE(G)+LE(\overline{G}) \leq \sqrt{2n^2(n-1)}.$$

Proof. Denote by $d_1, d_2, \cdots, d_n$ the vertex-degrees of G. Then $d_i = r$ $(i = 1, 2, \cdots, n)$ and

$$\begin{aligned}
M(G) + M(\overline{G}) &= m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2 + \overline{m} + \frac{1}{2}\sum_{i=1}^{n}\left(\overline{d_i} - \frac{2\overline{m}}{n}\right)^2 \\
&= \frac{1}{2}n(n-1) + \frac{1}{2}\sum_{i=1}^{n} d_i^2 - \frac{2m}{n}\sum_{i=1}^{n} d_i + \frac{2m^2}{n} + \frac{1}{2}\sum_{i=1}^{n}\overline{d_i}^2 - \frac{2\overline{m}}{n}\sum_{i=1}^{n}\overline{d_i} + \frac{2\overline{m}^2}{n} \\
&= \frac{1}{2}n(n-1) + \frac{1}{2}nr^2 - \frac{2m}{n}nr + \frac{2m^2}{n} + \frac{1}{2}n(n-1-r)^2 \\
&\quad -\frac{2}{n}\left[\frac{1}{2}n(n-1) - m\right]n(n-1-r) + \frac{2}{n}\left[\frac{1}{2}n(n-1) - m\right]^2 \\
&= \frac{4m^2}{n} + nr^2 + \frac{1}{2}n(n-1) - 4mr.
\end{aligned}$$

Due to $m = \frac{nr}{2}$, we have

$$M(G) + M(\overline{G}) = \frac{4m^2}{n} + nr^2 + \frac{1}{2}n(n-1) - 4mr = \frac{1}{2}n(n-1).$$

As $LE(G) \leq \sqrt{2nM}$, we obtain

$$\begin{aligned}
LE(G) + LE(\overline{G}) &\leq \sqrt{2nM(G)} + \sqrt{2nM(\overline{G})} \\
&= \sqrt{2n}\left(\sqrt{M(G)} + \sqrt{M(\overline{G})}\right) \\
&\leq \sqrt{2}\sqrt{2n}\sqrt{M(G) + M(\overline{G})} \\
&= \sqrt{2n^2(n-1)}.
\end{aligned}$$

■

From Theorem 3.6, we get

Theorem 3.7. *Let G be an r-regular L-borderenergetic graph with n vertices. Then*

$$LE(\overline{G}) \leq \sqrt{2n^2(n-1)} - 2(n-1).$$

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