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Some Spectral Characterizations of Equienergetic Regular Graphs and Their Complements

Irena M. Jovanović[1], Emir Zogić[2]

[1]*School of Computing, Union University, Belgrade, Serbia*

[2]*State University of Novi Pazar, Serbia*

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Abstract

The energy $E(G)$ of a graph G is defined as $E(G) = \sum\limits_{i=1}^{n} |\lambda_i(G)|$, where $\lambda_i(G)$, for $i = 1, 2, \ldots, n$, are the adjacency eigenvalues of G. Two graphs with the same number of vertices are said to be equienergetic if they have the same energy. The spectral distance $\sigma(G_1, G_2)$ of two non-isomorphic graphs G_1 and G_2 of order n, is $\sigma(G_1, G_2) = \sum_{i=1}^{n} |\lambda_i(G_1) - \lambda_i(G_2)|$. In [H. S. Ramane, B. Parvathalu, D. D. Patil, K. Ashoka, Graphs Equienergetic with Their Complements, *MATCH Commun. Math. Comput. Chem.* **82** (2019) 471–480], the authors asked about spectral properties of graphs which are equienergetic with their complements. Using spectral distances of graphs, we give a necessary and sufficient condition for a regular graph to have the energy equal to the energy of its complement. Based on this result, strongly regular graphs equienergetic with their complements are characterized. A spectral property that two equienergetic regular graphs should possess in order for their complements to have equal energies is stated. Equienergetic regular graphs with respect to some graph operations are considered by spectral means, as well.

1 Introduction

Let G be a simple graph of order n, and let $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$ be the adjacency matrix of G. The *characteristic polynomial* $P_G(x) = \det(A - x\,I)$ of G is the characteristic polynomial of its adjacency matrix A, while the *(adjacency) eigenvalues* $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ of G are the eigenvalues of A. These eigenvalues form the *spectrum* of G. If $\lambda_i(G)$, for some i, is the eigenvalue of the multiplicity k, we will write $[\lambda_i(G)]^k$.

The *energy* $E(G)$ of G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

This graph invariant was introduced by I. Gutman in 1978 in his paper [11], and for the last twenty years it has been the subject of intensive research, especially in the field of mathematics and chemistry. For more details about graph energy, the reader is referred to the monographs [15] and [23], and the review papers [13] and [14].

If G_1 and G_2 are two non-isomorphic graphs on n vertices, then the *spectral distance* of G_1 and G_2 regarding the adjacency matrix, is [34]:

$$\sigma(G_1, G_2) = \sum_{i=1}^{n} |\lambda_i(G_1) - \lambda_i(G_2)|.$$

Certain results related to the spectral distances of graphs connected with the adjacency, the Laplacian and the signless Laplacian matrix, can be found in [1], [2], [19], [20] and [21].

The notation common for spectral graph theory (see, for example, [7] and [8]) is used in the paper. In that way, K_n is the complete graph on n vertices, K_{n_1,n_2} is the complete bipartite graph of order $n_1 + n_2$, while $K_{n_1,n_2,\ldots,n_p}$ is the complete multipartite graph on p parts and n_i, $i = 1, 2, \ldots, p$ vertices in each of them. But, if $n_1 = n_2 = \cdots = n_p = k$, we will use the label $K_{p\times k}$. By $k\,G$ we denote the disjoint union of k copies of a graph G, while the complement of G will be marked by $\overline{G}$. If G is isomorphic to its complement, we say that G is *self-complementary graph.*

The *line graph* $L(G)$ of a graph G is the graph whose vertices are the edges of G, with two vertices in $L(G)$ adjacent whenever the corresponding edges in G have exactly one vertex in common. The *iterated line graphs* of a graph G are defined recursively as: $L^2(G) = L(L(G)), L^3(G) = L(L^2(G)), \ldots, L^k(G) = L(L^{k-1}(G)), \ldots$. It is assumed that $L^0(G) \equiv G$ and $L^1(G) \equiv L(G)$.

Two graphs with the same number of vertices are said to be *equienergetic* if they have the same energy. Since it is obvious that two isomorphic or two cospectral (i.e. with the same spectra) graphs are equienergetic, it is of interest to consider only non-isomorphic and non-cospectral graphs. The concept of equienergetic graphs was put forward independently by Brankov et al. [5] and Balakrishnan [4], and since then, there are many published papers related to this topic: $[3, 6, 10, 12, 16–18, 22, 25–33, 35]$.

Recently, H. S. Ramane et al. in [29], presented several examples of non-self-complementary regular graphs satisfying $E(G) = E(\overline{G})$. Besides, the authors of [29] asked about structural and spectral properties of graphs, not necessary regular, which are equienergetic with their complements. Using spectral distances of graphs, we are going to expose some spectral characteristics of regular graphs which are equienergetic with their complements. Due to obtained results, an existing set of examples of equienergetic regular graphs from [26] and [29] will be supplemented with the new ones.

The paper is organized as follows. In Section 2, we give a necessary and sufficient condition for a regular graph to have the energy equal to the energy of its complement. Based on this result, strongly regular graphs equienergetic with their complements are characterized, and a spectral property that two equienergetic regular graphs should possess in order for their complements to have equal energies, is stated. In Section 3, equienergetic regular graphs with respect to some frequently used graph operations are discussed, and some appropriate examples are exposed. The paper is concluded with few remarks in Section 4.

2 Main results

Let G be a r-regular graph of order n whose spectrum regarding the adjacency matrix is: $r = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. In [8], the following theorem has been stated:

Theorem 2.1 *(Theorem 2.1.2 from [8]) If G is a regular graph of degree r with n vertices, then:*

$$P_{\overline{G}}(x) = (-1)^n \, \frac{x - n + r + 1}{x + r + 1} \, P_G(-x - 1).$$

This means that the complement $\overline{G}$ of G is $(n-1-r)$-regular, and that the adjacency spectrum of $\overline{G}$ is: $n - 1 - r \geq -1 - \lambda_n(G) \geq -1 - \lambda_{n-1}(G) \geq \cdots \geq -1 - \lambda_2(G)$.

Since the adjacency spectrum of the complete n-vertex graph K_n consists of the eigenvalues: $n - 1$ and $[-1]^{n-1}$, we find:

$$\sigma(G, K_n) = \sum_{i=1}^{n} |\lambda_i(G) - \lambda_i(K_n)| = |n - 1 - r| + \sum_{i=2}^{n} | - 1 - \lambda_i(G)| = E(\overline{G}), \qquad (1)$$

and

$$\sigma(\overline{G}, K_n) = \sum_{i=1}^{n} |\lambda_i(\overline{G}) - \lambda_i(K_n)| = |n - 1 - (n - 1 - r)| + \sum_{i=2}^{n} | - 1 - (-1 - \lambda_i(G))| = E(G). \qquad (2)$$

In [20], the following theorem has been proved:

Theorem 2.2 *(Theorem 3.4 from [20]) Let G be a n-vertex graph with n^* adjacency eigenvalues which are greater than or equal to -1. Then:*

$$\sigma(G, K_n) = 2\left(n^* - 1 + \sum_{i=2}^{n^*} \lambda_i(G)\right). \tag{3}$$

Now, we can prove the following statement:

Theorem 2.3 *Let G be a r-regular graph of order n, with the following adjacency eigenvalues: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let $n^* = n^*(G)$ be the number of eigenvalues of G which are greater than or equal to -1, among which there are $n_1 = n_1(G)$ non-negative eigenvalues. Then:*

$$E(G) = E(\overline{G}) \ \textit{if and only if}\ \ n^* - 1 + \sum_{i=n_1+1}^{n^*} \lambda_i(G) = r. \tag{4}$$

Proof. According to (1) and (2), we have:

$$E(G) = E(\overline{G}) \Leftrightarrow \sigma(\overline{G}, K_n) = \sigma(G, K_n).$$

Using Theorem 2.2, we further find:

$$r + \sum_{i=2}^{n} |\lambda_i(G)| = 2\left(n^* - 1 + \sum_{i=2}^{n^*} \lambda_i(G)\right) \Leftrightarrow$$

$$r + \sum_{i=2}^{n_1} \lambda_i(G) - \sum_{i=n_1+1}^{n^*} \lambda_i(G) - \sum_{i=n^*+1}^{n} \lambda_i(G) = 2n^* - 2 + 2\sum_{i=2}^{n_1} \lambda_i(G) + 2\sum_{i=n_1+1}^{n^*} \lambda_i(G) \Leftrightarrow$$

$$r = 2n^* - 2 + \sum_{i=2}^{n} \lambda_i(G) + 2\sum_{i=n_1+1}^{n^*} \lambda_i(G).$$

Since $\sum_{i=1}^{n} \lambda_i(G) = 0$, we finally obtain: $r = 2n^* - 2 - \lambda_1(G) + 2\sum_{i=n_1+1}^{n^*} \lambda_i(G)$, i.e. $r = n^* - 1 + \sum_{i=n_1+1}^{n^*} \lambda_i(G)$. ■

Remark 2.1 *If in the spectrum of G there are no eigenvalues in the interval $[-1, 0)$, the relation (4) reduces to: $E(G) = E(\overline{G})$ if and only if $n^* - 1 = r$.*

Comment 2.1 *For the complete multipartite graph $K_{n\times n}$ with n parts of size n, where $n \geq 2$, i.e. for the graph $\overline{n\,K_n}$, whose the eigenvalues are: $n^2 - n$, $[0]^{n^2-n}$ and $[-n]^{n-1}$, and which is inspected in Theorem 2.2 in [29], we have $n^* = n^2 - n + 1$, $n_1 = n^*$, $\sum_{i=n_1+1}^{n^*} \lambda_i(K_{n\times n}) = 0$ and $r = n^2 - n$. This means that the right hand side of (4) is satisfied, so $E(K_{n\times n}) = E(\overline{K_{n\times n}})$.*

The line graph $L(K_{p,q})$ of the complete bipartite graph $K_{p,q}$, for $p, q \geq 2$ is considered in Theorem 2.4 in [29]. The spectrum of this graph is: $p+q-2$, $[p-2]^{q-1}$, $[q-2]^{p-1}$ and $[-2]^{(p-1)(q-1)}$, so it holds $n^* = p+q-1$, $n_1 = n^*$, $\sum_{i=n_1+1}^{n^*} \lambda_i(L(K_{p,q})) = 0$ and $r = p+q-2$. Since the right hand side of (4) is satisfied, it follows $E(L(K_{p,q})) = E(\overline{L(K_{p,q})})$.

Recall that a *strongly regular graph* G with parameters (n, r, e, f) is a r-regular graph on n vertices in which any two adjacent vertices have exactly e common neighbours, and any two non-adjacent vertices have exactly f common neighbours. For more details about strongly regular graphs, see Section 3.6. in [8].

Let $G = SRG(n, r, e, f)$ be a connected strongly regular graph, different from the complete graph K_n. The adjacency spectrum of G (Theorem 3.6.5 from [8]) consists of: r, $[s]^k$ and $[t]^l$, where $s, t = \frac{1}{2}\left((e-f) \pm \sqrt{\Delta}\right)$; $k, l = \frac{1}{2}\left(n - 1 \mp \frac{2r+(n-1)(e-f)}{\sqrt{\Delta}}\right)$, and $\Delta = (e-f)^2 + 4(r-f)$. It holds that $s \geq 0$, and, since G is different from K_n, that $t < -1$. Therefore, we have $n^* = n^*(G) = k+1$ and $n_1 = n_1(G) = n^*$, wherefrom it follows $n^* - 1 + \sum_{i=n_1+1}^{n^*} \lambda_i(G) = k$. In that way, using Theorem 2.3, we immediately obtain the following statement:

Corollary 2.1 *Let $G = SRG(n, r, e, f)$ be a connected strongly regular graph different from the complete graph K_n, and with the spectrum: r, $[s]^k$ and $[t]^l$. Then,*

$$E(G) = E(\overline{G}) \;\; if\; and\; only\; if \;\; k = r.$$

Remark 2.2 *Corollary 2.1 is a generalization of results given by Theorem 2.10 and Theorem 2.11 in [29].*

Example 2.1 *Shrikhande graph $G_1 = SRG(16, 6, 2, 2)$ of order 16 and Hall-Janko graph $G_2 = SRG(100, 36, 14, 12)$ of order 100, are examples of strongly regular graphs which are equienergetic with their complements. Spectra of these graphs are $6, [2]^6, [-2]^9$ and 36, $[6]^{36}$, $[-4]^{63}$, respectively. Notice that these two graphs are not cospectral with their complements.*

Using Corollary 2.1, we can obtain the following:

Corollary 2.2 *Let G be a connected strongly regular graph different from the complete graph K_n, with parameters (n, r, e, f). Then,*

$$E(G) = E(\overline{G}) \;\; if\; and\; only\; if \;\; n = 1 + \frac{2r\,(\sqrt{\Delta}+1)}{\sqrt{\Delta} - e + f}.$$

Remark 2.3 *Somewhat different proof of the statement given by Corollary 2.2, with several appropriate examples, is exposed in [26] (see Theorem 3.2, Theorem 3.4, and Remark 3.1).*

Corollary 2.3 *Let G_i, $i = 1, 2$, be two r_i regular graphs of order n whose the adjacency spectra are $\lambda_1(G_i) \geq \lambda_2(G_i) \geq \cdots \geq \lambda_n(G_i)$. Let us suppose that:*

$$-r_1 + r_2 + n_1^* - n_2^* + \sum_{j=n_1^{**}+1}^{n_1^*} \lambda_j(G_1) - \sum_{j=n_2^{**}+1}^{n_2^*} \lambda_j(G_2) = 0, \tag{5}$$

where n_i^{**} are the numbers of non-negative eigenvalues of G_i, and n_i^* are the numbers of eigenvalues which are greater than or equal to -1. If $E(G_1) = E(G_2)$, then $E(\overline{G_1}) = E(\overline{G_2})$.

Proof. If $E(G_1) = E(G_2)$, then, according to (2), $\sigma(\overline{G_1}, K_n) = \sigma(\overline{G_2}, K_n)$. Using Theorem 2.1, for $i = 1, 2$, we compute:

$$\sigma(\overline{G_i}, K_n) = \sum_{j=1}^{n} |\lambda_j(\overline{G_i}) - \lambda_j(K_n)| = r_i + \sum_{j=2}^{n} |\lambda_j(G_i)| = r_i + \sum_{j=2}^{n_i^{**}} \lambda_j(G_i) - \sum_{j=n_i^{**}+1}^{n} \lambda_j(G_i).$$

Now, the equality $\sigma(\overline{G_1}, K_n) = \sigma(\overline{G_2}, K_n)$ becomes:

$$\sum_{j=2}^{n_1^{**}} \lambda_j(G_1) - \sum_{j=n_1^{**}+1}^{n} \lambda_j(G_1) = r_2 - r_1 + \sum_{j=2}^{n_2^{**}} \lambda_j(G_2) - \sum_{j=n_2^{**}+1}^{n} \lambda_j(G_2). \tag{6}$$

For $i = 1, 2$, we have:

$$\begin{aligned} \sigma(G_i, K_n) &= n - 1 - r_i + \sum_{j=2}^{n_i^*} (1 + \lambda_j(G_i)) + \sum_{j=n_i^*+1}^{n} (-1 - \lambda_j(G_i)) \\ &= -2 - r_i + 2n_i^* + \sum_{j=2}^{n_i^*} \lambda_j(G_i) - \sum_{j=n_i^*+1}^{n} \lambda_j(G_i) \\ &= -2 - r_i + 2n_i^* + \sum_{j=2}^{n_i^{**}} \lambda_j(G_i) + 2 \sum_{j=n_i^{**}+1}^{n_i^*} \lambda_j(G_i) - \sum_{j=n_i^{**}+1}^{n} \lambda_j(G_i). \end{aligned} \tag{7}$$

By substituting (6) into (7) for $i = 1$, we obtain:

$$\sigma(G_1, K_n) = -2 - 2r_1 + r_2 + 2n_1^* + 2 \sum_{j=n_1^{**}+1}^{n_1^*} \lambda_j(G_1) + \sum_{j=2}^{n_2^{**}} \lambda_j(G_2) - \sum_{j=n_2^{**}+1}^{n} \lambda_j(G_2). \tag{8}$$

Using the relation (7) for $i = 2$, we find:

$$\sum_{j=2}^{n_2^{**}} \lambda_j(G_2) - \sum_{j=n_2^{**}+1}^{n} \lambda_j(G_2) = \sigma(G_2, K_n) + 2 + r_2 - 2n_2^* - 2 \sum_{j=n_2^{**}+1}^{n_2^*} \lambda_j(G_2). \qquad (9)$$

By substituting (9) into (8), we finally get:

$$\sigma(G_1, K_n) = 2 \cdot \left(-r_1 + r_2 + n_1^* - n_2^* + \sum_{j=n_1^{**}+1}^{n_1^*} \lambda_j(G_1) - \sum_{j=n_2^{**}+1}^{n_2^*} \lambda_j(G_2) \right) + \sigma(G_2, K_n). \qquad (10)$$

Having in mind the assumption (5) of the statement, the relation (10) reduces to:

$$\sigma(G_1, K_n) = \sigma(G_2, K_n),$$

i.e. $E(\overline{G_1}) = E(\overline{G_2})$. ■

Remark 2.4 *If G_1 and G_2 are integral, then $\sum_{j=n_1^{**}+1}^{n_1^*} \lambda_j(G_1)$ and $\sum_{j=n_2^{**}+1}^{n_2^*} \lambda_j(G_2)$ represent the sums of eigenvalues which are equal to -1. In that case, the relation (5) is of the following form:*

$$-r_1 + n_1^{**} = -r_2 + n_2^{**}. \qquad (11)$$

The result given below by Proposition 2.2, will be used in the next section to demonstrate Corollary 2.3. But, before we prove it, we will submit two statements which are relevant for the proof.

Proposition 2.1 *(Corollary 3.2.2 from [8]) A graph G of order n with the adjacency eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, is regular (of degree λ_1) if and only if*

$$n\lambda_1 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2.$$

The statement expressed by Theorem 2.4 is known as the *Courant-Weyl inequalities.* As usual, we suppose that the eigenvalues are in non-increasing order.

Theorem 2.4 *(Theorem 1.3.15 from [8]) Let A and B be $n \times n$ Hermitian matrices. Then:*

$$\lambda_i(A + B) \leq \lambda_j(A) + \lambda_{i-j+1}(B), \; n \geq i \geq j \geq 1; \qquad (12)$$

$$\lambda_i(A + B) \geq \lambda_j(A) + \lambda_{i-j+n}(B), \; 1 \leq i \leq j \leq n. \qquad (13)$$

Proposition 2.2 *Let us denote by* $V\left(K_{\frac{n}{2},\frac{n}{2}}\right) = V_1 \cup V_2$, *where* $V_1 = \{x_i \,|\, 1 \leq i \leq \frac{n}{2}\}$ *and* $V_2 = \{y_i \,|\, 1 \leq i \leq \frac{n}{2}\}$, *the vertex set of the complete bipartite graph* $K_{\frac{n}{2},\frac{n}{2}}$ *of order* n, *such that* $n \geq 12$ *and* $n = 4\,q$, *for* $q \in \mathbb{Z}^+$. *Let* G *be the graph obtained by deleting* $\frac{n}{2}$ *perfect matching edges* $\{x_i y_i \,|\, 1 \leq i \leq \frac{n}{2}\}$ *from the complete bipartite graph* $K_{\frac{n}{2},\frac{n}{2}}$, *and let* H *be the graph obtained by deleting* $\frac{n}{2}$ *perfect matching edges* $\{x_i y_{i+1} : i - odd\} \cup \{x_i y_{i-1} : i - even\}$ *from* G *(see Figure 1). Then the adjacency spectrum of* H *is:* $\frac{n}{2} - 2$, $[2]^{\frac{n}{4}-1}$, $[0]^{\frac{n}{2}}$, $[-2]^{\frac{n}{4}-1}$, $-\frac{n}{2} + 2$.

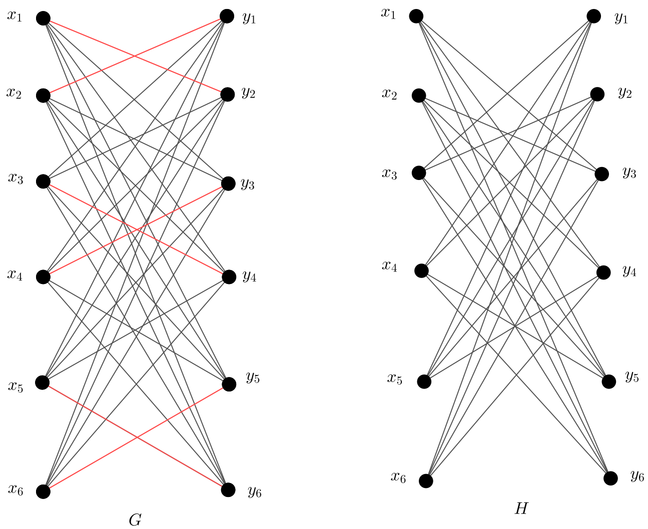

Figure 1. Graphs G and H of order $n = 12$

Proof. Let $\lambda_1(H) \geq \lambda_2(H) \geq \cdots \geq \lambda_n(H)$ be the adjacency eigenvalues of H. Obviously, H is regular and bipartite, so

$$\lambda_1(H) = -\lambda_n(H) = \frac{n}{2} - 2. \tag{14}$$

The adjacency matrix $A(H)$ of the graph H is $A(H) = A\left(K_{\frac{n}{2},\frac{n}{2}}\right) - M$, where $A\left(K_{\frac{n}{2},\frac{n}{2}}\right)$ is the adjacency matrix of the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$, while $M = \begin{pmatrix} O & N \\ N & O \end{pmatrix}$.

Here, N is the square matrix of order $\frac{n}{2}$ and of the following form:

$$N = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

It is not difficult to check that the spectrum of M is: $[2]^{\frac{n}{4}}$, $[0]^{\frac{n}{2}}$ and $[-2]^{\frac{n}{4}}$.

Having in mind that the eigenvalues of $K_{\frac{n}{2},\frac{n}{2}}$ are: $\frac{n}{2}$, $[0]^{n-2}$, $-\frac{n}{2}$, the Courant-Weyl inequality (12) for $j=2$ reduces to: $\lambda_i(H) \le \lambda_{i-1}(M),\ 2 \le i \le n$, wherefrom we find:

$$\begin{aligned} &\lambda_i(H) \le 2,\ 2 \le i \le \frac{n}{4}+1, \\ &\lambda_i(H) \le 0,\ \frac{n}{4}+2 \le i \le \frac{3n}{4}+1, \\ &\lambda_i(H) \le -2,\ \frac{3n}{4}+2 \le i \le n. \end{aligned} \tag{15}$$

The Courant-Weyl inequality (13) for $j = n-1$ becomes: $\lambda_i(H) \ge \lambda_{i+1}(M)$, where $1 \le i \le n-1$, wherefrom we get:

$$\begin{aligned} &\lambda_i(H) \ge 2,\ 1 \le i \le \frac{n}{4}-1, \\ &\lambda_i(H) \ge 0,\ \frac{n}{4} \le i \le \frac{3n}{4}-1, \\ &\lambda_i(H) \ge -2,\ \frac{3n}{4} \le i \le n-1. \end{aligned} \tag{16}$$

Now, from (15) and (16) it follows:

$$\begin{aligned} &\lambda_i(H) = 2,\ 2 \le i \le \frac{n}{4}-1, \\ &\lambda_i(H) = 0,\ \frac{n}{4}+2 \le i \le \frac{3n}{4}-1, \\ &\lambda_i(H) = -2,\ \frac{3n}{4}+2 \le i \le n-1. \end{aligned} \tag{17}$$

For the remaining four eigenvalues of H, we have:

$$\begin{aligned} &\lambda_{\frac{n}{4}}(H) = -\lambda_{\frac{3n}{4}+1}(H), \\ &\lambda_{\frac{n}{4}+1}(H) = -\lambda_{\frac{3n}{4}}(H). \end{aligned} \tag{18}$$

Since H is regular, using Proposition 2.1 and Equalities (14), (17) and (18), we find:

$$\lambda^2_{\frac{n}{4}}(H) + \lambda^2_{\frac{n}{4}+1}(H) = 4. \tag{19}$$

On the other hand, the adjacency matrix $A(H)$ can be considered as: $A(H) = \begin{pmatrix} O & P \\ P & O \end{pmatrix}$, where P is the square matrix of order $\frac{n}{2}$ and of the form as follows:

$$P = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

It is obvious that the rank of the matrix P is $\frac{n}{4}$, i.e. that the rank of the matrix $A(H)$ is $\frac{n}{2}$.

According to the well-known theorem from linear algebra, which says that if $\mathcal{M}$ is a square real and symmetric matrix, then the rank of $\mathcal{M}$ is equal to the total number of non-zero eigenvalues of $\mathcal{M}$, it follows that in the spectrum of H there are $\frac{n}{2}$ non-zero eigenvalues. Taking into account (14), (17) and (18), we conclude that $\lambda_{\frac{n}{4}}(H)$ and $\lambda_{\frac{3n}{4}+1}(H)$ are the remaining two non-zero eigenvalues of H. From (19) we get $\lambda_{\frac{n}{4}}(H) = -\lambda_{\frac{3n}{4}+1}(H) = 2$, and the proof follows. ■

Remark 2.5 *Let us notice that H is integral, and that the energy of H is $E(H) = 2n-8$. Following the labels as in Corollary 2.3, we have $r_H = \frac{n}{2} - 2$ and $n_H^{**} = \frac{3n}{4}$, and therefore $-r_H + n_H^{**} = \frac{n}{4} + 2$.*

3 Equienergetic regular graphs with respect to some graph operations

3.1 Sum of graphs

Let G_1 and G_2 be two graphs with vertex sets $V(G_1)$ and $V(G_2)$, respectively. The *sum* (i.e. *Cartesian product*) of G_1 and G_2 is the graph $G_1 + G_2$ whose the vertex set is $V(G_1) \times V(G_2)$, and in which two vertices (u_1, v_1) and (u_2, v_2), where $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$, are adjacent if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2, or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$. The sum of two graphs is a special case of a very general graph operation called NEPS, i.e. *non-complete extended p-sum* of graphs (for details, see Section 2.5 in [8]). The adjacency spectrum of $G_1 + G_2$ is given by the following theorem (see Theorem 2.5.4 in [8]):

Theorem 3.1 *If $\lambda_1(G_1), \lambda_2(G_1), \ldots, \lambda_n(G_1)$ and $\lambda_1(G_2), \lambda_2(G_2), \ldots, \lambda_m(G_2)$ are the eigenvalues of two graphs G_1 and G_2, respectively, then $\lambda_i(G_1)+\lambda_j(G_2)$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$ are the eigenvalues of the graph $G_1 + G_2$.*

Example 3.1 (for Corollary 2.3) *Let us denote by $G^\circ = 2\left(K_2 + K_{\frac{n}{4}}\right)$, where $n \geq 12$ and n is divisible by 4. Since the spectrum of K_n consists of: $n-1$ and $[-1]^{n-1}$, using Theorem 3.1, we easily find the spectrum of G°: $\left[\frac{n}{4}\right]^2$, $\left[\frac{n}{4}-2\right]^2$, $[0]^{\frac{n}{2}-2}$, $[-2]^{\frac{n}{2}-2}$, and its energy: $E(G^\circ) = 2n-8$. Following the labels as in Corollary 2.3, we also have $r_{G^\circ} = \frac{n}{4}$ and $n^{**}_{G^\circ} = \frac{n}{2}+2$, and therefore $-r_{G^\circ} + n^{**}_{G^\circ} = \frac{n}{4}+2$.*

The regular integral graphs G° and H (see Remark 2.5 and Figure 2) have equal energies and they satisfy the relation (11). Therefore, according to Corollary 2.3, it holds $E(\overline{H}) = E(\overline{G^\circ})$ (see Figure 3).

Indeed, the spectrum of $\overline{H}$ is: $\frac{n}{2}+1$, $\frac{n}{2}-3$, $[1]^{\frac{n}{4}-1}$, $[-1]^{\frac{n}{2}}$, $[-3]^{\frac{n}{4}-1}$, and therefore $E(\overline{H}) = \frac{5n}{2}-6$. Since the spectrum of $\overline{G^\circ}$ consists of: $\frac{3n}{4}-1$, $[1]^{\frac{n}{2}-2}$, $[-1]^{\frac{n}{2}-2}$, $\left[-\frac{n}{4}+1\right]^2$, $-\frac{n}{4}-1$, we calculate $E(\overline{G^\circ}) = \frac{5n}{2}-6$.

Notice that the complement $\overline{G^\circ}$ of the graph G° is isomorphic to the graph which is obtained by deleting appropriate $\frac{n}{2}$ perfect matching edges from the complete 4-partite graph $K_{4\times\frac{n}{4}}$, where $n \geq 12$ and $n = 4\,q$, $q \in \mathbb{Z}^+$. Also, if $n = 8$, H is cospectral with $2\,(K_2 + K_2)$.

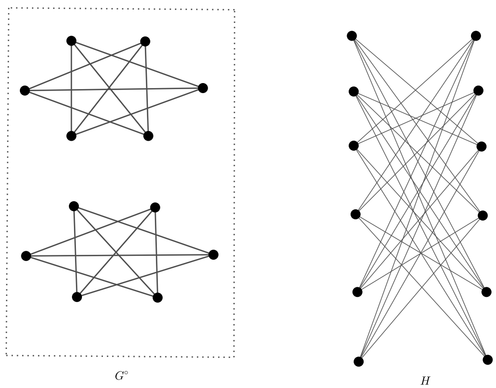

Figure 2. Equienergetic graphs G° and H of order $n = 12$

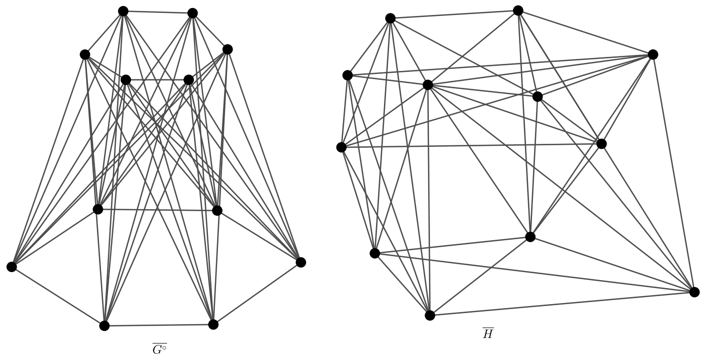

Figure 3. Equienergetic complements $\overline{G^\circ}$ and $\overline{H}$ for $n = 12$

In [29], some examples of regular graphs G with the property $E(G+K_2) = E(\overline{G+K_2})$ are exhibited. Using Theorem 2.3, we can obtain a spectral characterization of such graphs:

Corollary 3.1 *Let G be a r-regular graph of order n whose the adjacency eigenvalues are: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let us denote $I_{-1} = \{i \in \{1,2,\ldots,n\} : \lambda_i(G) \in [-2,-1)\}$ and $I_{+1} = \{i \in \{1,2,\ldots,n\} : \lambda_i(G) \in [0,1)\}$, and let us suppose that in the spectrum of G there are n_{-2} eigenvalues which are greater than or equal to -2, and that among them there are n_0 non-negative eigenvalues. Then $E(G+K_2) = E(\overline{G+K_2})$ if and only if*

$$n_{-2} + n_0 + \sum_{i \in I_{-1} \cup I_{+1}} \lambda_i(G) + |I_{-1}| - |I_{+1}| = r + 2, \tag{20}$$

where $|I|$ denotes the cardinality of the set I.

Proof. Using Theorem 3.1, we find the spectrum of $G+K_2$: $\lambda_i(G)+1$ and $\lambda_i(G)-1$, for $i = 1,2,\ldots,n$. Therefore, the number n^* of eigenvalues of this graph which are greater than or equal to -1 is $n^* = n_{-2} + n_0$. Eigenvalues of $G+K_2$ which belong to the interval $[-1,0)$ are related to eigenvalues $\lambda_i(G)$ for $i \in I_{-1} \cup I_{+1}$. Therefore, according to Theorem 2.3, the equality $E(G+K_2) = E(\overline{G+K_2})$ is equivalent to:

$$n^* - 1 + \sum_{i \in I} \lambda_i(G+K_2) = r + 1 \;\Leftrightarrow$$

$$n_{-2} + n_0 + \sum_{i \in I_{-1}} (\lambda_i(G)+1) + \sum_{i \in I_{+1}} (\lambda_i(G)-1) = r + 2 \;\Leftrightarrow$$

$$n_{-2} + n_0 + \sum_{i \in I_{-1} \cup I_{+1}} \lambda_i(G) + |I_{-1}| - |I_{+1}| = r + 2,$$

where $I = \{i \in \{1, 2, \ldots, 2n\} : \lambda_i(G + K_2) \in [-1, 0)\}$. ■

The following example is an addition to the examples exposed in [29].

Proposition 3.1 *If $G = SRG(d^2\,(d+2), d\,(d^2+d-1), d\,(d^2-1), d\,(d^2-1))$, where $d > 2$, then, $E(G + K_2) = E(\overline{G + K_2})$.*

Proof. By direct computation, we find the spectrum of G: $d\,(d^2+d-1)$, $[d]^{\frac{d}{2}\,(d^2+d-1)}$ and $[-d]^{\frac{1}{2}\,(d^3+3d^2+d-2)}$. Following the labels as in Corollary 3.1, since $d > 2$, we have $n_{-2} = 1 + \frac{d}{2}\,(d^2+d-1)$, $n_0 = n_{-2}$, $r = d\,(d^2+d-1)$, $|I_{-1}| = |I_{+1}| = 0$ and $\sum_{i \in I_{-1} \cup I_{+1}} \lambda_i(G) = 0$, i.e.

$$n_{-2} + n_0 + \sum_{i \in I_{-1} \cup I_{+1}} \lambda_i(G) + |I_{-1}| - |I_{+1}| = 2 + d\,(d^2 + d - 1) = r + 2,$$

which means, according to Corollary 3.1, that $E(G + K_2) = E(\overline{G + K_2})$. ■

Remark 3.1 *The energies of $G+K_2$ and $\overline{G + K_2}$ are equal to $E(G+K_2) = E(\overline{G + K_2}) = 2d\,(d^3+3d^2+d-2)$, since the adjacency eigenvalues of these graphs are: $d\,(d^2+d-1)+1$, $d\,(d^2+d-1)-1$, $[d+1]^{\frac{d}{2}(d^2+d-1)}$, $[d-1]^{\frac{d}{2}(d^2+d-1)}$, $[-d+1]^{\frac{1}{2}\,(d^3+3d^2+d-2)}$, $[-d-1]^{\frac{1}{2}\,(d^3+3d^2+d-2)}$, and d^3+3d^2+d-2, $-d\,(d^2+d-1)$, $[-d-2]^{\frac{d}{2}\,(d^2+d-1)}$, $[-d]^{\frac{d}{2}\,(d^2+d-1)}$, $[d-2]^{\frac{1}{2}\,(d^3+3d^2+d-2)}$, $[d]^{\frac{1}{2}\,(d^3+3d^2+d-2)}$, respectively. Notice that $G + K_2$ and $\overline{G + K_2}$ are not cospectral graphs.*

Example 3.2 *It was proved that for $d = 8$, the strongly regular graph G with parameters $(640, 568, 504, 504)$ exists (see [24]). In that case, we have $E(G + K_2) = E(\overline{G + K_2}) = 11360$.*

3.2 Line graph

In constructing pairs of equienergetic graphs, graph operation named line graph has been frequently used (see, for example, [27], [28], [29], [31]).

The proof of the following statement can be found in [8] (see Theorem 2.4.1):

Theorem 3.2 *If G is a regular graph of degree r, with n vertices, $m = \frac{n\,r}{2}$ edges, and the adjacency spectrum: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, then the adjacency spectrum of $L(G)$ is: $r - 2 + \lambda_i(G)$, $i = 1, 2, \ldots, n$, and $[-2]^{\frac{n\,(r-2)}{2}}$.*

Using Theorem 2.3, we can prove the following statement:

Corollary 3.2 *Let G be a r-regular graph of order n, where $n \geq 4$, and with the adjacency eigenvalues: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let us suppose that $\lambda_i(G) \geq -r+2$, for each $i = 1, 2, \ldots, n$. Then:*

$$E(L(G)) = E(\overline{L(G)}) \ \ if\ and\ only\ if \ \ r = \frac{n+1}{2}.$$

Proof. According to Theorem 3.2, $L(G)$ is $(2r-2)$-regular, with n non-negative eigenvalues, and without eigenvalues in the interval $[-1, 0)$, i.e. $\sum_{\lambda_i(L(G)) \in [-1,0)} \lambda_i(L(G)) = 0$. Hence, the relation (4) is of the following form:

$$E(L(G)) = E(\overline{L(G)}) \Leftrightarrow n - 1 + \sum_{\lambda_i(L(G)) \in [-1,0)} \lambda_i(L(G)) = 2r - 2,$$

i.e.

$$E(L(G)) = E(\overline{L(G)}) \Leftrightarrow r = \frac{n+1}{2}.$$

■

Various examples of regular graphs which satisfy Corollary 3.2 may be found. One such graph is the strongly regular graph $G = SRG(35, 18, 9, 9)$ (see [9]). The spectrum of G consists of: 18, $[3]^{14}$ and $[-3]^{20}$, which means that $E(L(G)) = E(\overline{L(G)}) = 1120$. Notice that $L(G)$ and $\overline{L(G)}$ are not cospectral graphs.

The similar situation is with the strongly regular graph $H = SRG(15, 8, 4, 4)$, whose spectrum is: 8, $[2]^5$, $[-2]^9$. Actually, H is the triangular graph T_6, which is isomorphic to $L(K_6)$.

In [28], it has been proved that if H^* is a r-regular graph of order n, where $r \geq 3$, then

$$E(L^2(H^*)) = E(\overline{L^2(H^*)}) \tag{21}$$

if and only if $H^* = K_6$.

The uniqueness of the graph which satisfies (21) can be proved using Theorem 2.3. Namely, according to Theorem 3.2, the spectrum of $L^2(H^*)$ is: $\lambda_i(H^*) + 3r - 6$, for $i = 1, 2, \ldots, n$, $[2r-6]^{\frac{n\,(r-2)}{2}}$ and $[-2]^{\frac{nr\,(r-2)}{2}}$. Since $r \geq 3$, then $2r - 6 \geq 0$. Eigenvalues $\lambda_i(H^*) + 3r - 6$ are in the interval $[2r-6, 4r-6]$, i.e. they are non-negative for each $i = 1, 2, \ldots, n$. Therefore, in the spectrum of $L^2(H^*)$ there are no eigenvalues in the interval $[-1, 0)$. Then, according to Theorem 2.3, Equality (21) is equivalent to

$$(n-8)\,r = -10. \tag{22}$$

Since, $r \geq 3$, Equality (22) is satisfied for $n = 6$ and $n = 7$. Because 10-regular graph of order 7 does not exist, it follows that K_6 is the unique regular graph which satisfies (22). Besides, it is such that $L^2(K_6)$ and $\overline{L^2(K_6)}$ are not cospectral.

4 Concluding remarks

The exposed results, in a certain way, represent a generalization of results exhibited in [29]. Although simple, we hope that the listed relations and examples may be useful in constructing new pairs of equienergetic regular graphs and that they will facilitate their further study. Namely, starting from an arbitrary regular graph G and applying on it some unary graph operations (such as complement and line graph), as well as some binary graph operations using other regular graphs (such as Cartesian product), we can identify whether thus obtained regular graphs are equienergetic (with G) just knowing the spectrum of the underlying graph G.

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